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Bézier variant of modified α-Bernstein operators

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Abstract
In the present paper, we introduce the Bézier variant of modified α-Bernstein operators and study the degree of approximation using second order modulus of continuity. We also establish a direct approximation theorem with the aid of Ditzian-Totik modulus of smoothness and the Peetre's K-functional. Further, we obtain a quantitative Voronovskaja type theorem and the rate of convergence for functions with a derivative of bounded variation on [0, 1]. Finally, we depict the rate of convergence of these operators for certain functions by graphical illustration using Matlab software.

Keywords Bézier operators · Modified α-Bernstein operators · Modulus of continuity · Ditizian–Totik modulus of smoothness · Rate of convergence · Bounded variation · Voronovskaja theorerm

Mathematics Subject Classification 41A10 · 41A25 · 41A30 · 41A63 · 26A15

1 Introduction

The most consequential result in approximation theory is the Weierstrass theorem, which affirmed that any continuous function f on the finite interval $[a, b]$ can be approximated with the help of sequence of polynomials which converges uniformly to f on $[a, b]$. There are a number of proofs of this theorem. Bernstein [4] gave the most simplest and constructive proof of Weierstrass theorem using Bernstein polynomials. For $f \in C(I)$, the space of all continuous functions on $I = [0, 1]$, the Bernstein polynomials are defined as follows:

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$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x), x \in I, \tag{1.1}$$

where

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, n = 1, 2, 3, \ldots$$

Bernstein operators exhibits many noticeable features but its efficient properties get overshadowed by its slow convergence which acts as a barrier to its use from the numerical point of view, inspite of its simplicity and popularity. Thus in order to improve the rate of approximation by these operators (1.1), Khosravian-Arab et al. [19] presented operators of Bernstein type with order of approximation one, two and three and studied some theoretical results concerning the rate of convergence and the asymptotic behaviour together with some applications of these operators. This new approach to Bernstein operators involving first-order approximation is given by:

$$B_n^{M,1}(f;x) = \sum_{k=0}^{n} b_{n,k}^{M,1}(x) f\left(\frac{k}{n}\right), x \in I, \tag{1.2}$$

where $b_{n,k}^{M,1}(x) = a(x,n) b_{n-1,k}(x) + a(1-x,n) b_{n-1,k-1}(x)$ and $a(x,n) = a_1(n)x + a_0(n)$, $n = 1, 2, \ldots$. $a_i(n), i = 0, 1, 2, \ldots$ are two unknown sequences which are to be determined later. If we put $a_1(n) = -1,\ a_0(n) = 1$, the operators (1.2) reduce to the classical Bernstein operators (1.1).

Chen et al. [10] modified Bernstein operators by including a real fixed parameter α as follows:

$$B_{n,\alpha}(f;x) = \sum_{k=0}^{n} b_{n,k,\alpha}(x) f\left(\frac{k}{n}\right),\ x \in I, \tag{1.3}$$

where $\alpha \in I$ and $b_{n,k,\alpha}(x)$ satisfies

$$\begin{aligned} b_{n,k,\alpha}(x) = & \binom{n-2}{k}(1-\alpha)x + \binom{n-2}{k-2}(1-\alpha)(1-x) \\ & + \binom{n}{k}\alpha x(1-x)x^{k-1}(1-x)^{n-k-1}, n \geq 2, \end{aligned}$$

with the recursion relation given by

$$b_{n,k,\alpha}(x) = (1-x) b_{n-1,k,\alpha}(x) + x b_{n-1,k-1,\alpha}(x), k = 0, 1, 2, \ldots n.$$

Various reseachers worked on the operators (1.3) and obtained several interesting results in local and global approximation (cf. [17, 21]). Note that for $\alpha = 1$, the α-Bernstein operators given by (1.3) include Bernstein operators.

Çetin and Radu [8] introduced a Stancu type generalization of the α-Bernstein operators defined by (1.3) and obtained the rate of convergence in terms of the modulus of continuity, Voronskaja and Grüss-Voronskaja asymptotic results and a Grüss-type inequality by means of second order modulus of continuity. Acu and Radu [3] constructed a new family of operators by linking the $\alpha-$Bernstein operators (1.3) and the genuine $\alpha-$Bernstein Durrmeyer operators and studied the approximation degree by means of modulus of continuity, a quantitative Voronskaja type theorem and some Grüss-type inequalities. Acar and Kajla [2] defined a bivariate extension of α-Bernstein operators and studied concerning convergence properties. Motivated by [19], Kajla and Acar [18] improved order of approximation of α-Bernstein operators defined in (1.3) by introducing modified α-Bernstein operators as:

$$B_{n,\alpha}^{M,1}(f;x) = \sum_{k=0}^{n} b_{n,k,\alpha}^{M,1}(x) f\left(\frac{k}{n}\right), x \in I, \tag{1.4}$$

where $b_{n,k,\alpha}^{M,1}(x) = a(x,n)b_{n-1,k,\alpha}(x) + a(1-x,n)b_{n-1,k-1,\alpha}(x)$ and $a(x,n)$ is already defined above.

Stancu [24] proposed the Bernstein type operators based on two parameters and gave its representation by a convex combination of second-order divided differences. In approximation theory, examining the smoothness of the approximating function from its order of approximation is as important as the rate of convergence of the approximation method for that function. In this context, Bézier of R$\acute{e}$gie Renault had played an important role. He was instrumental in developing a powerful system for designing free form curves and surfaces, popularly known as Bézier curves. These curves are generally used in computer graphics to produce smooth curves and are well suited to the geometric problems where smoothness is of utmost importance than any other consideration. Bézier curves make use of Bernstein polynomials as basis, so in this way Bézier described the mathematical basis for the system proposed by him and Bézier methods turned out to be an application of Bernstein polynomials (see [5, 6, 14]).

The aim of this paper is to introduce the Bézier variant of modified α-Bernstein operators (1.4) and to study its rate of convergence in terms of Ditizian-Totik modulus of smoothness and K-functional. We also obtain quantitative Voronovskaja type theorem and approximation of functions with a derivative of bounded variation on I. We also depict the rate of convergence of these operators for some function by graphical illustration using Matlab.

2 Construction of Bézier variant of modified α-Bernstein operators

Many researchers have worked on finding the Bézier variant of various important positive linear operators in approximation theory. The study in this direction began in 1972 when Bézier introduced a new basis function as $j_{n,k}(x) = \sum_{j=k}^{n} b_{n,j}(x)$. Chang [9] introduced Bernstein-Bézier operators and studied their convergence properties. Li and Gong [20] studied their convergence in terms of modulus of continuity. Tremendous contribution has been made by Zeng and his fellow workers (see e.g. [25, 26, 28]) in finding the Bézier variant of different operators. Zeng and Piriou [25] were the first, to study the rate of convergence of classical Bernstein polynomials and Bernstein-Kantorovich operators by considering better bounds. Zeng and Chen [27] introduced Bézier variant of Bernstein-Durrmeyer operators. Srivastava and Gupta [23] obtained the Bézier variant of Bleimann-Butzer-Hann operators defined in [7]. Zeng and Gupta [28] presented Bézier variant of well known Baskakov operators. Abel and Gupta [1] mentioned a Bézier variant of Baskakov-Kantorovich operators. Bézier variant of Baskakov-Durrmeyer operators was introduced by Gupta and Abel [15]. We refer the reader to the book [16] and the references therein, for a detailed study of the research work in this direction.

Inspired by the above research, we introduce the Bézier variant of modified α-Bernstein operators defined in (1.4), as follows:

$$B_{n,\alpha,\theta}^{M,1}(f;x)=\sum_{k=0}^{n}Q_{n,k,\alpha}^{(\theta)M,1}(x)f\left(\frac{k}{n}\right), \tag{2.1}$$

where $\theta\geq 1, Q_{n,k,\alpha}^{(\theta)M,1}(x)=\left[J_{n,k,\alpha}(x)\right]^{\theta}-\left[J_{n,k+1,\alpha}(x)\right]^{\theta}, J_{n,k,\alpha}(x)=\sum_{j=k}^{n}b_{n,j,\alpha}^{M,1}(x),$ for $k=0,1,\dots,n,$ are the Bézier basis functions.

3 Basic results

In order to prove the main results, we will need the following lemmas:

Lemma 1 [18] *Let* $e_r(x)=x^r, r\in\{0,1,2\}$. *For the operators* $B_{n,\alpha}^{M,1}(f;x)$ *defined in* (1.4), *we have*

(1) $B_{n,\alpha}^{M,1}\left(e_0;x\right)=2a_0(n)+a_1(n);$

(2) $B_{n,\alpha}^{M,1}\left(e_1;x\right)=x(2a_0(n)+a_1(n))+\dfrac{(1-2x)(a_0(n)+a_1(n))}{n};$

(3) $$B_{n,\alpha^{M,1}}\left(e_2;x\right)=x^2(2a_0(n)+a_1(n))+\frac{x(2a_0(n)(2-3x)+a_1(n)(3-5x))}{n}+\frac{(a_0(n)+a_1(n))-2x(1-x)(2\alpha a_0(n)+(1+\alpha)a_1(n))}{n^2}$$

Consequently,

Lemma 2 [18] *Let* $\lambda_{i,\alpha}(x)=B_{n,\alpha}^{M,1}\left((r-x)^i;x\right), i=1,2,4$. *For* $B_{n,\alpha}^{M,1}(f;x)$ *using Lemma* 1 , *we have*

(1) $\lambda_{1,\alpha}(x)=\frac{(1-2x)(a_0(n)+a_1(n))}{n};$

(2) $\lambda_{2,\alpha}(x)=\frac{x(2a_0(n)(2-3x)+a_1(n)(3-5x))-2x(1-2x)(a_0(n)+a_1(n))}{n}+\frac{(a_0(n)+a_1(n))-2x(1-x)(2\alpha a_0(n)+(1+\alpha)a_1(n))}{n^2};$

(3) $$\begin{aligned}\lambda_{4,\alpha}(x)=&\frac{3x^2\left(1-x^2\right)\left(2a_0(n)+a_1(n)\right)}{n^2}\\&+\frac{x(1-x)\left(12a_0(n)+11a_1(n)\right)-2x^2\left(1-x^2\right)\left(2a_0(n)(7+6\alpha)+(17+6\alpha)a_1(n)\right)}{n^3}\\&+\frac{\left(a_0(n)+a_1(n)\right)+x(1-x)\left(16(4\alpha-3)a_0(n)+2(17-25\alpha)a_1(n)\right)}{n^4}\\&+\frac{3x^2\left(1-x^2\right)\left((5-6\alpha)a_0(n)+8(9\alpha-7)a_1(n)\right)}{n^4}.\end{aligned}$$

Remark 1 In approximation theory, our first concern is to ensure the uniform convergence of the linear positive operators under consideration which requires to verify the conditions of the well-known Korovkin theorem. So, keeping in view this aspect, we assume that the sequences $a_i(n), i=0,1$ occuring in the operators (1.4), verify the condition that $2a_0(n)+a_1(n)=1$.

In our further study throughout, by $\|.\|$ we mean the uniform norm on $C(I)$.

Lemma 3 *For every $f \in C(I)$, we have* $\left\| B_{n,\alpha}^{M,1}(f;.) \right\| \leq \|f\|$.

Proof Applying the definition (1.4) and Lemma 1, the result is obvious. □

Lemma 4 [18] *For $B_{n,\alpha}^{M,1}(f;x)$, there hold the following results:*

$$(1) \lim_{n\to\infty} n\lambda_{2,\alpha}(x) = x(1-x);$$
$$(2) \lim_{n\to\infty} n^2\lambda_{4,\alpha}(x) = 3x^2(1-x)^2.$$

Consequently, for sufficiently large n, we have

$$\lambda_{2,\alpha}(x) \leq \frac{C_0 x(1-x)}{n}; \tag{3.1}$$

$$\lambda_{4,\alpha}(x) \leq \frac{C_0 x^2(1-x)^2}{n^2}, \tag{3.2}$$

where C_0 is a positive constant not necessarily the same.

Lemma 5 *If $f \in C(I)$, then for the modified α-Bernstein- Bézier operators defined by (2.1), for each $x \in I$ we have* $|B_{n,\alpha,\theta}^{M,1}(f;x)| \leq \|f\|$.

Proof From Lemma 1 and our assumption that $2a_0(n) + a_1(n) = 1$, we obtained $B_{n,\alpha,\theta}^{M,1}(e_0;x) = 1$. Hence,

$$\begin{aligned}
B_{n,\alpha,\theta}^{M,1}(e_0;x) &= \sum_{k=0}^{n} Q_{n,k,\alpha}^{(\theta)M,1}(x) \\
&= \sum_{k=0}^{n} \left\{ \left[J_{n,k,\alpha}(x)\right]^\theta - \left[J_{n,k+1,\alpha}(x)\right]^\theta \right\} \\
&= \left[J_{n,0,\alpha}(x)\right]^\theta \\
&= \left(\sum_{j=0}^{n} b_{n,j,\alpha}^{M,1}(x) \right)^\theta \\
&= 1.
\end{aligned}$$

$$\begin{aligned}
\textit{Thus,}\quad \left| B_{n,\alpha,\theta}^{M,1}(f;x) \right| &\leq \sum_{k=0}^{n} \left| f\left(\frac{k}{n}\right) \right| Q_{n,k,\alpha}^{(\theta)M,1}(x) \\
&\leq \|f\| \sum_{k=0}^{n} Q_{n,k,\alpha}^{(\theta)M,1}(x) \\
&= \|f\| B_{n,\alpha,\theta}^{M,1}(e_0;x) \\
&= \|f\|.
\end{aligned}$$

This completes the proof. □

Lemma 6 *If $f \in C(I)$, then for each $x \in I$ we have*

$$\left|B^{M,1}_{n,\alpha,\theta}(f;x)\right| \leq \theta\left|B^{M,1}_{n,\alpha}(f;x)\right| \leq \theta\|f\|.$$

Proof From (2.1), we know that $Q^{(\theta)M,1}_{n,k,\alpha}(x) = \left[J_{n,k,\alpha}(x)\right]^{\theta} - \left[J_{n,k+1,\alpha}(x)\right]^{\theta}$.

Using the fact that $0 \leq \left[J_{n,k,\alpha}(x)\right]^{\theta} - \left[J_{n,k+1,\alpha}(x)\right]^{\theta}$ and the well known inequality $|a^{\alpha} - b^{\alpha}| \leq \alpha|a-b|$, where $0 \leq a, b \leq 1$ and $\alpha \geq 1$, we get

$$\begin{aligned}\left[J_{n,k,\alpha}(x)\right]^{\theta} - \left[J_{n,k+1,\alpha}(x)\right]^{\theta} &\leq \theta\left[J_{n,k,\alpha}(x) - J_{n,k+1,\alpha}(x)\right]\\ &= \theta b^{M,1}_{n,k,\alpha}(x).\end{aligned}$$

Hence, by the definition (2.1), we have

$$\begin{aligned}\left|B^{M,1}_{n,\alpha,\theta}(f;x)\right| &\leq \sum_{k=0}^{n}\left|f\left(\frac{k}{n}\right)\right| Q^{(\theta)M,1}_{n,k,\alpha}(x)\\ &\leq \|f\|\sum_{k=0}^{n} Q^{(\theta)M,1}_{n,k,\alpha}(x)\\ &\leq \theta\|f\|\sum_{k=0}^{n} b^{M,1}_{n,k,\alpha}(x).\end{aligned}$$

Applying Lemma 1, we obtain $|B^{M,1}_{n,\alpha,\theta}(f;x)| \leq \theta\|f\|$. □

4 Rate of approximation

Before discussing about the rate of convergence of the operators (2.1) , let us recall some important definitions which are needed to establish the main results.

For $f \in C(I)$, the appropriate Peetre's K-functional [22] is given by

$$K_2(f;\delta) = \inf\left\{\|f-h\| + \delta\|h'\| + \delta^2\|h''\| : h \in W^2\right\}, \delta > 0, \tag{4.1}$$

where $W^2 = \{h \in C(I) : h', h'' \in C(I)\}$. From [11], the relation between the K-functional and the second order modulus of continuity is given by

$$K_2(f,\delta) \leq C_0\omega_2\left(f, \sqrt{\delta}\right), \tag{4.2}$$

where

$$\omega_2(f,\delta) = \sup_{0<h\leq\delta}\ \sup_{x,x+h,x+2h\in I}|f(x+2h) - 2f(x+h) + f(x)|. \tag{4.3}$$

The usual modulus of continuity is given by

$$\omega(f,\delta) = \sup_{0<h\leq\delta}\ \sup_{x,x+h\in I}|f(x+h) - f(x)|. \tag{4.4}$$

Now, we establish an approximation theorem for the operators (2.1), using classical modulus of continuity and Peetre's K-functional.

Theorem 2 *For $f \in C(I)$, $\alpha \in I$, there exists a positive constant $C_0(\theta)$ such that*

$$\left|B^{M,1}_{n,\alpha,\theta}(f;x) - f(x)\right| \le C_0(\theta)\omega_2\left(f;\sqrt{\theta^{1/2}\chi^*_{n,\alpha}(x)}\right),$$

where $\chi^*_{n,\alpha}(x) = \left(\lambda_{2,\alpha}(x)\right)^{\frac{1}{2}}$.

Proof Let $h \in W^2$. By Taylor's formula, we have

$$h(r) = h(x) + h'(x)(r-x) + \int_x^r (r-\gamma)h''(\gamma)d\gamma. \tag{4.5}$$

Applying $B^{M,1}_{n,\alpha,\theta}(.;x)$ to both sides of (4.5) and using the linearity of the operators, we get

$$B^{M,1}_{n,\alpha,\theta}(h;x) = h(x)B^{M,1}_{n,\alpha,\theta}(1;x) + h'(x)B^{M,1}_{n,\alpha,\theta}(r-x,x) + B^{M,1}_{n,\alpha,\theta}\left(\int_x^r (r-\gamma)h''(\gamma)d\gamma;x\right).$$

As $B^{M,1}_{n,\alpha,\theta}(1;x) = 1$, we have $B^{M,1}_{n,\alpha,\theta}(h;x) = h(x) + h'(x)B^{M,1}_{n,\alpha,\theta}(r-x;x) + B^{M,1}_{n,\alpha,\theta}\left(\int_x^r (r-\gamma)h''(\gamma)d\gamma;x\right)$.

Using Cauchy-Schwarz inequality and Lemma 6, we have

$$\begin{aligned}\left|B^{M,1}_{n,\alpha,\theta}(h;x) - h(x)\right| \le& \left|h'(x)\right|\left|B^{M,1}_{n,\alpha,\theta}(|r-x|;x)\right| \\ &+ \left|B^{M,1}_{n,\alpha,\theta}\left(\int_x^r (r-\gamma)h''(\gamma)d\gamma;x\right)\right| \le \|h'\|B^{M,1}_{n,\alpha,\theta}(|r-x|;x) \\ &+ \frac{\|h''\|}{2}B^{M,1}_{n,\alpha,\theta}\left((r-x)^2;x\right) \le \|h'\|\left[B^{M,1}_{n,\alpha,\theta}\left((r-x)^2;x\right)\right]^{\frac{1}{2}} \\ &+ \frac{\|h''\|}{2}B^{M,1}_{n,\alpha,\theta}\left((r-x)^2;x\right) \le \|h'\|\sqrt{\theta}\left[\lambda_{2,\alpha}(x)\right]^{\frac{1}{2}} + \theta\frac{\|h''\|}{2}\lambda_{2,\alpha}(x) \\ &\le \|h'\|\sqrt{\theta}\chi^*_{n,\alpha}(x) + \theta\frac{\|h''\|}{2}\chi^{*2}_{n,\alpha}(x).\end{aligned} \tag{4.6}$$

As $\left|B^{M,1}_{n,\alpha,\theta}(f;x) - f(x)\right| \le \left|B^{M,1}_{n,\alpha,\theta}(f-h;x)\right| + \left|B^{M,1}_{n,\alpha,\theta}(h;x) - h(x)\right| + |f(x) - h(x)|$, using Lemma 6 and (4.6), we have

$$\left|B^{M,1}_{n,\alpha,\theta}(f;x) - f(x)\right| \le (\theta+1)\|f-h\| + \sqrt{\theta}\|h'\|\chi^*_{n,\alpha}(x) + \theta\frac{\|h''\|}{2}\chi^{*2}_{n,\alpha}(x).$$

Taking infimum over all $h \in W^2$ on the right side of the above inequality, we get

$$\left|B^{M,1}_{n,\alpha,\theta}(f;x) - f(x)\right| \le (\theta+1)K_2\left(f;\sqrt{\theta}\chi^*_{n,\alpha}\right).$$

Using relation (4.2), there exists a constant $C_0(\theta)$, such that

$$\left|B^{M,1}_{n,\alpha,\theta}(f;x)-f(x)\right| \leq C_0(\theta)\omega_2\Big(f;\sqrt{\theta^{1/2}\chi^*_{n,\alpha}(x)}\Big).$$

This completes the proof. □

Remark 3 Using Lemma 4, for sufficiently large n, the above result can also be expressed as

$$\left|B^{M,1}_{n,\alpha,\theta}(f;x)-f(x)\right| \leq C_0(\theta)\omega_2\left(f;\left(\frac{\theta x(1-x)}{n}\right)^{\frac{1}{4}}\right).$$

In our next result, we obtain an estimate of error for the operators (2.1) using Ditzian-Totik first order modulus of smoothness and the associated K-functional.

The Ditzian-Totik first order modulus of smoothness [12] is given by

$$\omega_\phi(f;r) = \sup_{0<h\leq r}\left\{\left|f\left(x+\frac{h\phi(x)}{2}\right)-f\left(x-\frac{h\phi(x)}{2}\right)\right|, x\pm\frac{h\phi(x)}{2}\in I\right\}, \tag{4.7}$$

where $\phi(x)=\sqrt{x(1-x)}$ and $f\in C(I)$. The K-functional corresponding to $\omega_\phi(f;r)$ is denoted by $K_\phi(f;r)$ and is defined as

$$K_\phi(f;r) = \inf\left\{\|f-h\|+r\|\phi h'\| \,:\, h\in H_\phi(I)\right\}. \tag{4.8}$$

where $r>0$ and $H_\phi(I)=\left\{h : h\in AC(I), \|\phi h'\|<\infty\right\}$, $AC(I)$ is the class of absolutely continuous functions on I. Further from [12], the relation between K-functional and the Ditzian-Totik first order modulus of smoothness is given by

$$K_\phi(f;r) \leq C_0\omega_\phi(f;r), C_0>0. \tag{4.9}$$

Theorem 4 *For $f\in C(I), x\in(0,1)$ and $\phi(x)=\sqrt{x(1-x)}$, we have*

$$\left|B^{M,1}_{n,\alpha,\theta}(f;x)-f(x)\right| \leq C_0(\theta)\omega_\phi\left(f;\frac{2\sqrt{2\theta}}{\phi(x)}\chi^*_{n,\alpha}\right),$$

where $C_0(\theta)$ is a positive constant depending on θ.

Proof Let $h\in H_\phi(I)$ be arbitrary. Then,

$$h(r) = h(x)+\int_x^r h'(\gamma)d\gamma.$$

Applying $B^{M,1}_{n,\alpha,\theta}(.;x)$ to both sides of the above equation, we get

$$\left|B^{M,1}_{n,\alpha,\theta}(h;x)-h(x)\right| = \left|B^{M,1}_{n,\alpha,\theta}\left(\int_x^r h'(\gamma)d\gamma;x\right)\right|. \tag{4.10}$$

Now we will estimate the value of $\int_x^r h'(\gamma)d\gamma$. For any $x, r\in(0,1)$, we have

$$
\begin{aligned}
\left|\int_x^r h'(\gamma)d\gamma\right| &\le \|\phi h'\|\left|\int_x^r \frac{1}{\phi(\gamma)}d\gamma\right| \\
&= \|\phi h'\|\left|\int_x^r \frac{1}{\sqrt{\gamma(1-\gamma)d\gamma}}\right| \\
&\le \|\phi h'\|\left|\int_x^r \left(\frac{1}{\sqrt{\gamma}}+\frac{1}{\sqrt{1-\gamma}}\right)d\gamma\right| \\
&\le 2\|\phi h'\|\left(\left|\sqrt{r}-\sqrt{x}\right|+\left|\sqrt{1-r}-\sqrt{1-x}\right|\right) \\
&= 2\|\phi h'\||r-x|\left(\frac{1}{\sqrt{r}+\sqrt{x}}+\frac{1}{\sqrt{1-r}+\sqrt{1-x}}\right) \\
&\le 2\|\phi h'\||r-x|\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{1-x}}\right) \\
&\le 2\sqrt{2}\|\phi h'\|\frac{|r-x|}{\phi(x)}.
\end{aligned}
$$

$$
Thus, \left|B^{M,1}_{n,\alpha,\theta}(h;x)-h(x)\right| \le \frac{2\sqrt{2}\|\phi h'\|}{\phi(x)}B^{M,1}_{n,\alpha,\theta}(|r-x|;x).
$$

Using Cauchy-Schwarz inequality and Lemma 6, we have

$$
\begin{aligned}
\left|B^{M,1}_{n,\alpha,\theta}(h;x)-h(x)\right| &\le \frac{2\sqrt{2}\|\phi h'\|}{\phi(x)}\sqrt{B^{M,1}_{n,\alpha,\theta}\left((r-x)^2;x\right)} \\
&\le \frac{2\sqrt{2\theta}\|\phi h'\|\sqrt{\lambda_{2,\alpha}(x)}}{\phi(x)}
\end{aligned}
$$

$$
\le \frac{2\sqrt{2\theta}\|\phi h'\|\chi^*_{n,\alpha}(x)}{\phi(x)}. \tag{4.11}
$$

As $\left|B^{M,1}_{n,\alpha,\theta}(f;x)-f(x)\right| \le \left|B^{M,1}_{n,\alpha,\theta}(f-h;x)\right| + |f(x)-h(x)| + \left|B^{M,1}_{n,\alpha,\theta}(h;x)-h(x)\right|$,
using (4.11) and Lemma 6, we have

$$
\left|B^{M,1}_{n,\alpha,\theta}(f;x)-f(x)\right| \le (\theta+1)\|f-h\| + \frac{2\sqrt{2\theta}\|\phi h'\|}{\phi(x)}\chi^*_{n,\alpha}.
$$

Taking infimum on the right side of the above inequality over all $h \in H_\phi(I)$, we get

$$
\left|B^{M,1}_{n,\alpha,\theta}(f;x)-f(x)\right| \le (\theta+1)K_\phi\left(f;\frac{2\sqrt{2\theta}}{\phi(x)}\chi^*_{n,\alpha}\right).
$$

Using relation (4.9), we get

$$
\left|B^{M,1}_{n,\alpha,\theta}(f;x)-f(x)\right| \le C_0(\theta)\omega_\phi\left(f;\frac{2\sqrt{2\theta}}{\phi(x)}\chi^*_{n,\alpha}\right).
$$

This completes the proof. □

Remark 5 Using Remark 1 and Lemma 4, for sufficiently large n, the result of the above theorem can also be expressed as

$$\left|B^{M,1}_{n,\alpha,\theta}(f;x)-f(x)\right| \leq C_0(\theta)\omega_\phi\left(f;\frac{2\sqrt{\frac{2\theta x(1-x)}{n}}}{\phi(x)}\right) \leq C_0(\theta)\omega_\phi\left(f;2\sqrt{\frac{2\theta}{n}}\right).$$

5 Quantitative Voronovskaja type theorem

In this section we establish a quantitative Voronovskaja type theorem for the operators (2.1) with the aid of first order Ditzian Totik modulus of smoothness.

Theorem 6 *Let $f \in W^2$ and $x \in (0,1)$, then for sufficiently large n, we have following results:*

$$(1)\ \left|n\left\{B^{M,1}_{n,\alpha,\theta}(f;x)-f(x)-f'(x)B^{M,1}_{n,\alpha,\theta}(r-x;x)-\frac{1}{2}f''(x)B^{M,1}_{n,\alpha,\theta}\left((r-x)^2;x\right)\right\}\right| \leq C_0\omega_\phi\left(f'',\phi(x)n^{-1/2}\right);$$

$$(2)\ \left|n\left\{B^{M,1}_{n,\alpha,\theta}(f;x)-f(x)-f'(x)B^{M,1}_{n,\alpha,\theta}(r-x;x)-\frac{1}{2}f''(x)B^{M,1}_{n,\alpha,\theta}\left((r-x)^2;x\right)\right\}\right| \leq C_0\phi^2(x)\omega_\phi\left(f'';n^{-1/2}\right);$$

where C_0 denotes a positive constant.

Proof Let $r \in I$ and $x \in (0,1)$ be arbitrary. Since $f \in W^2$, by Taylor's formula we have

$$f(r)-f(x)=(r-x)f'(x)+\int_x^r (r-\gamma)f''(\gamma)d\gamma. \tag{5.1}$$

Equation (5.1) can also be written as:

$$f(r)-f(x)-(r-x)f'(x)-\frac{1}{2}(r-x)^2f''(x)=\int_x^r (r-\gamma)\left[f''(\gamma)-f''(x)\right]d\gamma. \tag{5.2}$$

Applying $B^{M,1}_{n,\alpha,\theta}(.;x)$ to both sides of (5.2), we get

$$\begin{aligned}&|B^{M,1}_{n,\alpha,\theta}(f;x)-f(x)-f'(x)B^{M,1}_{n,\alpha,\theta}(r-x;x)-\frac{1}{2}f''(x)B^{M,1}_{n,\alpha,\theta}\left((r-x)^2;x\right)|\\&\leq B^{M,1}_{n,\alpha,\theta}\left(\left|\int_x^r |r-\gamma||f''(\gamma)-f''(x)|d\gamma\right|;x\right).\end{aligned} \tag{5.3}$$

Now, we will estimate the value of $\left|\int_x^r |r-\gamma||f''(\gamma)-f''(x)|d\gamma\right|$.

From [13, p. 337], the estimated value can be found and is given by

$$\left|\int_x^r |r-\gamma||f''(\gamma)-f''(x)|d\gamma\right| \le 2\|f''-h\|(r-x)^2 + 2\|\phi h'\|\phi^{-1}(x)|r-x|^3, \quad (5.4)$$

where $h \in H_\phi(I)$.

Using Cauchy-Schwarz inequality, Eqs. (3.1, 3.2, 5.3, 5.4) and Lemma 6 we have

$$\begin{aligned}
&\left|B^{M,1}_{n,\alpha,\theta}(f;x) - f(x) - f'(x)B^{M,1}_{n,\alpha,\theta}(r-x;x) - \frac{1}{2}f''(x)B^{M,1}_{n,\alpha,\theta}\left((r-x)^2;x\right)\right| \\
&\le 2\|f''-h\|B^{M,1}_{n,\alpha,\theta}\left((r-x)^2;x\right) + 2\|\phi h'\|\phi^{-1}(x)B^{M,1}_{n,\alpha,\theta}\left(|r-x|^3;x\right) \\
&\le 2\|f''-h\|\theta\lambda_{2,\alpha}(x) + 2\theta\|\phi h'\|\phi^{-1}(x)\left\{\lambda_{2,\alpha}(x)\right\}^{\frac{1}{2}}\left\{\lambda_{4,\alpha}(x)\right\}^{\frac{1}{2}} \\
&\le C_0\left\{\frac{\phi^2(x)}{n}\|f''-h\| + \frac{1}{n}\frac{\phi^2(x)}{\sqrt{n}}\|\phi h'\|\right\} \\
&= \frac{C_0}{n}\left\{\phi^2(x)\|f''-h\| + \frac{\phi^2(x)}{\sqrt{n}}\|\phi h'\|\right\}.
\end{aligned}$$

Since, $\phi^2(x) \le \phi(x) \le 1$, for all $x \in (0,1)$, we have

$$\begin{aligned}
&\left|B^{M,1}_{n,\alpha,\theta}(f;x) - f(x) - f'(x)B^{M,1}_{n,\alpha,\theta}(r-x;x) - \frac{1}{2}f''(x)B^{M,1}_{n,\alpha,\theta}\left((r-x)^2;x\right)\right| \\
&\le \frac{C_0}{n}\left\{\|f''-h\| + n^{-\frac{1}{2}}\phi(x)\|\phi h'\|\right\}.
\end{aligned}$$

The above inequality can also be written as

$$\begin{aligned}
&|B^{M,1}_{n,\alpha,\theta}(f;x) - f(x) - f'(x)B^{M,1}_{n,\alpha,\theta}(r-x;x) - \frac{1}{2}f''(x)B^{M,1}_{n,\alpha,\theta}\left((r-x)^2;x\right)| \\
&\le \frac{C_0}{n}\phi^2(x)\left\{\|f''-h\| + n^{-\frac{1}{2}}\|\phi h'\|\right\}.
\end{aligned}$$

Taking infimum on the right side of the above inequalities over $h \in H_\phi(I)$, we have

$$\begin{aligned}
&\left|n\left\{B^{M,1}_{n,\alpha,\theta}(f;x) - f(x) - f'(x)B^{M,1}_{n,\alpha,\theta}(r-x;x) - \frac{1}{2}f''(x)B^{M,1}_{n,\alpha,\theta}\left((r-x)^2;x\right)\right\}\right| \\
&\le \begin{cases} C_0K_\phi\left(f'';\phi(x)n^{-\frac{1}{2}}\right) \\ C_0\phi^2(x)K_\phi\left(f'';n^{-\frac{1}{2}}\right). \end{cases}
\end{aligned}$$

Using the relation (4.9), the theorem is proved. □

6 Rate of approximation for functions with derivatives of bounded variation

In this section, we estimate the convergence rate of the operators (2.1) for functions with a derivative of bounded variation on I. Let $DBV(I)$ denote the class of functions with derivatives of bounded variation on the interval I and let $f \in DBV(I)$ then f can be represented as:

$$f(x) = \int_0^x \varsigma(t)dt + f(0),$$

where $\varsigma \in BV(I)$, i.e. ς is a function of bounded variation on I.

For the operators (2.1), Lesbesgue-Stieltjes integral representation is given by:

$$B^{M,1}_{n,\alpha,\theta}(f;x) = \int_0^1 f(r)d_r T^{M,1}_{n,\alpha,\theta}(x;r), \tag{6.1}$$

where

$$T^{M,1}_{n,\alpha,\theta}(x;r) = \begin{cases} \sum\limits_{k\le nr} Q^{(\theta)M,1}_{n,k,\alpha}(x), & if\ \ 0 < r \le 1 \\ 0, & if\ \ r = 0. \end{cases} \tag{6.2}$$

Let $\overset{b}{\underset{a}{V}}(f)$ denote the total variation of f on $[a,b]$ and f'_x is given by

$$f'_x(\gamma) = \begin{cases} f'(\gamma) - f'(x+)\ \ amp;if\ \ amp;\gamma \quad gt;x \\ 0 \quad amp;if\ \ amp;\gamma = x \\ f'(\gamma) - f'(x-)\ \ amp;if\ \ amp;0 \le \gamma\ \ lt;x. \end{cases} \tag{6.3}$$

We shall need the following lemmas to prove the main result of this section:

Lemma 7 *Let* $x \in (0,1)$, *then for sufficiently large* n, *we have the following results:*

$$(1)\varrho_{n,\alpha,\theta}(x,\gamma) = \int_0^\gamma d_r T^{M,1}_{n,\alpha,\theta}(x;r) \le \frac{C_0\theta x(1-x)}{n(x-\gamma)^2}, \qquad 0 \le \gamma < x; \tag{6.4}$$

$$(2)1 - \varrho_{n,\alpha,\theta}(x,y) = \int_y^1 d_r T^{M,1}_{n,\alpha,\theta}(x;r) \le \frac{C_0\theta x(1-x)}{n(x-y)^2}, \qquad x < y < 1. \tag{6.5}$$

Proof Using Lemma 6, we have

$$\begin{aligned} \varrho_{n,\alpha,\theta}(x,\gamma) &\le \int_0^\gamma \left(\frac{x-r}{x-\gamma}\right)^2 d_r T^{M,1}_{n,\alpha,\theta}(x;r) \\ &\le \frac{1}{(x-\gamma)^2}\int_0^1 (x-r)^2 d_r T^{M,1}_{n,\alpha,\theta}(x;r) \\ &\le \frac{\theta}{(x-\gamma)^2}\lambda_{2,\alpha}(x) \\ &\le \frac{C_0\theta x(1-x)}{n(x-\gamma)^2}. \end{aligned}$$

This proves part (i).

Similarly, we can prove part (ii). ☐

Lemma 8 *Let* $f \in DBV(I)$ *and* $x \in (0,1)$, *then for sufficiently large* n, *the following inequalities hold.*

$$(1)\ L_1 = \left| \int_0^x \left(\int_\gamma^x f_x'(w)dw \right) d_\gamma T_{n,\alpha,\theta}^{M,1}(x;\gamma) \right|$$
$$\leq \frac{C_0\theta(1-x)}{n} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x-\frac{x}{k}}^{x} f_x' \right) + \frac{x}{\sqrt{n}} \left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f_x' \right);$$
$$(2)\ L_2 = \left| \int_x^1 \left(\int_x^\gamma f_x'(w)dw \right) d_\gamma T_{n,\alpha,\theta}^{M,1}(x;\gamma) \right|$$
$$\leq \frac{1-x}{\sqrt{n}} \left(\bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}} f_x' \right) + \frac{C_0\theta x}{n} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x}^{x+\frac{1-x}{k}} f_x' \right).$$

Proof In order to estimate the value of L_1, we will decompose the interval $[0, x]$ into two parts.

$I_1^* = \left[0, x - \frac{x}{\sqrt{n}}\right]$ and $I_2^* = \left[x - \frac{x}{\sqrt{n}}, x\right]$.

Using integration by parts, we get

$$\begin{aligned} &\left| \int_0^x \left(\int_\gamma^x f_x{}'(w)dw \right) d_\gamma T_{n,\alpha,\theta}^{M,1}(x;\gamma) \right| \\ &= \left| \int_0^x \left(\int_\gamma^x f_x'(w)dw \right) \frac{\partial}{\partial\gamma} \left(\varrho_{n,\alpha,\theta}(x,\gamma) \right) d\gamma \right| \\ &= \left| - \int_0^x f_x'(\gamma)\varrho_{n,\alpha,\theta}(x,\gamma)d\gamma \right| \\ &\leq \int_0^x \left| f_x'(\gamma) \right| \varrho_{n,\alpha,\theta}(x,\gamma)d\gamma \\ &\leq \int_{I_1^*} \left| f_x'(\gamma) \right| \varrho_{n,\alpha,\theta}(x,\gamma)d\gamma + \int_{I_2^*} \left| f_x'(\gamma) \right| \varrho_{n,\alpha,\theta}(x,\gamma)d\gamma \\ &= P_1 + P_2. \end{aligned} \tag{6.6}$$

Using $f_x'(x) = 0$, Lemma 7 and putting $\gamma = x - \frac{x}{w}$, we get

$$\begin{aligned} P_1 &= \int_{I_1^*} \left| f_x'(\gamma) \right| \varrho_{n,\alpha,\theta}(x,\gamma)d\gamma \\ &\leq \frac{C_0\theta x(1-x)}{n} \int_{I_1^*} \left| f_x'(\gamma) - f_x'(x) \right| \frac{d\gamma}{(x-\gamma)^2} \\ &\leq \frac{C_0\theta x(1-x)}{n} \int_{I_1^*} \left(\bigvee_{\gamma}^{x} f_x' \right) \frac{d\gamma}{(x-\gamma)^2} \\ &= \frac{C_0\theta(1-x)}{n} \int_1^{\sqrt{n}} \left(\bigvee_{x-\frac{x}{w}}^{x} f_x' \right) dw \\ &\leq \frac{C_0\theta(1-x)}{n} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x-\frac{x}{k}}^{x} f_x' \right). \end{aligned} \tag{6.7}$$

As $f_x'(x) = 0$ and $\varrho_{n,\alpha,\theta}(x,\gamma) \leq 1$, we have

$$\begin{aligned} P_2 &= \int_{I_2^*} \left| f_x'(\gamma) \right| \varrho_{n,\alpha,\theta}(x,\gamma) d\gamma \\ &= \int_{I_2^*} \left| f_x'(\gamma) - f_x'(x) \right| \varrho_{n,\alpha,\theta}(x,\gamma) d\gamma \\ &\leq \int_{I_2^*} \left(\bigvee_{\gamma}^{x} f_x' \right) d\gamma \\ &\leq \left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f_x' \right) \int_{I_2^*} d\gamma. \end{aligned}$$

Hence,

$$P_2 = \frac{x}{\sqrt{n}} \left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f_x' \right). \tag{6.8}$$

Combining equations (6.6) - (6.8), we get

$$L_1 \leq \frac{C_0\theta(1-x)}{n} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x-\frac{x}{k}}^{x} f_x' \right) + \frac{x}{\sqrt{n}} \left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f_x' \right),$$

which proves part (1).

Now, we will estimate the value of $L_2 = \left| \int_x^1 \left(\int_x^\gamma f_x'(w)dw \right) d_\gamma T_{n,\alpha,\theta}^{M,1}(x;\gamma) \right|$.

Using integration by parts, we have

$$\begin{aligned} L_2 &= \left| \int_x^y \left(\int_x^\gamma f_x'(w)dw \right) \frac{\partial}{\partial\gamma} \left(1 - \varrho_{n,\alpha,\theta}(x,\gamma)\right) d\gamma \right. \\ &\quad \left. + \int_y^1 \left(\int_x^\gamma f_x'(w)dw \right) \frac{\partial}{\partial\gamma} \left(1 - \varrho_{n,\alpha,\theta}(x,\gamma)\right) d\gamma \right| \\ &= \left| \int_x^y f_x'(\gamma)\left(1 - \varrho_{n,\alpha,\theta}(x,\gamma)\right) d\gamma + \int_y^1 f_x'(\gamma)\left(1 - \varrho_{n,\alpha,\theta}(x,\gamma)\right) d\gamma \right| \\ &\leq \int_x^y \left(\bigvee_x^\gamma f_x' \right) d\gamma + \frac{C_0\theta x(1-x)}{n} \int_y^1 \left(\bigvee_x^\gamma f_x' \right) (\gamma - x)^{-2} d\gamma. \end{aligned}$$

Now, put $y = x + \frac{1-x}{\sqrt{n}}$ and $w = \frac{1-x}{\gamma-x}$, we get

$$\begin{aligned} L_2 &\leq \frac{1-x}{\sqrt{n}} \left(\bigvee_x^{x+\frac{1-x}{\sqrt{n}}} f_x' \right) + \frac{C_0\theta x(1-x)}{n} \int_{x+\frac{1-x}{\sqrt{n}}}^1 \left(\bigvee_x^\gamma f_x' \right) (\gamma - x)^{-2} d\gamma \\ &\leq \frac{1-x}{\sqrt{n}} \left(\bigvee_x^{x+\frac{1-\gamma}{\sqrt{n}}} f_x' \right) + \frac{C_0\theta x}{n} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_x^{x+\frac{1-x}{k}} f_x' \right), \end{aligned}$$

which proves part (2) . □

Theorem 7 *Let* $f \in DBV(I)$ *and* $x \in (0,1)$, *then for sufficiently large* n, *we have*

$$\begin{aligned}\left|B^{M,1}_{n,\alpha,\theta}(f;x) - f(x)\right| \leq & \sqrt{\frac{C_0\theta x(1-x)}{n}}\left|\frac{f'(x+)+\theta f'(x-)}{\theta+1}\right| \\ & + \sqrt{\frac{C_0\theta x(1-x)}{n}}\left|f'(x+)-f'(x-)\right| \\ & + \frac{C_0\theta(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x} f'_x\right) \\ & + \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f'_x\right) + \frac{1-x}{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}} f'_x\right) + \frac{C_0\theta x}{n}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{1-x}{k}} f'_x\right).\end{aligned}$$

Proof Using (6.3), we can write

$$\begin{aligned}f'(\gamma) =& f'_x(\gamma) + \frac{1}{\theta+1}\left(f'(x+)+\theta f'(x-)\right) \\ & + \frac{1}{2}\left(f'(x+)-f'(x-)\right)\left(sgn(\gamma-x)+\frac{\theta-1}{\theta+1}\right) \\ & + \chi_x(\gamma)\left[f'(\gamma)-\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right].\end{aligned} \tag{6.9}$$

Using $B^{M,1}_{n,\alpha,\theta}(1;x) = 1$ and (6.1), for every $x \in (0,1)$ we have

$$\begin{aligned}B^{M,1}_{n,\alpha,\theta}(f;\gamma) - f(x) &= \int_0^1 f(\gamma) d_\gamma T^{M,1}_{n,\alpha,\theta}(x;\gamma) - f(x) \\ &= \int_0^1 (f(\gamma)-f(\gamma)) d_\gamma T^{M,1}_{n,\alpha,\theta}(x;\gamma) \\ &= \int_0^x (f(\gamma)-f(x)) d_\gamma T^{M,1}_{n,\alpha,\theta}(x;\gamma) \\ &\quad + \int_x^1 (f(\gamma)-f(x)) d_\gamma T^{M,1}_{n,\alpha,\theta}(x;\gamma) \\ &= -\int_0^x \left(\int_\gamma^x f'(w)dw\right) d_\gamma T^{M,1}_{n,\alpha,\theta}(x;\gamma) \\ &\quad + \int_x^1 \left(\int_x^\gamma f'(w)dw\right) d_\gamma T^{M,1}_{n,\alpha,\theta}(x;\gamma) \\ &= -A_1 + A_2.\end{aligned} \tag{6.10}$$

Now, we will calculate the values of A_1 and A_2.

Using (6.9), we have

$$
\begin{aligned}
A_1 &= \int_0^x \left(\int_\gamma^x f'(w)dw \right) d_\gamma T_{n,\alpha,\theta}^{M,1}(x;\gamma) \\
&= \int_0^x \left(\int_\gamma^x f_x'(w)dw \right) d_\gamma T_{n,\alpha,\theta}^{M,1}(x;\gamma) + \frac{f'(x+) + \theta f'(x-)}{\theta+1} \int_0^x (x-\gamma) d_\gamma T_{n,\alpha,\theta}^{M,1}(x;\gamma) \\
&\quad - \frac{f'(x+) - f'(x-)}{\theta+1} \int_0^x (x-\gamma) d_\gamma T_{n,\alpha,\theta}^{M,1}(x;\gamma).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
A_2 &= \int_x^1 \left(\int_x^\gamma f'(w)dw \right) d_\gamma T_{n,\alpha,\theta}^{M,1}(x,\gamma) \\
&= \int_x^1 \left(\int_x^\gamma f_x'(w)dw \right) d_\gamma T_{n,\alpha,\theta}^{M,1}(x,\gamma) \\
&\quad + \frac{f'(x+) + \theta f'(x-)}{\theta+1} \int_x^1 (\gamma - x) d_\gamma T_{n,\alpha,\theta}^{M,1}(x,\gamma) \\
&\quad + \frac{\theta}{\theta+1} \left(f'(x+) - f'(x-) \right) \int_x^1 (\gamma - x) d_\gamma T_{n,\alpha,\theta}^{M,1}(x,\gamma).
\end{aligned}
$$

Using the values of A_1 and A_2, we get

$$
\begin{aligned}
B_{n,\alpha,\theta}^{M,1}(f;x) - f(x) &= \frac{f'(x+) + \theta f'(x-)}{\theta+1} \int_0^1 (\gamma - x) d_\gamma T_{n,\alpha,\theta}^{M,1}(x,\gamma) \\
&\quad + \int_x^1 \left(\int_x^\gamma f_x{}'(w)dw \right) d_\gamma T_{n,\alpha,\theta}^{M,1}(x,\gamma) \\
&\quad - \int_0^x \left(\int_\gamma^x f_x{}'(w)dw \right) d_\gamma T_{n,\alpha,\theta}^{M,1}(x,\gamma) \\
&\quad + \frac{\theta}{\theta+1} \left(f'(x+) - f'(x-) \right) \int_x^1 (\gamma - x) d_\gamma T_{n,\alpha,\theta}^{M,1}(x,\gamma) \\
&\quad + \frac{1}{\theta+1} \left(f'(x+) - f'(x-) \right) \int_0^x (x - \gamma) d_\gamma T_{n,\alpha,\theta}^{M,1}(x,\gamma).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left| B_{n,\alpha,\theta}^{M,1}(f;x) - f(x) \right| &\le \left| \frac{f'(x+) + \theta f'(x-)}{\theta+1} \right| \left(B_{n,\alpha,\theta}^{M,1}(|\gamma - x|;x) \right) \\
&\quad + \left| f'(x+) - f'(x-) \right| \left(B_{n,\alpha,\theta}^{M,1}(|\gamma - x|;x) \right) \\
&\quad + \left| \int_x^1 \left(\int_x^\gamma f_x'(w)dw \right) d_\gamma T_{n,\alpha,\theta}^{M,1}(x,\gamma) \right| \\
&\quad + \left| \int_0^x \left(\int_\gamma^x f_x'(w)dw \right) d_\gamma T_{n,\alpha,\theta}^{M,1}(x,\gamma) \right|.
\end{aligned}
$$

Applying Cauchy-Schwarz inequality, Lemma 6 and Lemma 4, we have

$$\begin{aligned}\left|B^{M,1}_{n,\alpha,\theta}(f;x)-f(x)\right| \le & \left|\frac{f'(x+)+\theta f'(x-)}{\theta+1}\right|\left(B^{M,1}_{n,\alpha,\theta}\left((\gamma-x)^2;x\right)\right)^{\frac{1}{2}}\\ &+\left|f'(x+)-f'(x-)\right|\left(B^{M,1}_{n,\alpha,\theta}\left((\gamma-x)^2;x\right)\right)^{\frac{1}{2}}\\ &+\left|\int_x^1\left(\int_x^\gamma f'_x(w)dw\right)d_\gamma T^{M,1}_{n,\alpha,\theta}(x,\gamma)\right|\\ &+\left|\int_0^x\left(\int_\gamma^x f'_x(w)dw\right)d_\gamma T^{M,1}_{n,\alpha,\theta}(x,\gamma)\right|.\end{aligned}$$

Thus,

$$\begin{aligned}\left|B^{M,1}_{n,\alpha,\theta}(f;x)-f(x)\right| \le & \sqrt{\frac{C_0\theta x(1-x)}{n}}\left|\frac{f'(x+)+\theta f'(x-)}{\theta+1}\right|\\ &+\sqrt{\frac{C_0\theta x(1-x)}{n}}\left|f'(x+)-f'(x-)\right|\\ &+\left|\int_x^1\left(\int_x^\gamma f'_x(w)dw\right)d_\gamma T^{M,1}_{n,\alpha,\theta}(x,\gamma)\right|\\ &+\left|\int_0^x\left(\int_\gamma^x f'_x(w)dw\right)d_\gamma T^{M,1}_{n,\alpha,\theta}(x,\gamma)\right|.\end{aligned}$$

Using Lemma 8, we get the desired result. □

7 Graphical analysis

In order to exemplify the theoretical results of previous sections, we exhibit the convergence and error of approximation of (2.1) by choosing different functions with the help of Matlab software.

In Fig. 1, we illustrate the convergence of our operators to the function $f(x)=-1+sin\left(-9x^2\right)$ for $n=30,60,80,130,160$, $\theta=2$ and $\alpha=0.9$. In Table 1, the absolute error $B_\theta=\left|B^{M,1}_{n,\alpha,\theta}(f;x)-f(x)\right|$ of the above function is computed for certain values of x in interval I and is shown graphically in Fig. 2. that error decreases to zero with increase in parameter θ i.e operator is converging towards function with a faster rate as we increase the value of θ.

We also signify the convergence of our operators by taking another function $f(x)=x^2+cos(2\pi x), n=30,60,80,100,120, \theta=2$ and $\alpha=0.9$ which is illustrated in Fig. 3. In Table 2, the absolute error of the operators (2.1) with the function $f(x)=x^2+cos(2\pi x)$ is computed and is shown graphically in Fig. 4.

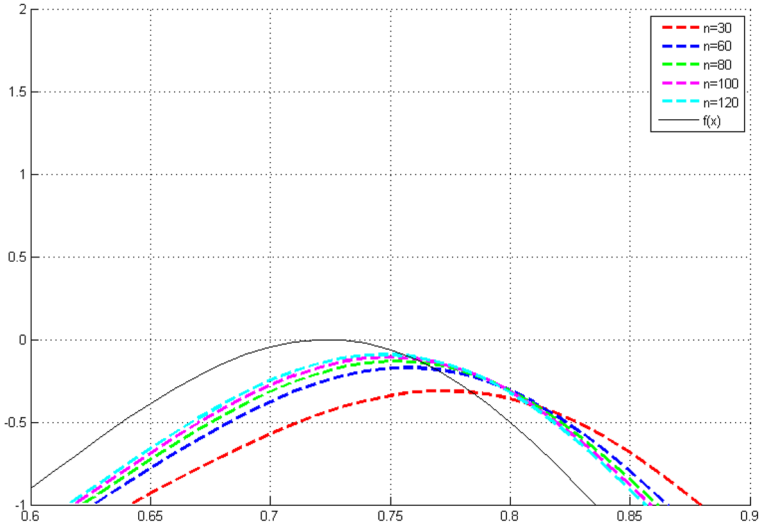

Fig. 1 Graph showing convergence of Bézier variant of modified α- Bernstein operators to the function $f(x) = -1 + sin\left(-9x^2\right)$

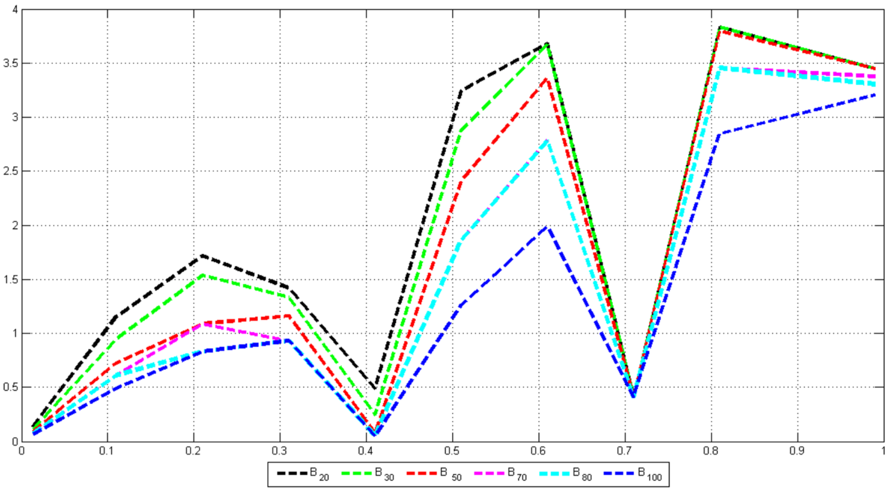

Fig. 2 Graph showing absolute error of Bézier variant of modified α- Bernstein operators for $f(x) = -1 + sin\left(-9x^2\right)$

Table 1 Absolute error of Bézier variant of modified α-Bernstein operators with function $f(x) = -1 + sin\left(-9x^2\right)$ for $\theta = 20, 30, 50, 70, 80, 100, n = 100$ and $\alpha = 0.5$.

x	B_{20}	B_{30}	B_{50}	B_{70}	B_{80}	B_{100}
0.01	0.1085	0.0887	0.0713	0.0572	0.0537	0.0501
0.11	1.1539	0.9507	0.7261	0.6088	0.6088	0.4892
0.31	1.4201	1.3294	1.1613	0.9289	0.9289	0.9289
0.41	0.4877	0.2461	0.0775	0.0513	0.0513	0.0513
0.71	0.4078	0.4078	0.4078	0.4078	0.4078	0.4078

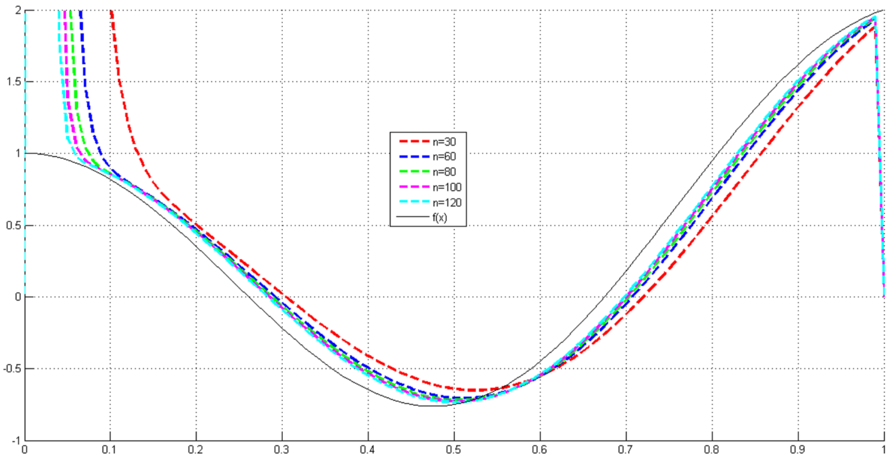

Fig. 3 Graph showing the convergence of Bézier variant of modified α- Bernstein operators to the function $f(x) = x^2 + cos(2\pi x)$

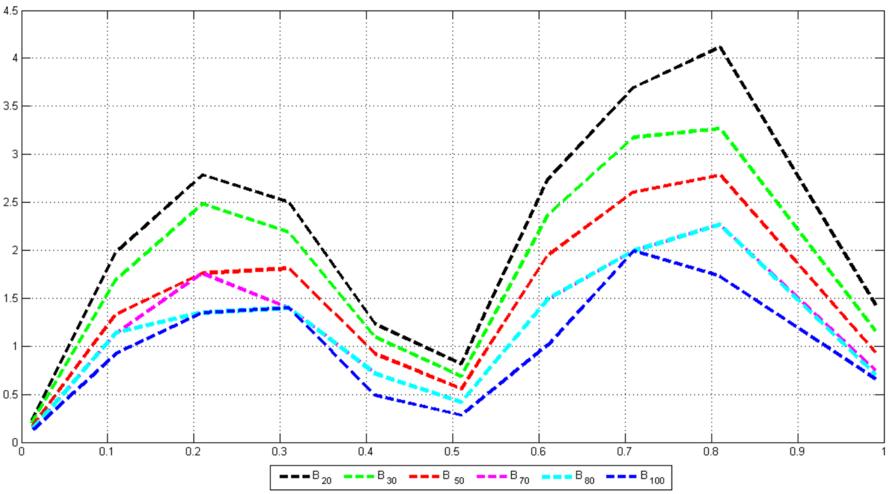

Fig. 4 Graph showing absolute error of Bézier variant of modified α- Bernstein operators for $f(x) = x^2 + cos(2\pi x)$

Table 2 Absolute error of Bézier variant of modified α- Bernstein operators with function $f(x) = x^2 + cos(2\pi x)$ for $\theta = 20, 30, 50, 70, 80, 100, n = 100$ and $\alpha = 0.5$.

x	B_{20}	B_{30}	B_{50}	B_{70}	B_{80}	B_{100}
0.01	0.1947	0.1669	0.1390	0.1141	0.1075	0.1009
0.11	1.9841	1.6945	1.3355	1.1345	1.1345	0.9218
0.41	1.2304	1.0927	0.9193	0.7163	0.7163	0.4905
0.51	0.8150	0.6875	0.5555	0.4199	0.4199	0.2816
0.99	1.4285	1.1619	0.9350	0.7509	0.7045	0.6581

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Declarations

Conflict of interest The authors declare that they have no conflict of interests.

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