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Identities involving generalized derivations in prime rings

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Abstract
Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, $f(x_1, \dots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that d is a nonzero derivation of R and G is a generalized derivation of R. If $G^2(u)d(u) = 0$ for all $u \in f(R)$, then one of the following holds:

(i) there exists $a \in U$ such that $G(x) = ax$ for all $x \in R$ with $a^2 = 0$,
(ii) there exists $a \in U$ such that $G(x) = xa$ for all $x \in R$ with $a^2 = 0$.

Keywords Prime ring · Derivation · Generalized derivation · Extended centroid · Utumi quotient ring

Mathematics Subject Classification 16W25 · 16N60

1 Introduction

Throughout this paper, unless specifically stated, R always denotes a prime ring of characteristic different from 2. Let U be a Utumi ring of quotients and C be its center known as the extended centroid of R. An additive mapping $d : R \to R$ is said to be a derivation on R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Motivated by elementry operators in the theory of operators Algebra, Bresar [3] has introduced the concept of generalized derivations, which is a generalization of derivation. A generalized derivation F is an additive mapping on R with $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, where d is a derivation on R. Clearly, every derivation is generalized derivation but not conversely. A polynomial $f = f(x_1, \dots, x_n) \in \mathbb{Z} < X >$ is said to be multilinear if it is linear in every x_i, $1 \le i \le n$, where $\mathbb{Z}$ is the set of integers.

In [11], Giambruno and Herstien proved that if R is a prime ring and d is a derivation on R such that $d(x)^n = 0$ for all $x \in R$, where n is a fixed positive integer, then $d = 0$. Bresar et al. [2] has extended Herstien result by taking a sequence of different derivations in place

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of single derivation. Precisely, it is proved that, Let R be a prime ring with infinite extended centroid. If derivations $d_1, d_2, d_3, \ldots, d_n$ of R satisfy $d_1(x)d_2(x)...d_n(x) = 0$, for all $x \in R$, then $d_i = 0$ for some i. Later, similar situations considered in [18, 19].

In this sequence, Fosner and Vukman [10], have proved that if F_1 and F_2 are generalized derivations of a prime ring R of characteristic different from 2, such that$F_1(x)F_2(x) = 0$ for all $x \in R$, then there exist elements p, q of the Martindale quotient ring Q of R such that $F_1(x) = xp$ and $F_2(x) = qx$ for all $x \in R$ and $pq = 0$ except when at least one F_i is zero. Moreover, above identity studied by Carini et al [5] by taking multilinear polynomial. They have proved the following:

Let R be a non-commutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, $f(x_1, x_2, \ldots, x_n)$ a multilinear polynomial over C which is not an identity for R, F and G two non-zero generalized derivations on R. If $F(u)G(u) = 0$ for all $u \in f(R) = \{f(r_1, r_2, \ldots, r_n) : r_i \in R\}$, then one of the following holds:

(1) There exist $a, c \in U$ such that $ac = 0$ and $F(x) = xa, G(x) = cx$ for all $x \in R$;
(2) $f(x_1, x_2, \ldots, x_n)^2$ is central valued on R and there exist $a, c \in U$ such that $ac = 0$ and $F(x) = ax, G(x) = xc$ for all $x \in R$;
(3) $f(x_1, x_2, \ldots, x_n)$ is central valued on R and there exist $a, b, c, q \in U$ such that $(a + b)(c + q) = 0$ and $F(x) = ax + xb, G(x) = cx + xq$ for all $x \in R$.

Here in this article, we have studied the identity $G^2(u)d(u) = 0$, for all $u \in f(R) = \{f(r_1, r_2, \ldots, r_n) : r_i \in R\}$, where G is a generalized derivation and d is a non zero derivation on prime ring R of characteristic different from 2. More preisely, we have proved the following:

Theorem 1.1 *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that d is a nonzero derivation of R and G is a generalized derivation on R. If $G^2(u)d(u) = 0$ for all $u \in f(R)$, then one of the following holds*:

(i) *there exists $a \in U$ such that $G(x) = ax$ for all $x \in R$ with $a^2 = 0$,*
(ii) *there exists $a \in U$ such that $G(x) = xa$ for all $x \in R$ with $a^2 = 0$.*

2 Preliminaries

We will use frequently some important theory of generalized polynomial identities and differential identities. We recall some of the facts.

Fact-1: Every derivation d of R can be uniquely extended to a derivation of U (see Proposition 2.5.1 [1]).

Fact-2: If I is a two-sided ideal of R, then R, I and U satisfy the same differential identities ([15]).

Fact-3: If I is a two-sided ideal of R, then R, I and U satisfies the same generalized polynomial identities with coefficients in U ([6]).

Fact-4: (Kharchenko [13, Theorem 2]) Let R be a prime ring, d a nonzero derivation on R and I a nonzero ideal of R. If I satisfies the differential identity

$$f(r_1, r_2, \dots, r_n, d(r_1), d(r_2), \dots, d(r_n)) = 0$$

for any $r_1, r_2, \dots, r_n \in I$, then either

(i) I satisfies the generalized polynomial identity

$$f(r_1, r_2, \dots, r_n, x_1, x_2, \dots, x_n) = 0$$

or

(ii) d is Q-inner i.e., for some $q \in Q$, $d(x) = [q, x]$ and I satisfies the generalized polynomial identity

$$f(r_1, r_2, \dots, r_n, [q, r_1], [q, r_2], \dots, [q, r_n]) = 0.$$

Fact-5: We shall use the following notation:

$$f(x_1, \dots, x_n) = x_1 x_2 \dots x_n + \sum_{\sigma \in S_n, \sigma \neq id} \alpha_\sigma x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(n)}$$

for some $\alpha_\sigma \in C$ and S_n the symmetric group of degree n.

Let d be a derivation. We denote by $f^d(x_1, \dots, x_n)$, $f^{d^2}(x_1, \dots, x_n)$ the polynomials obtained from $f(x_1, \dots, x_n)$ replacing each coefficients α_σ with $d(\alpha_\sigma)$ and $d^2(\alpha_\sigma)$ respectively. Then we have

$$d(f(x_1, \dots, x_n)) = f^d(x_1, \dots, x_n) + \sum_i f(x_1, \dots, d(x_i), \dots, x_n)$$

and

$$d^2(f(x_1, \dots, x_n)) = f^{d^2}(x_1, \dots, x_n) + 2 \sum_i f^d(x_1, \dots, d(x_i), \dots, x_n)$$
$$+ \sum_i f(x_1, \dots, d^2(x_i), \dots, x_n) + \sum_{i \neq j} f(x_1, \dots, d(x_i), \dots, d(x_j), \dots, x_n).$$

3 The case when *d* and *G* are an inner

First, we study the situation when both d and G are an inner. Let $d(x) = [P, x]$ for all $x \in R$ be an inner derivation on R and $G(x) = ax + xb$ for all $x \in R$ be an inner generalized derivation on R for some $P, a, b \in U$. Then $G^2(f(r))d(f(r)) = 0$ for all $r = (r_1, \dots, r_n) \in R^n$ implies

$$(a^2 f(r) + 2af(r)b + f(r)b^2)Pf(r) - (a^2 f(r) + 2af(r)b + f(r)b^2)f(r)P = 0.$$

This gives

$$a' f(r)Pf(r) + 2af(r)b' f(r) + f(r)b'' f(r) - a' f(r)^2 P$$
$$- 2af(r)bf(r)P - f(r)cf(r)P = 0$$

for any $r = (r_1, \dots, r_n) \in R^n$, where $a' = a^2, b' = bP, b'' = b^2 P, c = b^2$.

To prove main results, we need the following.

Lemma 3.1 [7, *Lemma* 1] *Let C be an infinite field and $m \geq 2$. If $A_1, \ldots, A_k$ are not scalar matrices in $M_m(C)$ then there exists some invertible matrix $B \in M_m(C)$ such that any matrices $BA_1B^{-1}, \ldots, BA_kB^{-1}$ have all non-zero entries.*

The following lemma is a particular case of Theorem 1.1 of [4].

Lemma 3.2 *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring, and C its extended centroid. Suppose that F is a generalized derivation and d is a non zero derivation on R and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C with n noncommuting variables, such that $F(f(r_1, \ldots, r_n))d(f(r_1, \ldots, r_n)) = 0$ for all $r_1, r_2, \ldots, r_n \in R$, then $F = 0$.*

Proposition 3.3 *Let $R = M_m(C)$ be the ring of all $m \times m$ matrices over the field C, $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over C and $a, b, c, P, a', b', b'' \in R$. If $a'f(r)Pf(r) + 2af(r)b'f(r) + f(r)b''f(r) - a'f(r)^2P - 2af(r)bf(r)P - f(r)cf(r)P = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$, then either P or a or b is central.*

Proof By our assumption, R satisfies the generalized identity

$$\begin{aligned} &a'f(r_1, \ldots, r_n)Pf(r_1, \ldots, r_n) + 2af(r_1, \ldots, r_n)b'f(r_1, \ldots, r_n) \\ &\quad + f(r_1, \ldots, r_n)b''f(r_1, \ldots, r_n) - a'f(r_1, \ldots, r_n)^2P - 2af(r_1, \ldots, r_n)bf(r_1, \ldots, r_n)P \\ &\quad - f(r_1, \ldots, r_n)cf(r_1, \ldots, r_n)P = 0. \end{aligned} \tag{1}$$

We shall prove it by contradiction. Suppose that $a \notin Z(R)$, $b \notin Z(R)$ and $P \notin Z(R)$.

Case-I: Suppose that C is infinite field. Since $a \notin Z(R)$, $b \notin Z(R)$ and $P \notin Z(R)$, by Lemma 3.1 there exists a C-automorphism ϕ of $M_m(C)$ such that $a_1 = \phi(a)$, $b_1 = \phi(b)$ and $P_1 = \phi(P)$ have all non-zero entries. Clearly a_1, b_1, P_1, $c_1 = \phi(c)$, $a'_1 = \phi(a')$, $b'_1 = \phi(b')$ and $b''_1 = \phi(b'')$ must satisfy the condition (1). Without loss of generality we may replace a, b, c, P, a', b', b'' with $a_1, b_1, c_1, P_1, a'_1, b'_1, b''_1$ respectively.

Here e_{ij} denotes the matrix whose (i, j)-entry is 1 and rest entries are zero. Since $f(x_1, \ldots, x_n)$ is not central, by [15] (see also [16]), there exist $u_1, \ldots, u_n \in M_m(C)$ and $\gamma \in C - \{0\}$ such that $f(u_1, \ldots, u_n) = \gamma e_{st}$, with $s \neq t$. Moreover, since the set $\{f(r_1, \ldots, r_n) : r_1, \ldots, r_n \in M_m(C)\}$ is invariant under the action of all C-automorphisms of $M_m(C)$, then for any $i \neq j$ there exist $r_1, \ldots, r_n \in M_m(C)$ such that $f(r_1, \ldots, r_n) = e_{ij}$. Hence by (1) we have

$$a'e_{ij}Pe_{ij} + 2ae_{ij}b'e_{ij} + e_{ij}b''e_{ij} - 2ae_{ij}be_{ij}P - e_{ij}ce_{ij}P = 0.$$

Right and left multiplying by e_{ij}, we obtain $2a_{ji}b_{ji}P_{ji}e_{ij} = 0$. Since char $(R) \neq 2$, thus we have $a_{ji}b_{ji}P_{ji}e_{ij} = 0$. It implies either $a_{ji} = 0$ or $b_{ji} = 0$ or $P_{ji} = 0$. By Lemma 3.1, it gives a contradiction, since a, b and P have all non-zero entries. Thus we conclude that either a or b or P is central.

Case-II: Suppose C is finite field. Let K be an infinite field which is an extension of the field C. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1, \ldots, x_n)$ is central-valued on R if and only if it is central-valued on $\overline{R}$. Suppose that the generalized polynomial $Q(r_1, \ldots, r_n)$ such that

$$Q(r_1,\dots,r_n) = a'f(r_1,\dots,r_n)Pf(r_1,\dots,r_n) + 2af(r_1,\dots,r_n)b'f(r_1,\dots,r_n)$$
$$+f(r_1,\dots,r_n)b''f(r_1,\dots,r_n) - a'f(r_1,\dots,r_n)^2P - 2af(r_1,\dots,r_n) \quad (2)$$
$$bf(r_1,\dots,r_n)P - f(r_1,\dots,r_n)cf(r_1,\dots,r_n)P$$

is a generalized polynomial identity for R.

Moreover, it is a multi-homogeneous of multi-degree $(2,\dots,2)$ in the indeterminates $r_1,\dots,r_n$. Hence the complete linearization of $Q(r_1,\dots,r_n)$ is a multilinear generalized polynomial $\Theta(r_1,\dots,r_n,x_1,\dots,x_n)$ in $2n$ indeterminates, moreover

$$\Theta(r_1,\dots,r_n,r_1,\dots,r_n) = 2^nQ(r_1,\dots,r_n).$$

It is clear that the multilinear polynomial $\Theta(r_1,\dots,r_n,x_1,\dots,x_n)$ is a generalized polynomial identity for both R and $\overline{R}$. For assumption $char(R) \neq 2$ we obtain $Q(r_1,\dots,r_n) = 0$ for all $r_1,\dots,r_n \in \overline{R}$ and then conclusion follows from Case-I.

Lemma 3.4 *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and $f(x_1,\dots,x_n)$ a multilinear polynomial over C, which is not central valued on R. Suppose that for some $a,b,c,P,a',b',b'' \in R$, $a'f(r)Pf(r) + 2af(r)b'f(r) + f(r)b''f(r) - a'f(r)^2P - 2af(r)bf(r)P - f(r)cf(r)P = 0$ for all $r = (r_1,\dots,r_n) \in R^n$, then either a or b or P is central.*

Proof Let $P \notin C$, $a \notin C$ and $b \notin C$. By hypothesis, we have

$$h(x_1,\dots,x_n) = a'f(x_1,\dots,x_n)Pf(x_1,\dots,x_n) + 2af(x_1,\dots,x_n)b'f(x_1,\dots,x_n)$$
$$+f(x_1,\dots,x_n)b''f(x_1,\dots,x_n) - a'f(x_1,\dots,x_n)^2P - 2af(x_1,\dots,x_n) \quad (3)$$
$$bf(x_1,\dots,x_n)P - f(x_1,\dots,x_n)cf(x_1,\dots,x_n)P = 0$$

for all $x_1,\dots,x_n \in R$. Since R and U satisfy same generalized polynomial identity (GPI) (see [6]), U satisfies $h(x_1,\dots,x_n) = 0$. Suppose that $h(x_1,\dots,x_n)$ is a trivial GPI for U. Let $T = U *_C C\{x_1,x_2,\dots,x_n\}$, the free product of U and $C\{x_1,\dots,x_n\}$, the free C-algebra in noncommuting indeterminates $x_1,x_2,\dots,x_n$. Then, $h(x_1,\dots,x_n)$ is zero element in $T = U *_C C\{x_1,\dots,x_n\}$. Since $P \notin C$, $a \notin C$ and $b \notin C$, the term $2af(x_1,\dots,x_n)bf(x_1,\dots,x_n)P$ appears nontrivially in $h(x_1,\dots,x_n)$. This gives a contradiction.

Next, suppose that $h(x_1,\dots,x_n)$ is a non-trivial GPI for U. In case C is infinite, we have $h(x_1,\dots,x_n) = 0$ for all $x_1,\dots,x_n \in U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U \otimes_C \overline{C}$ are prime and centrally closed [8, Theorems 2.5 and 3.5], we may replace R by U or $U \otimes_C \overline{C}$ according to C finite or infinite. Then R is centrally closed over C and $h(x_1,\dots,x_n) = 0$ for all $x_1,\dots,x_n \in R$. By Martindale's theorem [17], R is then a primitive ring with nonzero socle $soc(R)$ and with C as its associated division ring. Then, by Jacobson's theorem [12, p.75], R is isomorphic to a dense ring of linear transformations of a vector space V over C. Assume first that V is finite dimensional over C, that is, $\dim_C V = m$. By density of R, we have $R \cong M_m(C)$. Since $f(r_1,\dots,r_n)$ is not central valued on R, R must be noncommutative and so $m \geq 2$. In this case, by Proposition ??, we get that either a or b or P is in C, a contradiction. If V is infinite dimensional over C, then for any $e^2 = e \in soc(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. Since P, a and b are not in C, there exist $h_1,h_2,h_3 \in soc(R)$ such that $[P,h_1] \neq 0$, $[a,h_2] \neq 0$ and $[b,h_3] \neq 0$. By Litoff's Theorem [9], there exists idempotent $e \in soc(R)$ such that $Ph_1, h_1P, ah_2, h_2a, bh_3, h_3b, h_1, h_2, h_3 \in eRe$. Since R satisfies generalized identity

$$\begin{aligned}&e\{a'f(ex_1e,\dots,ex_ne)Pf(ex_1e,\dots,ex_ne)+2af(ex_1e,\dots,ex_ne)b'f(ex_1e,\dots,ex_ne)\\&+f(ex_1e,\dots,ex_ne)b''f(ex_1e,\dots,ex_ne)-a'f(ex_1e,\dots,ex_ne)^2P\\&-2af(ex_1e,\dots,ex_ne)bf(ex_1e,\dots,ex_ne)P\\&-f(ex_1e,\dots,ex_ne)cf(ex_1e,\dots,ex_ne)P\}e,\end{aligned}$$

the subring eRe satisfies

$$\begin{aligned}&ea'ef(x_1,\dots,x_n)ePef(x_1,\dots,x_n)+2eaef(x_1,\dots,x_n)eb'ef(x_1,\dots,x_n)\\&\quad+f(x_1,\dots,x_n)eb''ef(x_1,\dots,x_n)-ea'ef(x_1,\dots,x_n)^2ePe-2eaef(x_1,\dots,x_n)\\&\quad ebef(x_1,\dots,x_n)ePe-f(x_1,\dots,x_n)ecef(x_1,\dots,x_n)ePe=0.\end{aligned}$$

Then by the above finite dimensional case, either ePe or eae or ebe is central element of eRe. This leads a contradiction, since $Ph_1=(ePe)h_1=h_1ePe=h_1P$, $ah_2=(eae)h_2=h_2(eae)=h_2a$ and $bh_3=(ebe)h_3=h_3(ebe)=h_3b$. Thus, we have proved that either P or a or b is in C.

Lemma 3.5 *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, $f(x_1,\dots,x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that for some $P,a,b\in U$, $d(x)=[P,x]$ for all $x\in R$ is a nonzero inner derivation of R and $G(x)=ax+xb$ for all $x\in R$ is an inner generalized derivation of R. If $G^2(f(r))d(f(r))=0$ for all $r=(r_1,\dots,r_n)\in R^n$, then one of the following holds*:

(i) $G(x)=(a+b)x$ *for all* $x\in R$ *with* $(a+b)^2=0$,
(ii) $G(x)=x(a+b)$ *for all* $x\in R$ *with* $(a+b)^2=0$.

Proof By hypothesis, we have

$$\big(a^2f(r)+2af(r)b+f(r)b^2\big)Pf(r)-\big(a^2f(r)+2af(r)b+f(r)b^2\big)f(r)P=0. \tag{4}$$

That is

$$\begin{aligned}&a^2f(r)Pf(r)+2af(r)bPf(r)+f(r)b^2Pf(r)-a^2f(r)^2P\\&-2af(r)bf(r)P-f(r)b^2f(r)P=0\end{aligned}$$

for all $r=(r_1,\dots,r_n)\in R^n$. Since $d\neq 0$, so $P\notin C$, then by Lemma 3.4, either $a\in C$ or $b\in C$.

If $a\in C$, then $G(x)=x(a+b)$ for all $x\in R$. Then by hypothesis, we have

$$f(r)(a+b)^2[P,f(r)]=0$$

for all $r=(r_1,\dots,r_n)\in R^n$. Since $d\neq 0$ so $P\notin C$, from Lemma 3.2, it implies that $(a+b)^2=0$, which is our conclusion (*ii*).

If $b\in C$, then $G(x)=(a+b)x$. Hence hypothesis becomes

$$(a+b)^2f(r)[P,f(r)]=0$$

for all $r=(r_1,\dots,r_n)\in R^n$. Since $d\neq 0$ so $P\notin C$, from Lemma 3.2, it implies that $(a+b)^2=0$, which gives our conclusion (*i*).

Lemma 3.6 *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, $f(x_1, \dots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that for some $a, b \in U$, d is a nonzero derivation of R, and $G(x) = ax + xb$ for all $x \in R$ is an inner generalized derivation of R. If $G^2(f(r))d(f(r)) = 0$ for all $r = (r_1, \dots, r_n) \in R^n$, then one of the following holds*:

(i) $G(x) = (a + b)x$ *for all* $x \in R$ *with* $(a + b)^2 = 0$,
(ii) $G(x) = x(a + b)$ *for all* $x \in R$ *with* $(a + b)^2 = 0$.

Proof If d is an inner derivation, then by Lemma 3.5 we get our conclusions. Suppose d is not an inner derivation. Then hypothesis implies that

$$G^2(f(r_1, \dots, r_n))d(f(r_1, \dots, r_n)) = 0.$$

That is

$$\left(a^2f(r_1, \dots, r_n) + 2af(r_1, \dots, r_n)b + f(r_1, \dots, r_n)b^2\right)d(f(r_1, \dots, r_n)) = 0. \tag{5}$$

Since

$$d(f(r_1, \dots, r_n)) = f^d(r_1, \dots, r_n) + \sum_i f(r_1, \dots, d(r_i), \dots, r_n),$$

by applying Kharchenko's theorem (see Fact 4) to (5), we can replace $d(f(r_1, \dots, r_n))$ with $f^d(r_1, \dots, r_n) + \sum_i f(r_1, \dots, y_i, \dots, r_n)$ and then U satisfies

$$\begin{aligned}&\left(a^2f(r_1, \dots, r_n) + 2af(r_1, \dots, r_n)b + f(r_1, \dots, r_n)b^2\right)\Big(f^d(r_1, \dots, r_n)\\ &+ \sum_i f(r_1, \dots, y_i, \dots, r_n)\Big) = 0.\end{aligned} \tag{6}$$

Hence U satisfies blended component

$$\begin{aligned}&\left(a^2f(r_1, \dots, r_n) + 2af(r_1, \dots, r_n)b + f(r_1, \dots, r_n)b^2\right)\\ &\Big(\sum_i f(r_1, \dots, y_i, \dots, r_n)\Big) = 0.\end{aligned} \tag{7}$$

Replacing y_i with $[q, r_i]$ for some $q \notin C$, U satisfies

$$\begin{aligned}&\left(a^2f(r_1, \dots, r_n) + 2af(r_1, \dots, r_n)b + f(r_1, \dots, r_n)b^2\right)\\ &\left[q, f(r_1, \dots, r_n)\right] = 0.\end{aligned} \tag{8}$$

Equation (8) is same as Eq. (4). Hence from Lemma 3.5, we conclude our results. □

Theorem 3.7 *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, $f(x_1, \dots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that d is a nonzero derivation of R and G is a generalized derivation of R. If $G^2(u)d(u) = 0$ for all $u \in f(R)$, then one of the following holds*:

(i) *there exists* $a \in U$ *such that* $G(x) = ax$ *for all* $x \in R$ *with* $a^2 = 0$,
(ii) *there exists* $a \in U$ *such that* $G(x) = xa$ *for all* $x \in R$ *with* $a^2 = 0$.

Proof If G is an inner generalized derivation, then by Lemma 3.6 we get desired results.

Next we assume that G is not an inner generalized derivation. By [14, Theorem 3], we may assume that there exist derivations δ on U, $a \in U$ such that $G(x) = ax + \delta(x)$. Since R and U satisfy the same generalized polynomial identities (see [6]) as well as the same differential identities (see [15]), without loss of generality, we have

$$\Big((a^2 + \delta(a))f(r_1, \dots, r_n) + 2a\delta(f(r_1, \dots, r_n)) + \delta^2(f(r_1, \dots, r_n))\Big) d(f(r_1, \dots, r_n)) = 0 \tag{9}$$

for all $r_1, \dots, r_n \in U$. Now we consider two cases:

Cases-I: Let d and δ be C-dependent modulo inner derivations of U, that is $\alpha d + \beta\delta = ad_q$, where $\alpha, \beta \in C$, $q \in U$ and $ad_q(x) = [q, x]$ for all $x \in U$. If $\alpha = 0$, then $\delta = [q', x]$ for all $x \in R$, where $q' = \beta^{-1}q$, which implies that δ is an inner derivation. It implies that G is an inner generalized derivation, a contradiction. Hence $\alpha \neq 0$, and hence $d = \lambda\delta + ad_p$, where $\lambda = \alpha^{-1}\beta$ and $p = \alpha^{-1}q$. Then by hypothesis, we have

$$\Big((a^2 + \delta(a))f(r_1, \dots, r_n) + 2a\delta(f(r_1, \dots, r_n)) + \delta^2(f(r_1, \dots, r_n))\Big) \Big(\lambda\delta(f(r_1, \dots, r_n)) + \big[p, f(r_1, \dots, r_n)\big]\Big) = 0 \tag{10}$$

for all $r_1, \dots, r_n \in U$.

Since $\delta(f(r_1, \dots, r_n)) = f^{\delta}(r_1, \dots, r_n) + \sum_i f(r_1, \dots, \delta(r_i), \dots, r_n)$ and $\delta^2(f(r_1, \dots, r_n)) = f^{\delta^2}(r_1, \dots, r_n) + 2\sum_i f^{\delta}(r_1, \dots, \delta(r_i), \dots, r_n) + \sum_i f(r_1, \dots, \delta^2(r_i), \dots, r_n) + \sum_{i \neq j} f(r_1, \dots, \delta(r_i), \dots, \delta(r_j), \dots, r_n)$.
Hence our hypothesis becomes

$$\begin{aligned}\Big(&(a^2 + \delta(a))f(r_1, \dots, r_n) + 2af^{\delta}(r_1, \dots, r_n) + 2a\sum_i f(r_1, \dots, \delta(r_i), \dots, r_n) \\ &+ f^{\delta^2}(r_1, \dots, r_n) + 2\sum_i f^{\delta}(r_1, \dots, \delta(r_i), \dots, r_n) + \sum_i f(r_1, \dots, \delta^2(r_i), \dots, r_n) \\ &+ \sum_{i \neq j} f(r_1, \dots, \delta(r_i), \dots, \delta(r_j), \dots, r_n)\Big)\Big(\lambda f^{\delta}(r_1, \dots, r_n) \\ &+ \lambda\sum_i f(r_1, \dots, \delta(r_i), \dots, r_n) + \big[p, f(r_1, \dots, r_n)\big]\Big) = 0.\end{aligned} \tag{11}$$

By Kharchenko's theorem (see Fact-4), we can replace $\delta(r_i)$ with y_i, $\delta^2(r_i)$ with z_i in (10), then U satisfies the blended component

$$\Big(\sum_i f(r_1, \dots, z_i, \dots, r_n)\Big)\Big(\lambda f^{\delta}(r_1, \dots, r_n) + \lambda\sum_i f(r_1, \dots, y_i, \dots, r_n) + \big[p, f(r_1, \dots, r_n)\big]\Big) = 0. \tag{12}$$

In particular for $y_i = 0$ for all $i = 1, 2, \dots, n$, U satisfies the blended component

$$\Big(\sum_i f(r_1,\dots,z_i,\dots,r_n)\Big)\Big(\lambda\sum_i f(r_1,\dots,y_i,\dots,r_n)\Big)=0. \tag{13}$$

Replacing y_i with $[q,r_i]$ for some $q\notin C$ and $z_1=r_1$, $z_2=\cdots=z_n=0$, we have that U satisfies

$$f(r_1,\dots,r_n)\lambda\Big[q,f(r_1,\dots,r_n)\Big]=0. \tag{14}$$

Since $q\notin C$, it implies that $\lambda=0$. Hence (11) reduces to

$$\begin{aligned}\Big(&(a^2+\delta(a))f(r_1,\dots,r_n)+2af^{\delta}(r_1,\dots,r_n)+2a\sum_i f(r_1,\dots,\delta(r_i),\dots,r_n)\\ &+f^{\delta^2}(r_1,\dots,r_n)+2\sum_i f^{\delta}(r_1,\dots,\delta(r_i),\dots,r_n)+\sum_i f(r_1,\dots,\delta^2(r_i),\dots,r_n)\\ &+\sum_{i\neq j} f(r_1,\dots,\delta(r_i),\dots,\delta(r_j),\dots,r_n)\Big)\Big[p,f(r_1,\dots,r_n)\Big]=0.\end{aligned} \tag{15}$$

Again using Kharchenko's theorem (see Fact-4) and using Fact-5, U satisfies the blended component

$$\sum_i f(r_1,\dots,z_i,\dots,r_n)\Big[p,f(r_1,\dots,r_n)\Big]=0. \tag{16}$$

Replacing z_i with $[q,r_i]$ for some $q\notin C$, we have that U satisfies

$$\Big[q,f(r_1,\dots,\dots,r_n)\Big]\Big[p,f(r_1,\dots,r_n)\Big]=0. \tag{17}$$

Since $q\notin C$, by Lemma 3.6, it gives $p\in C$. It implies $d=0$, a contradiction.

Case-II: Let d and δ be C-independent modulo inner derivations of U. By Kharchenko's theorem (see Fact-4) and using Fact-5, we can replace $\delta(f(r_1,\dots,r_n))$ with $f^{\delta}(r_1,\dots,r_n)+\sum_i f(r_1,\dots,z_i,\dots,r_n)$, $d(f(r_1,\dots,r_n))$ with $f^{d}(r_1,\dots,r_n)+\sum_i f(r_1,\dots,y_i,\dots,r_n)$ and $\delta^2(f(r_1,\dots,r_n))$ with $f^{\delta^2}(r_1,\dots,r_n)+2\sum_i f^{\delta}(r_1,\dots,z_i,\dots,r_n)+\sum_i f(r_1,\dots,x_i,\dots,r_n)$ $+\sum_{i\neq j} f(r_1,\dots,z_i,\dots,z_j,\dots,r_n)$, where $\delta(r_i)=z_i$, $d(r_i)=y_i$ and $\delta^2(r_i)=x_i$ in (9) and then U satisfies

$$\begin{aligned}\Big(&(a^2+\delta(a))f(r_1,\dots,r_n)+2af^{\delta}(r_1,\dots,r_n)+2a\sum_i f(r_1,\dots,z_i,\dots,r_n)\\ &+f^{\delta^2}(r_1,\dots,r_n)+2\sum_i f^{\delta}(r_1,\dots,z_i,\dots,r_n)+\sum_i f(r_1,\dots,x_i,\dots,r_n)\\ &+\sum_{i\neq j} f(r_1,\dots,z_i,\dots,z_j,\dots,r_n)\Big)\Big(f^{d}(r_1,\dots,r_n)\\ &+\sum_i f(r_1,\dots,y_i,\dots,r_n)\Big)=0\end{aligned} \tag{18}$$

for all $r_1,\dots,r_n\in U$. In particular U satisfies the blended component

$$\sum_i f(r_1,\ldots,x_i,\ldots,r_n)\sum_i f(r_1,\ldots,y_i,\ldots,r_n)=0. \tag{19}$$

Replacing x_i with $[q, r_i]$ and y_i with $[p, r_i]$ for some $q \notin C$ and $p \notin C$, we have that U satisfies

$$\Big[q, f(r_1,\ldots,\ldots,r_n)\Big]\Big[p, f(r_1,\ldots,\ldots,r_n\Big]=0. \tag{20}$$

This is same as equation (17). In this case, we get a contradiction. □

In particular for $G = I$ in theorem 3.7, I denotes an identity function on R, we have the following corollaries.

Corollary 3.8 *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, $f(x_1,\ldots,x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that d is a derivation on R such that $f(x_1,\ldots,x_n)d(f(x_1,\ldots,x_n)) = 0$ for all $(x_1,\ldots,x_n) \in R^n$, then $d = 0$*

Corollary 3.9 *Let R be a prime ring of characteristic different from 2, I a non zero ideal of R and d be a non zero derivation on R. If $xd(x) = 0$ for all $x \in I$, then R is a commutative.*

Again for $G = g$, where g is a derivation on R, we have the following.

Corollary 3.10 *Let R be a prime ring of characteristic different from 2, I a non zero ideal of R. Suppose that g and d a non zero derivations on R. If $g^2(x)d(x) = 0$ for all $x \in I$, then R is a commutative.*

4 Open problems

In this section, we will give some open problems. In the Theorem 3.7, we have studied the identity $F^2(u)g(u) = 0$ for all $u \in f(R)$, where F is a generalized derivation and g is a derivation on prime ring R. The natural question will arise that what will happen if we replace derivation g with generalized derivation G on prime ring R? More precisely, the statement is given below.

Proposition 4.1 *Let R be a prime ring and G and F are two generalized derivations on R. Let U be Utumi ring of quotient of R with extended centroid C. Suppose $f(x_1,\ldots,x_n)$ is a multilinear polynomial over C which is not central valued on R such that $F^2(u)G(u) = 0$ for all $u \in f(R)$. Then find the structure of these additive mappings as well as prime ring R.*

If we replace generalized derivation G with G^2 in above problem, we have the following.

Proposition 4.2 *Let R be a prime ring and G and F are two generalized derivations on R. Let U be Utumi ring of quotient of R with extended centroid C. Suppose $f(x_1,\ldots,x_n)$ is a multilinear polynomial over C which is not central valued on R such that $F^2(u)G^2(u) = 0$ for all $u \in f(R)$. Then find the structure of these additive mappings as well as prime ring R.*

Proposition 4.3 *Let R be a prime ring and G and F are two generalized derivations on R. Let U be Utumi ring of quotient of R with extended centroid C. Suppose $f(x_1, \dots, x_n)$ is a multilinear polynomial over C which is not central valued on R such that $F^2(u)G(u) \in C$ for all $u \in f(R)$. Then find the structure of these additive mappings as well as prime ring R.*

Since we know that identity mapping is a generalized derivation on R. If we replace $G = id$, where id is the identity mapping on R, in problem 4.3, then it will be [Eroğlu and Argaç **Canad. Math. Bull. 2017; 60: 721–735**].

Proposition 4.4 *Let R be a prime ring and G and F are two generalized derivations on R. Let U be Utumi ring of quotient of R with extended centroid C. Suppose $f(x_1, \dots, x_n)$ is a multilinear polynomial over C which is not central valued on R such that $F^n(u)G^m(u) \in C$ (or $F^n(u)G^m(u) = 0$) for all $u \in f(R)$, where m and n are positive integers. Then find the structure of these additive mappings as well as prime ring R.*

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