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Existence and uniqueness of entropy solution of a nonlinear elliptic equation in anisotropic Sobolev–Orlicz space

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Abstract
Our objective in this paper is to study a certain class of anisotropic elliptic equations with the second term, which is a low-order term and non-polynomial growth; described by an N-uplet of N-function satisfying the Δ_2-condition in the framework of anisotropic Orlicz spaces. We prove the existence and uniqueness of entropic solution for a source in the dual or in L^1, without assuming any condition on the behaviour of the solutions when x tends towards infinity. Moreover, we are giving an example of an anisotropic elliptic equation that verifies all our demonstrated results.

Keywords Anisotropic elliptic equation · Entropy solution · Sobolev–Orlicz anisotropic spaces · Unbounded domain

Mathematics Subject Classification MSC 35J47 · MSC 35J60

1 Introduction

In this paper, we focused on the study of existence and uniqueness solution to anisotropic elliptic non-linear equation, driven by low-order term and non-polynomial growth; described by n-uplet of N-function satisfying the Δ_2-condition, in Sobolev–Orlicz anisotropic space $\mathring{W}^1_B(\Omega) = \overline{C^\infty(\Omega)}^{\mathring{W}^1_B(\Omega)}$. To be more precise, Ω is an unbounded domain of $\mathbb{R}^N$, $N \geq 2$, we study the following equation:

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$$(\mathcal{P})\begin{cases} A(u)+\sum_{i=1}^{N} b_i(x,u,\nabla u)=f(x) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases}$$

where $A(u)=\sum_{i=1}^{N}(a_i(x,u,\nabla u))_{x_i}$ is a Leray–Lions operator defined from $\mathring{W}^1_B(\Omega)$ into its dual, $B(\theta)=(B_1(\theta),\dots,B_N(\theta))$ are N-uplet Orlicz functions that satisfy the Δ_2-condition, and for $i=1,\dots,N$, $b_i(x,u,\nabla u):\Omega\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}$ the Carathéodory functions that do not satisfy any sign condition and the growth described by the vector N-function $B(\theta)$. In the recent studies, specifically the case of bounded domain Ω which is a well known for operators with polynomial, non-standard and non-polynomial growth (described by N-function). We refer the reader to [13–18, 28, 33] for the classical case, and for the Sobolev-Spaces with variable exponents Mihăilescu, M. et al. in [35]; were they proved the existence of solutions on the following nonhomogeneous anisotropic eigenvalue problem:

$$(\mathcal{P})\begin{cases} \sum_{i=1}^{N}\partial_{x_i}(|\,\partial_{x_i}u\,|^{p_i(x)-2}\partial_{x_i}u)=\lambda|\,u\,|^{q(x)-2}u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded domain with smooth boundary, λ is a positive number and p_i, q are continuous functions on $\bar{\Omega}$ such as $2\leq p_i(x)<N$ and $q(x)>1$ for any $x\in\bar{\Omega}$ and $i=\{1,\dots,N\}$. For more detail we refer the reader to [36, 37], and [2, 3, 5, 9, 10, 25–27, 32, 34, 38, 39] for Orlicz Spaces.

In the case where Ω is an unbounded domain, without any assumption on the behaviour of solution where $|x|\longrightarrow+\infty$. The existing result has been established by Brézis [19] for the semi-linear equation:

$$-\Delta u+|u|^{p_0-2}u=f(x).$$

Where $x\in\mathbb{R}^N$, $p_0>2$, $f\in L_{1,loc}(\mathbb{R}^N)$. Karlson and Bendahmane in [8] solved the problem $\Leftarrow\mathcal{P}\Rightarrow$ in the classic case such as $b(x,u,\nabla u)=\operatorname{div}(g(u))$, with $g(u)$ has a growth like $|\,u\,|^{q-1}$, $q\in(1,p_0-1)$. For more result we refer to [24]. In the Sobolev-Spaces with variable exponent, in [20] have demonstrated the existence of solutions to the following problem: $\Delta_{p(x)}u+|\,u\,|^{p(x)-2}u=f(x,u)$ in $\Omega=\mathbb{R}^N$, in both situations were $p:\Omega\longrightarrow\mathbb{R}$ is a log-Hölder continuous functions satisfying

$$1<p^-=\inf_{x\in\Omega}p(x)\leq p^+=\sup_{x\in\Omega}p(x)<\min\{\,n,\ \frac{np^-}{n-p}\}$$

and $f(x,u)=\lambda f_1(x,u)-\delta f_2(x,u)+\eta f_3(x,u)$ with λ,δ,η as real positive parameters, f_1,f_2,$f_3:\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions with subcritical growth. The dependence among the parameters makes f_1 a perturbation of f_3 and, in turn, f_2 a perturbation of f_1. For more result we refer to the work of Aharrouch Benali and al. [6], for the Orlicz-Anisotropic Spaces L. M. Kozhevnikova [30] solved the problem $\Leftarrow\mathcal{P}\Rightarrow$ without the lower order $b_i(x,u,\nabla u)$ and $f(x)=0$, we also cite [7, 23, 29, 31] for more detail.

Our goal, in this paper, is to show the existence and uniqueness of entropy solution for the equations $(\mathcal{P})$; governed with growth and described by an N-uplet of N-functions satisfying the Δ_2-condition. The function $b_i(x,u,\nabla u)$ does not satisfy any sign condition and the source f is merely integrable, within the fulfilling of anisotropic Orlicz spaces. An approximation procedure and some a priori estimates are used to solve the problem, the challenges that we had were due to behaviour of solution near infinity.

Definition 1.1 A measurable function $u : \Omega \longrightarrow \mathbb{R}$ is called an entropy solution of the problem $(\mathcal{P})$ if it satisfies the following conditions: 1/ $u \in \mathcal{T}_0^{1,B}(\Omega) = \{ u : \Omega \longrightarrow \mathbb{R}$ measurable , $T_k(u) \in \mathring{W}_B^1(\Omega)$ for any $k > 0\}$
2/ $b(x, u, \nabla u) \in L^1(\Omega)$ 3/ For any $k > 0$

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla T_k(u - \xi)\, dx + \int_\Omega b(x, u, \nabla u) \cdot T_k(u - \xi)\, dx$$
$$\leq \int_\Omega f(x) \cdot T_k(u - \xi)\, dx \quad \forall \xi \in \mathring{W}_B^1(\Omega) \cap L^\infty(\Omega).$$

The paper is organized as follows: in Sect. 2, we recall the most important and relevant properties and notation about N-functions and the space of Sobolev–Orlicz anisotropic, in Sect. 3, we show the existence of entropy solutions for the problem $(\mathcal{P})$ in an unbounded domain, in Sect. 4, we demonstrate the uniqueness of the solution to the problem $(\mathcal{P})$ in an unbounded domain and in Sect. 5 appendix.

2 Framework space: notations and basic properties

In this section, we briefly review some basic facts about Sobolev–Orlicz anisotropic space which we will need in our analysis of the problem $\mathcal{P}$. A comprehensive presentation of Sobolev–Orlicz anisotropic space can be found in the work of M.A Krasnoselskii and Ja. B. Rutickii [32] and [23].

Definition 2.1 We say that $B : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a N-function if B is continuous, convex, with $B(\theta) > 0$ for $\theta > 0$, $\frac{B(\theta)}{\theta} \to 0$ when $\theta \to 0$ and $\frac{B(\theta)}{\theta} \to \infty$ when $\theta \to \infty$. This N-function B admit the following representation: $B(\theta) = \int_0^\theta b(t)\, dt$, with $b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ which is an increasing function on the right, with $b(0) = 0$ in the case $\theta > 0$ and $b(\theta) \longrightarrow \infty$ when $\theta \longrightarrow \infty$. Its conjugate is noted by $\bar{B}(\theta) = \int_0^{|\theta|} q(t)\, dt$ with q also satisfies all the properties already quoted from b, with

$$\bar{B}(\theta) = \sup_{\mu \geq 0} (\, \mu\, |\, \theta\, | - B(\mu)\,), \quad \theta > 0. \tag{1}$$

The Young's inequality is given as follow

$$\forall \theta,\ \mu > 0 \quad \theta\, \mu \leq B(\mu) + \bar{B}(\theta). \tag{2}$$

Definition 2.2 The N-function $B(\theta)$ satisfies the Δ_2-condition if $\exists c > 0,\ \theta_0 \geq 0$ such as

$$B(2\,\theta) \leq c\, B(\theta) \quad |\,\theta\,| \geq \theta_0. \tag{3}$$

This definition is equivalent to, $\forall k > 1,\ \exists\ c(k) > 0$ such as

$$B(K\,\theta) \leq c(K)\, B(\theta) \quad \text{for} \quad |\,\theta\,| \geq \theta_0. \tag{4}$$

Definition 2.3 The N-function $B(\theta)$ satisfies the Δ_2-condition as long as there exists positive numbers $c > 1$ and $\theta_0 \geq 0$ such as for $\theta \geq \theta_0$ we have

$$\theta\, b(\theta) \leq c\, B(\theta). \tag{5}$$

Also, each N-function $B(\theta)$ satisfies the inequality

$$B(\mu + \theta) \leq c\, B(\theta) + c\, B(\mu) \quad \theta,\ \mu \geq 0. \tag{6}$$

We consider the Orlicz space $L_B(\Omega)$ provided with the norm of Luxemburg given by

$$||\, u\, ||_{B,\Omega} = \inf \{\, k > 0 \,/\, \int_\Omega B\left(\frac{u(x)}{k} \right) dx \leq 1 \,\}. \tag{7}$$

According to [32] we obtain the inequalities

$$\int_\Omega B\left(\frac{u(x)}{||\, u\, ||_{B,\Omega}} \right) dx \leq 1 \tag{8}$$

and

$$||\, u\, ||_{B,\Omega} \leq \int_\Omega B(u)\ dx + 1. \tag{9}$$

Moreover, the Hölder's inequality holds and we have for all $u \in L_B(\Omega)$ and $v \in L_{\bar{B}}(\Omega)$

$$\Big| \int_\Omega u(x)\, v(x)\ dx \Big| \leq 2\, ||\, u\, ||_{B,\Omega} \cdot ||\, v\, ||_{\bar{B},\Omega}. \tag{10}$$

In [32] and [23], if $P(\theta)$ and $B(\theta)$ are two N-functions such as $P(\theta) \ll B(\theta)$ and meas $\Omega < \infty$, then $L_B(\Omega) \subset L_P(\Omega)$, furthermore

$$||\, u\, ||_{P,\Omega} \leq A_0\, (\text{ meas } \Omega\,)\, ||\, u\, ||_{B,\Omega} \quad u \in L_B(\Omega). \tag{11}$$

And for all N-functions $B(\theta)$, if meas $\Omega < \infty$, then $L_\infty(\Omega) \subset L_B(\Omega)$ with

$$||\, u\, ||_{B,\Omega} \leq A_1\, (\text{ meas } \Omega\,)\, ||\, u\, ||_{\infty,\Omega} \quad u \in L_B(\Omega). \tag{12}$$

Also for all N-functions $B(\theta)$, if meas $\Omega < \infty$, then $L_B(\Omega) \subset L^1(\Omega)$ with

$$||\, u\, ||_{1,\Omega} \leq A_2\, ||\, u\, ||_{B,\Omega} \quad u \in L_B(\Omega). \tag{13}$$

We define for all N-functions $B_1(\theta), \dots, B_N(\theta)$ the space of Sobolev–Orlicz anisotropic $\mathring{W}^1_B(\Omega)$ as the adherence space $C_0^\infty(\Omega)$ under the norm

$$||\, u\, ||_{\mathring{W}^1_B(\Omega)} = \sum_{i=1}^{N} ||\, u_{x_i}\, ||_{B_i,\Omega}. \tag{14}$$

Definition 2.4 A sequence $\{\, u_m \,\}$ is said to converge modularly to u in $\mathring{W}^1_B(\Omega)$ if for some $k > 0$ we have

$$\int_\Omega B\left(\frac{u_m - u}{k} \right) dx \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty. \tag{15}$$

Remark 2.5 Since B satisfies the Δ_2-condition, then the modular convergence coincide with the norm convergence.

Proposition 2.6

$$\theta B'(\theta) = \bar{B}(B'(\theta)) + B(\theta), \theta > 0, \tag{16}$$

with B' is the right derivative of the N-function $B(\theta)$.

Proof By (2), we take $\mu = B'(\theta)$, then we obtain

$$B'(\theta)\theta \le B(\theta) + \bar{B}(B'(\theta))$$

and by Ch. I [32], we get the result. □

Let $\omega \subset \Omega$, be a bounded domain in $\mathbb{R}^N$. The following Lemmas are true:

Lemma 2.7 [27] *For all $u \in \mathring{W}^1_{L_B}(\omega)$ with* meas $\omega < \infty$, *we have*

$$\int_\omega B\left(\frac{|u|}{\lambda}\right) dx \le \int_\omega B(|\nabla u|)\, dx$$

where $\lambda =$ diam (ω), *is the diameter of ω.*

Note by $h(t) = \left(\prod_{i=1}^N \frac{B_i^{-1}(t)}{t}\right)^{\frac{1}{N}}$ and we assume that $\int_0^1 \frac{h(t)}{t}\, dt$ converge, so we consider the N-functions $B^*(z)$ defined by $(B^*)^{-1}(z) = \int_0^{|z|} \frac{h(t)}{t}\, dt$.

Lemma 2.8 [29] *Let $u \in \mathring{W}^1_B(\omega)$. If*

$$\int_1^\infty \frac{h(t)}{t}\, dt = \infty, \tag{17}$$

then, $\mathring{W}^1_B(\omega) \subset L_{B^*}(\omega)$ and $||u||_{B^*,\omega} \le \frac{N-1}{N} ||u||_{\mathring{W}^1_B(\omega)}$. If

$$\int_1^\infty \frac{h(t)}{t}\, dt \le \infty,$$

then, $\mathring{W}^1_B(\omega) \subset L_\infty(\omega)$ and $||u||_{\infty,\omega} \le \beta ||u||_{\mathring{W}^1_B(\omega)}$, with $\beta = \int_0^\infty \frac{h(t)}{t}\, dt$.

Lemma 2.9 *Suppose that conditions (20)–(23) are satisfied, and let $(u^m)_{m\in\mathbb{N}}$ be sequence in $\mathring{W}^1_B(\omega)$ such as*

(a) $u^m \rightharpoonup u$ in $\mathring{W}^1_B(\omega)$.
(b) $a^m(x, u^m, \nabla u^m)$ is bounded in $L_{\bar{B}}(\omega)$.
(c) $\sum_{i=1}^N \int_\omega \left[a_i^m(x, u^m, \nabla u^m) - a_i^m(x, u^m, \nabla u \chi_s)\right] \cdot (\nabla u^m - \nabla u\, \chi_s)\, dx \longrightarrow 0$ as $m \to +\infty,\ s \to \infty$.
Where χ_s is the characteristic function of $\omega^s = \{x \in \omega : |\nabla u| \le s\}$. Then,

$$\nabla u^m \longrightarrow \nabla u \text{ a.e in } \omega, \tag{18}$$

and

$$B(|\nabla u^m|) \longrightarrow B(|\nabla u|) \text{ in } L^1(\omega). \tag{19}$$

Proof Let $\vartheta > 0$ fixed and $s > \vartheta$, then from (21) we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^{N} \int_{\omega^{\vartheta}} \left[a_i^m(x, u^m, \nabla u^m) - a_i^m(x, u^m, \nabla u) \right] \cdot (\nabla u^m - \nabla u) dx \\ &= \sum_{i=1}^{N} \int_{\omega^{s}} \left[a_i^m(x, u^m, \nabla u^m) - a_i^m(x, u^m, \nabla u\, \chi_s) \right] \cdot (\nabla u^m - \nabla u\, \chi_s) dx \\ &\leq \sum_{i=1}^{N} \int_{\omega} \left[a_i^m(x, u^m, \nabla u^m) - a_i^m(x, u^m, \nabla u\, \chi_s) \right] \cdot (\nabla u^m - \nabla u\, \chi_s) dx. \end{aligned}$$

According to (c), we get

$$\lim_{m\to\infty} \sum_{i=1}^{N} \int_{\omega^{\vartheta}} \left[a_i^m(x, u^m, \nabla u^m) - a_i^m(x, u^m, \nabla u) \right] \cdot (\nabla u^m - \nabla u)\; dx = 0.$$

Proceeding as in [4], we obtain

$$\nabla u^m \longrightarrow \nabla u \;\text{ a.e in }\; \omega.$$

On the other hand, we have

$$\begin{aligned} \sum_{i=1}^{N} \int_{\omega} a_i^m(x, u^m, \nabla u^m) \cdot \nabla u^m dx = &\sum_{i=1}^{N} \int_{\omega} \left[a_i^m(x, u^m, \nabla u^m) - a_i^m(x, u^m, \nabla u\, \chi_s) \right] \\ &\times (\nabla u^m - \nabla u\, \chi_s)\; dx \\ &+ \sum_{i=1}^{N} \int_{\omega} a_i^m(x, u^m, \nabla u\, \chi_s) \cdot (\nabla u^m - \nabla u\, \chi_s) \cdot dx \\ &+ \sum_{i=1}^{N} \int_{\omega} a_i^m(x, u^m, \nabla u^m) \cdot \nabla u \chi_s dx, \end{aligned}$$

using (b) and (18), we obtain

$$\sum_{i=1}^{N} a_i^m(x, u^m, \nabla u^m) \rightharpoonup \sum_{i=1}^{N} a_i(x, u, \nabla u) \;\text{ weakly in }\; (L_{\bar{B}}(\omega))^N.$$

Therefore

$$\sum_{i=1}^{N} \int_{\omega} a_i^m(x, u^m, \nabla u^m) \nabla u\, \chi_s\; dx \longrightarrow \sum_{i=1}^{N} \int_{\omega} a_i(x, u, \nabla u) \cdot \nabla u$$

as $m \to \infty$, $s \to \infty$. So,

$$\sum_{i=1}^{N} \int_{\omega} \left[a_i^m(x, u^m, \nabla u^m) - a_i^m(x, u^m, \nabla u\, \chi_s) \right] \cdot (\nabla u^m - \nabla u\, \chi_s)\; dx \longrightarrow 0,$$

and

$$\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u^m,\nabla u\,\chi_s)\cdot(\nabla u^m-\nabla u\,\chi_s)\cdot dx \longrightarrow 0.$$

Thus,

$$\lim_{m\to\infty}\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u^m,\nabla u^m)\cdot\nabla u^m\ dx=\sum_{i=1}^{N}\int_{\omega} a_i(x,u,\nabla u)\cdot\nabla u\ dx,$$

from (22) and vitali's Theorem, we get

$$\bar a\sum_{i=1}^{N}\int_{\omega} B_i(|\,\nabla u^m\,|)\ dx-\int_{\omega}\phi(x)\ dx\ge \bar a\sum_{i=1}^{N}\int_{\omega} B_i(|\,\nabla u\,|)\ dx-\int_{\omega}\phi(x)\ dx.$$

Consequently, by Lemma 2.6 in [27], we get

$$B(|\,\nabla u^m\,|)\longrightarrow B(|\,\nabla u\,|)\ \text{ in }\ \mathring{W}_B^1(\omega).$$

Thanks to Lemma 1 in [29], we have

$$B(|\,\nabla u^m\,|)\longrightarrow B(|\,\nabla u\,|)\ \text{ in }\ L^1(\omega).$$

□

3 Existence result in unbounded domain

In this section, we assume they have non-negative measurable functions ϕ, $\varphi \in L^1(\Omega)$ and $\bar a$, $\tilde a$ are two positive constants such that

$$\sum_{i=1}^{N}|\,a_i(x,s,\xi)\,|\le \tilde a\sum_{i=1}^{N}\bar B_i^{-1}B_i(|\,\xi\,|)+\varphi(x), \tag{20}$$

$$\sum_{i=1}^{N}\big(a_i(x,s,\xi)-a_i(x,s,\xi')\big)\cdot(\xi_i-\xi_i')>0, \tag{21}$$

$$\sum_{i=1}^{N}a_i(x,s,\xi)\cdot\xi_i>\bar a\sum_{i=1}^{N}B_i(|\,\xi\,|)-\phi(x), \tag{22}$$

and there exists $h\in L^1(\Omega)$ and $l:\mathbb{R}\longrightarrow\mathbb{R}^+$ a positive continuous functions such that $l\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$.

$$\sum_{i=1}^{N}|\,b_i(x,s,\xi)\,|\le l(s)\cdot\sum_{i=1}^{N}B_i(|\,\xi\,|)+h(x). \tag{23}$$

Theorem 3.1 *Let Ω be an unbounded domain of $\mathbb{R}^N$. Under assumptions (20)–(23), there exists a least one entropy solution of the problem $(\mathcal{P})$ on the sense of Definition 1.1.*

Proof Let $\Omega(m) = \{ x \in \Omega : |x| \le m \}$ and $f^m(x) = \frac{f(x)}{1+\frac{1}{m}|f(x)|} \cdot \chi_{\Omega(m)}$.

We have $f^m \longrightarrow f$ in $L^1(\Omega)$, $m \to \infty$, $|f^m(x)| \le |f(x)|$ and $|f^m| \le m\chi_{\Omega(m)}$.

$$a^m(x,s,\xi) = (a_1^m(x,s,\xi), \dots, a_N^m(x,s,\xi))$$

where $a_i^m(x,s,\xi) = a_i(x, T_m(s), \xi)$ for $i = 1, \dots, N$.

$$b^m(x,s,\xi) = T_m(b(x,s,\xi)) \cdot \chi_{\Omega(m)}$$

and for any $v \in \mathring{W}_B^1(\Omega)$, we consider the following approximate equations

$$(\mathcal{P}_m) : \int_\Omega a(x, T_m(u^m), \nabla u^m) \nabla v \, dx + \int_\Omega b^m(x, u^m, \nabla u^m) v \, dx = \int_\Omega f^m v \, dx.$$

For the proof. See Appendix 5. We divide our proof in six steps.

Step 1 A priori estimate of $\{ u^m \}$.

Proposition 3.2 *Suppose that the assumptions (20)–(23) hold true, and let $(u^m)_m$ be a solution of the approximate problem $(\mathcal{P}_m)$. Then, for all $k > 0$, there exists a constant $c \cdot k$ (not depending on m), such that*

$$\int_\Omega B(|\nabla T_k(u^m)|) \le c \cdot k$$

Proof Taking $v = \exp(G(u^m)) \cdot T_k(u^m)$, as a test function with $G(s) = \int_0^s \frac{l(t)}{\bar{a}} \, dt$ and $\bar{a}$ is the coercivity constant, we obtain

$$\sum_{i=1}^N \int_\Omega a_i^m(x, u^m, \nabla u^m) \cdot \nabla(\exp(G(u^m)) \cdot T_k(u^m)) \, dx$$
$$+ \sum_{i=1}^N \int_\Omega b_i^m(x, u^m, \nabla u^m) \cdot \exp(G(u^m)) \cdot T_k(u^m) \, dx$$
$$\le \int_\Omega f^m \cdot \exp(G(u^m)) \cdot T_k(u^m) \, dx.$$

Then,

$$\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u^m,\nabla u^m)\exp(G(u^m))\,\nabla T_k(u^m))\,dx$$
$$+\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u^m,\nabla u^m)\cdot\nabla u^m\cdot\frac{l(u^m)}{\bar a}.\exp(G(u^m))T_k(u^m)dx$$
$$\le\sum_{i=1}^{N}\int_{\Omega}|\,b_i^m(x,u^m,\nabla u^m)\,|\cdot\exp(G(u^m))\cdot T_k(u^m)\,dx+\int_{\Omega} f^m\cdot\exp(G(u^m))\cdot T_k(u^m)\,dx$$
$$\le\sum_{i=1}^{N}\int_{\Omega}\big[h(x)+l(u^m)\cdot B_i(\nabla u^m)\big]\cdot\exp(G(u^m))\cdot T_k(u^m)\,dx$$
$$+\int_{\Omega} f^m\cdot\exp(G(u^m))\times T_k(u^m)\,dx$$
$$\le\sum_{i=1}^{N}\int_{\Omega} l(u^m)\cdot B_i(\nabla u^m)\cdot\exp(G(u^m))\cdot T_k(u^m)\,dx$$
$$+\int_{\Omega}\big(f^m+h(x)\big)\cdot\exp(G(u^m))\cdot T_k(u^m)\,dx,$$

so,

$$\sum_{i=1}^{N}\int_{\{\,\Omega:|u^m|<k\,\}} a_i^m(x,u^m,\nabla u^m)\cdot\nabla u^m\cdot\exp(G(u^m))\,dx$$
$$\le\int_{\Omega}\Big[f^m(x)+h(x)+\phi(x)\,\frac{l(u^m)}{\bar a}\,\Big]\cdot\exp(G(u^m))\,T_k(u^m)\,dx$$

by (22), we get

$$\bar a\sum_{i=1}^{N}\int_{\{\Omega\,:\,|\,u^m\,|\le k\}} B_i(\nabla u^m)\,\exp(G(u^m))\,dx$$
$$\le\int_{\{\Omega\,:\,|\,u^m\,|\le k\}}\phi(x)\,\exp(G(u^m))\,dx$$
$$+\int_{\Omega}\Big[f^m(x)+h(x)+\phi(x)\,\frac{l(u_m)}{\bar a}\,\Big]\cdot\exp(G(u^m))\,T_k(u^m)\,dx,$$

since ϕ, h and $f^m\in L^1(\Omega)$, and the fact that $\exp(G(\pm\infty))\le\exp\left(\frac{||\,l\,||_{L^1(\Omega)}}{\bar a}\right)$, we deduce that,

$$\int_{\{\,\Omega:\,|u^m|<k\,\}} B(\nabla T_k(u^m))\,dx\le k\cdot c\quad k>0.$$

Finally

$$\int_{\Omega} B(\nabla T_k(u^m))\,dx\le k\cdot c\quad k>0.$$

□

Step 2 Almost everywhere convergence of $\{ u^m \}$.

Lemma 3.3 *For all u^m measurable function on Ω, we have*

$$\text{meas}\,\{ x \in \Omega,\ |\, u^m \,| > k \} \longrightarrow 0.$$

Proof According to Lemma 2.7 and Lemma 2.8, we have

$$\begin{aligned} ||\, T_k(u^m) \,||_{B^*} &\le A \cdot ||\, \nabla T_k(u^m) \,||_B \\ &\le A \cdot \epsilon(k) \int_\omega B(\nabla T_k(u^m) \ dx \\ &\le c \cdot k \cdot \epsilon(k) \quad \text{ for } \ k > 1 \end{aligned} \tag{24}$$

with $\epsilon(k) \longrightarrow 0$ as $k \longrightarrow \infty$.

Form (24) we have

$$\begin{aligned} B^*\left(\frac{k}{||\, T_k(u^m) \,||_{B^*}} \right) \text{meas}\,\{ x \in \Omega \,:\ |\, u^m \,| \ge k \} &\le \int_\Omega B^*\left(\frac{T_k(u^m)}{||\, T_k(u^m) \,||_{B^*}} \right) dx \\ &\le \int_\Omega B^*\left(\frac{k}{||\, T_k(u^m) \,||_{B^*}} \right) dx \end{aligned}$$

by (24) again, we obtain

$$B^*\left(\frac{k}{||\, T_k(u^m) \,||_{B^*}} \right) \longrightarrow \infty \ \text{ as } \ k \longrightarrow \infty.$$

Hence,

$$\text{meas}\,\{ x \in \Omega \,:\ |\, u^m \,| \ge k \} \longrightarrow 0 \ \text{ as } \ k \longrightarrow \infty \quad \text{for all} \quad m \in \mathbb{N}.$$

□

Lemma 3.4 *For all u^m measurable function on Ω, such that*

$$T_k(u^m) \in \mathring{W}^1_B(\Omega) \quad \forall k \ge 1.$$

We have,

$$\text{meas}\,\{ \Omega \,:\ B(\nabla u^m) \ge r \} \longrightarrow 0 \ \text{ as } \ r \longrightarrow \infty.$$

$$\begin{aligned} \text{meas}\,\{ x \in \Omega \,:\ B(\nabla u^m) \ge 0 \} = \ &\text{meas}\,\{ \{ x \in \Omega \,:\ |\, u^m \,| \ge k \ \ B(\nabla u^m) \ge r \} \\ &\cup \{ x \in \Omega \,:\ |\, u^m \,| < k \ \ B(\nabla u^m) \ge r \} \} \end{aligned}$$

Proof

if we denote

$$g(r, k) = \ \text{meas}\,\{ x \in \Omega \,:\ |\, u^m \,| \ge k,\ B(\nabla u^m) \ge r \}$$

we have

$$\text{meas}\,\{ x \in \Omega \,:\ |\, u^m \,| < k \ \ B(\nabla u^m) \ge r \} = g(r, 0) - g(r, k).$$

Then,

$$\int_{\{x\in\Omega:\,|u^m|<k\}} B(\nabla u^m)\,dx = \int_0^\infty \big(g(r,0)-g(r,k)\big)\,dr \le c\cdot k \tag{25}$$

with $r \longrightarrow g(r,k)$ is a decreasing map. Then,

$$\begin{aligned} g(r,0) &\le \frac{1}{r}\int_0^r g(r,0)\,dr \\ &\le \frac{1}{r}\int_0^r \big(g(r,0)-g(r,k)\big)\,dr + \frac{1}{r}\int_0^r g(r,k)\,dr \\ &\le \frac{1}{r}\int_0^r \big(g(r,0)-g(r,k)\big)\,dr + g(0,k) \end{aligned} \tag{26}$$

combining (25) and (26), we obtain

$$g(r,0) \le \frac{c\cdot k}{r} + g(0,k)$$

by Lemma 2.7,

$$\lim_{k\to\infty} g(0,k) = 0.$$

Thus

$$g(r,0) \longrightarrow 0 \quad\text{as}\quad r \longrightarrow \infty.$$

□

We have now to prove the almost everywhere convergence of $\{u^m\}$

$$u^m \longrightarrow u \quad \text{a.e in} \quad \Omega. \tag{27}$$

Let $g(k) = \sup_{m\in\mathbb{N}} \text{ meas }\{x\in\Omega : |u^m| > k\} \longrightarrow 0$ as $k \longrightarrow \infty$.

Since Ω is unbounded domain in $\mathbb{R}^N$, we define η_R as

$$\eta_R(r) = \begin{cases} 1 & \text{if } r < R, \\ R+1-r & \text{if } R \le r < R+1, \\ 0 & \text{if } r \ge R+1. \end{cases}$$

For R, $k > 0$, we have by (6)

$$\begin{aligned} \int_\Omega B(\nabla\eta_R(|x|)\cdot T_k(u^m))\,dx &\le c\int_{\{x\in\Omega:|u^m|<k\}} B(\nabla u^m)\,dx \\ &\quad + c\int_\Omega B(T_k(u^m)\cdot\nabla\eta_R(|x|)\,dx \\ &\le c(k,\,R), \end{aligned}$$

which implies that the sequence $\{\eta_R(|x|)\,T_K(u^m)\}$ is bounded in $\mathring{W}^1_B(\Omega(R+1))$ and by embedding Theorem, for $P \ll B$ we have

$$\mathring{W}^1_B(\Omega(R+1)) \hookrightarrow L_P(\Omega(R+1)),$$

and since $\eta_R = 1$ in $\Omega(R)$, we have

$$\eta_R\, T_k(u^m) \longrightarrow v_k \ \text{ in } \ L_P(\Omega(R+1)) \ \text{ as } \ m \longrightarrow \infty.$$

For $k = 1, \dots,$

$$T_k(u^m) \longrightarrow v_k \ \text{ in } \ L_P(\Omega(R+1)) \ \text{ as } \ m \longrightarrow \infty,$$

by diagonal process, we prove that there is $u : \Omega \longrightarrow \mathbb{R}$ measurable such that $u^m \longrightarrow u$ a.e in Ω. This implies the (27).

Lemma 3.5 *Let an N-functions $\bar{B}(t)$ satisfy the Δ_2-condition and u^m, $m = 1, \dots, \infty$, and u be two functions of $L_B(\Omega)$ such as*

$$|| \, u^m \, ||_B \leq c \quad m = 1, 2, \dots .$$
$$u^m \longrightarrow u \ \text{ almost everywhere in } \ \Omega, \ m \longrightarrow \infty.$$

Then,

$$u^m \rightharpoonup u \ \text{ weakly in } \ L_B(\Omega) \ \text{ as } \ m \to \infty.$$

Proof See Lemma 1.3 in [34]. □

Step 3 Weak convergence of the gradient.
Since $\mathring{W}^1_B(\Omega)$ reflexive, then, there exists a subsequence

$$T_k(u^m) \rightharpoonup v \ \text{ weakly in } \ \mathring{W}^1_B(\Omega), \ m \to \infty.$$

And since,

$$\mathring{W}^1_B(\Omega) \hookrightarrow L_B(\Omega),$$

we have

$$\nabla T_k(u^m) \rightharpoonup \nabla v \ \text{ in } \ L_B(\Omega) \ \text{ as } \ m \to \infty,$$

since

$$u^m \longrightarrow u \ \text{ a.e in } \ \Omega \ \text{ as } \ m \to \infty,$$

we get

$$\nabla u^m \longrightarrow \nabla u \ \text{ a.e in } \ \Omega \ \text{ as } \ m \to \infty.$$

Then, we obtain for any fixed $k > 0$

$$\nabla T_k(u^m) \longrightarrow \nabla T_k(u) \ \text{ a.e in } \ \Omega.$$

Applying Lemma 3.5, we have the following weak convergence

$$\nabla T_k(u^m) \rightharpoonup \nabla T_k(u) \ \text{ in } \ L_B(\Omega) \ \text{ as } \ m \to \infty,$$

for more detail see page 11 in [10].
Step 4 Strong convergence of the gradient.
For $j > k > 0$, we introduce the following function defined as

$$h_j(s)=\begin{cases}1 & \text{if } |s|\le j,\\ 1-|\,s-j\,| & \text{if } j\le |s|\le j+1,\\ 0 & \text{if } s\ge j+1.\end{cases}$$

and we show that the following assertions are true:

Assertion 1

$$\lim_{j\to\infty}\lim_{m\to\infty}\sum_{i=1}^{N}\int_{\{\,j\le|\,u^m\,|\le j+1\,\}} a_i^m(x,u^m,\nabla u^m)\cdot\nabla u^m\cdot\eta_R(|\,x\,|)\;dx=0. \tag{28}$$

Assertion 2

$$\nabla u^m \longrightarrow \nabla u \;\text{ a.e in }\; \Omega(m). \tag{29}$$

Proof We take $v=\exp(G(u^m))\,T_{1,j}(u^m)\,\eta_R(|\,x\,|)=\exp(G(u^m))\,T_1(u^m-T_j(u^m))\,\eta_R(|\,x\,|)$ as a test function in the problem $(\mathcal{P}_m)$, we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i^m(x,u^m,\nabla u^m)\cdot\nabla\Big(\exp(G(u^m))\cdot T_1(u^m-T_j(u^m))\cdot\eta_R(|\,x\,|)\Big)\;dx\\ &\quad\le\sum_{i=1}^{N}\int_\Omega|\,b_i^m(x,u^m,\nabla u^m)\,|\cdot\exp(G(u^m))\cdot T_1(u^m-T_j(u^m))\cdot\eta_R(|\,x\,|)\;dx\\ &\qquad+\int_\Omega f^m(x)\cdot\exp(G(u^m))\cdot T_1(u^m-T_j(u^m))\cdot\eta_R(|\,x\,|)\;dx\end{aligned}$$

according to (22) and (23) we deduce that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{\,j<|\,u^m\,|<j+1\,\}} a_i^m(x,u^m,\nabla u^m)\cdot\nabla u^m\cdot\exp(G(u^m))\cdot\eta_R(|\,x\,|)\;dx\\ &\quad\le\int_\Omega\left[f^m(x)+h(x)+\phi(x)\cdot\frac{l(u^m)}{\bar a}\right]\cdot\exp(G(u^m))\cdot T_1(u^m-T_j(u^m))\cdot\eta_R(|\,x\,|)\;dx\end{aligned}$$

since $\phi\in L^1(\Omega)$, $h\in L^1(\Omega)$, $f^m\in(L^1(\Omega))^N$, and the fact that $\exp(G(\pm))\le\exp\left(\frac{||\,l\,||_{L^1(\mathbb{R})}}{\bar a}\right)$, we deduce from vitali's Theorem that

$$\begin{aligned}&\lim_{j\to\infty}\lim_{m\to\infty}\int_\Omega\left[f^m(x)+h(x)+\phi(x)\cdot\frac{l(u^m)}{\bar a}\right]\cdot\exp(G(u^m))\cdot T_1(u^m-T_j(u^m))\\ &\quad\times\eta_R(|\,x\,|)\;dx=0.\end{aligned}$$

Hence,

$$\lim_{j\to\infty}\lim_{m\to\infty}\int_{\{\,j<|\,u^m\,|<j+1\,\}} a_i^m(x,u^m,\nabla u^m)\cdot\nabla u^m\cdot\eta_R(|\,x\,|)\;dx=0.$$

And to show that assertion 2 is true, we take

$$v = \exp(G(u^m))\,(T_k(u^m) - T_k(u))\,h_j(u^m)\,\eta_R(|\,x\,|),$$

as a test function in the problem $(\mathcal{P}_m)$. We have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i^m(x,u^m,\nabla u^m)\cdot\nabla\Big(\exp(G(u^m))\cdot(T_k(u^m)-T_k(u))\cdot h_j(u^m)\cdot\eta_R(|\,x\,|)\Big)\,dx\\
&\qquad+\sum_{i=1}^{N}\int_\Omega b_i^m(x,u^m,\nabla u^m)\cdot\exp(G(u^m))\cdot(T_k(u^m)-T_k(u))\cdot h_j(u^m)\cdot\eta_R(|\,x\,|)\,dx\\
&\quad\le\int_\Omega f^m(x)\cdot\exp(G(u^m))\cdot(T_k(u^m)-T_k(u))\cdot h_j(u^m)\cdot\eta_R(|\,x\,|)\,dx,
\end{aligned}$$

which implies

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i^m(x,u^m,\nabla u^m)\cdot\nabla u^m\cdot\frac{l(u^m)}{\bar a}\cdot\exp(G(u^m))\cdot(T_k(u^m)-T_k(u))\cdot h_j(u^m)\\
&\quad\times\eta_R(|\,x\,|)\,dx\\
&\quad+\sum_{i=1}^{N}a_i^m(x,u^m,\nabla u^m)\cdot\exp(G(u^m))\cdot(\nabla T_k(u^m)-\nabla T_k(u))\cdot h_j(u^m)\cdot\eta_R(|\,x\,|)\,dx\\
&\quad+\sum_{i=1}^{N}\int_\Omega a_i^m(x,u^m,\nabla u^m)\cdot\exp(G(u^m))\cdot(T_k(u^m)-T_k(u))\cdot\nabla h_j(u^m)\cdot\eta_R(|\,x\,|)\,dx\\
&\quad+\sum_{i=1}^{N}\int_\Omega a_i^m(x,u^m,\nabla u^m)\cdot\exp(G(u^m))\cdot(T_k(u^m)-T_k(u))\cdot h_j(u^m)\cdot\nabla\eta_R(|\,x\,|)\,dx\\
&\quad\le\sum_{i=1}^{N}\int_\Omega|\,b_i^m(x,u^m,\nabla u^m)\,|\cdot\exp(G(u^m))\cdot(T_k(u^m)-T_k(u))\cdot h_j(u^m)\cdot\eta_R(|\,x\,|)\,dx\\
&\qquad+\int_\Omega f^m(x)\cdot\exp(G(u^m))\cdot(T_k(u^m)-T_k(u))\cdot h_j(u^m)\cdot\eta_R(|\,x\,|)\,dx,
\end{aligned}$$

thanks to (22) and (23), we obtain

$$\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u^m,\nabla u^m)\cdot\exp(G(u^m))\cdot(\nabla T_k(u^m)-\nabla T_k(u))\cdot h_j(u^m)\cdot\eta_R(|x|)\,dx$$
$$+\sum_{i=1}^{N}\int_{\{\Omega:\,j\le|u^m|\le j+1\}} a_i^m(x,u^m,\nabla u^m)\cdot\nabla u^m\cdot\exp(G(u^m))$$
$$\times(T_k(u^m)-T_k(u))\cdot\eta_R(|x|)\,dx$$
$$+\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u^m,\nabla u^m)\cdot\exp(G(u^m))\cdot(T_k(u^m)-T_k(u))\cdot h_j(u^m)$$
$$\times\nabla\eta_R(|x|)\,dx$$
$$\le\int_{\Omega}\left[f^m(x)+h(x)+\phi(x)\cdot\frac{l(u^m)}{\bar a}\right]\cdot\exp(G(u^m))\cdot(T_k(u^m)-T_k(u))\cdot h_j(u^m)$$
$$\times\eta_R(|x|)\,dx$$

sine $h_j\ge 0$, $\eta_R(|x|)\ge 0$ and $u^m\,(T_k(u^m)-T_k(u))\ge 0$ we have

$$\sum_{i=1}^{N}\int_{\{\Omega:\,|u^m|\le k\}} a_i(x,T_k(u^m),\nabla T_k(u^m))\exp(G(u^m))\cdot(\nabla T_k(u^m)-\nabla T_k(u))$$
$$\times\eta_R(|x|)\,dx$$
$$+\int_{\{\Omega:\,j\le|u^m|\le j+1\}} a_i^m(x,u^m,\nabla u^m)\,\nabla u^m\ \exp(G(u^m))\,(T_k(u^m)-T_k(u))\,\eta_R(|x|)\,dx$$
$$+\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u^m,\nabla u^m)\cdot\exp(G(u^m))\cdot(T_k(u^m)-T_k(u))\cdot\nabla\eta_R(|x|)\,dx$$
$$\le\int_{\Omega}\left[f^m(x)+h(x)+\phi(x)\cdot\frac{l(u^m)}{\bar a}\right]\cdot\exp(G(u^m))\cdot(T_k(u^m)-T_k(u))\cdot\eta_R(|x|)\,dx$$
$$+\sum_{i=1}^{N}\int_{\{\Omega:\,k\le|u^m|\le j+1\}} a_i(x,T_{j+1}(u^m),\nabla T_{j+1}(u^m))\cdot\exp(G(u^m))\cdot|\nabla T_k(u)|$$
$$\times\eta_R(|x|)\,dx$$
$$+\sum_{i=1}^{N}\int_{\{\Omega:j\le|u^m|\le j+1\}} a_i^m(x,u^m,\nabla u^m)\cdot\nabla u^m\cdot\exp(G(u^m))\cdot|T_k(u^m)-T_k(u)|$$
$$\times\eta_R(|x|)\,dx.$$

The first term in the right hand side goes to zero as m tend to ∞, since $T_k(u^m)\rightharpoonup T_k(u)$ weakly in $\mathring{W}_B^1(\Omega(m))$.

Since $a_i^m(x,T_{j+1}(u^m),\nabla T_{j+1}(u^m))$ is bounded in $L_{\bar B}(\Omega(m))$, there exists $\tilde a^m\in L_{\bar B}(\Omega(m))$ such as

$$|a_i^m(x,T_{j+1}(u^m),\nabla T_{j+1}(u^m))|\rightharpoonup\tilde a^m\ \text{ in }\ L_{\bar B}(\Omega(m)).\tag{30}$$

Thus, the second term of the right hand side goes also to zero.

Since $T_k(u^m)\longrightarrow T_K(u)$ strongly in $\mathring{W}_{B,loc}^1(\Omega(m))$. The third term of the left hand side increased by a quantity that tends to zero as m tend to zero, and according to (28) we deduce that

$$\sum_{i=1}^{N}\int_{\{\Omega:\,|u^m|\leq k\}} a_i(x,T_k(u^m),\nabla T_k(u^m))\cdot\exp(G(u^m))\cdot|\nabla T_k(u^m)-\nabla T_k(u)|$$
$$\times\ \eta_R(|x|)\,dx$$
$$\leq \epsilon(j,m).$$

Then,

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}\Big[a_i(x,T_k(u^m),\nabla T_k(u^m))-a_i(x,T_k(u^m),\nabla T_k(u))\Big]\cdot(\nabla T_k(u^m)-T_K(u))\\
&\quad\times\eta_R(|x|)\,dx\\
&\leq -\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_k(u^m),\nabla T_k(u))\cdot\exp(G(u^m))\cdot|\nabla T_k(u^m)-\nabla T_k(u)|\\
&\quad\times\ \eta_R(|x|)\,dx\\
&\quad-\sum_{i=1}^{N}\int_{\{\Omega:\,|u^m|\leq k\}}a_i(x,T_k(u^m),\nabla T_k(u^m))\cdot\exp(G(u^m))\cdot\nabla T_k(u)\cdot\eta_R(|x|)\,dx\\
&\quad+\ \epsilon(j,m).
\end{aligned}\tag{31}$$

According to Lebesgue dominated convergence Theorem, we have $T_k(u^m)\longrightarrow T_k(u)$ in $\mathring{W}^1_{B,loc}(\Omega)$ and $\nabla T_k(u^m)\rightharpoonup\nabla T_k(u)$ in $\mathring{W}^1_B(\Omega)$, then the terms on the right had side of (31) goes to zero as m and j tend to infinity. Which implies that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}\Big[a_i(x,T_k(u^m),\nabla T_k(u^m))-a_i(x,T_k(u^m),\nabla T_k(u))\Big]\\
&\quad\times(\nabla T_k(u^m)-T_K(u))\,dx\longrightarrow 0.
\end{aligned}\tag{32}$$

Thanks to Lemma 2.9, we have for $k=1,\dots,$

$$\nabla T_k(u^m)\longrightarrow\nabla T_k(u)\ \text{ a.e in }\ \Omega(m)\tag{33}$$

and by diagonal process, we prove that

$$\nabla u^m\longrightarrow\nabla u\ \text{ a.e in }\ \Omega(m).$$

□

Step 5 Equi-integrability of $b^m(x,u^m,\nabla u^m)$.

Let $v=\exp(2\,G(|u^m|))\cdot T_1(u^m-T_R(u^m))\cdot\eta_R(|x|)$ as a test function in the problem $(\mathcal{P}_m)$, we obtain

$$\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u^m,\nabla u^m)\cdot\nabla\big(\exp(2\,G(|\,u^m\,|))\cdot T_1(u^m-T_R(u^m))\cdot\eta_R(|\,x\,|)\big)\,dx$$
$$+\sum_{i=1}^{N}\int_{\Omega} b_i^m(x,u^m,\nabla u^m)\cdot\exp(2\,G(|\,u^m\,|))\cdot T_1(u^m-T_R(u^m))\cdot\eta_R(|\,x\,|)\,dx$$
$$\le\int_{\Omega} f^m(x)\cdot\exp(2\,G(|\,u^m\,|))\cdot T_1(u^m-T_R(u^m))\cdot\eta_R(|\,x\,|)\,dx,$$

which implies that

$$\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u^m,\nabla u^m)\cdot\nabla u^m\cdot\frac{l(u^m)}{\bar a}\cdot\exp(2\,G(|\,u^m\,|))\cdot T_1(u^m-T_R(u^m))$$
$$\times\,\eta_R(|\,x\,|)\,dx$$
$$+\sum_{i=1}^{N}\int_{\{\,\Omega:\,R\le|\,u^m\,|\le R+1\,\}} a_i^m(x,u^m,\nabla u^m)\cdot\nabla u^m\cdot\exp(2\,G(|\,u^m\,|))\cdot\eta_R(|\,x\,|)\,dx$$
$$+\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u^m,\nabla u^m)\cdot\exp(2\,G(|\,u^m\,|)\cdot T_1(u^m-T_R(u^m))\cdot\nabla\eta_R(|\,x\,|)\,dx$$
$$\le\sum_{i=1}^{N}\int_{\Omega}|\,b_i^m(x,u^m,\nabla u^m)\,|\cdot\exp(2\,G(|\,u^m\,|))\cdot T_1(u^m-T_R(u^m))\cdot\eta_R(|\,x\,|)\,dx$$
$$+\int_{\Omega} f^m(x)\cdot\exp(2\,G(|\,u^m\,|))\cdot T_1(u^m-T_R(u^m))\cdot\eta_R(|\,x\,|)\,dx,$$

by (22) and (23), we obtain

$$\bar a\sum_{i=1}^{N}\int_{\{\,\Omega:\,R\le|\,u^m\,|\le R+1\,\}} B_i(|\,\nabla u^m\,|)\cdot\exp(2\,G(|\,u^m\,|)\cdot\eta_R(|\,x\,|)\,dx$$
$$+\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u^m,\nabla u^m)\cdot\exp(2\,G(|\,u^m\,|)\cdot T_1(u^m-T_R(u^m))\cdot\nabla\eta_R(|\,x\,|)\,dx$$
$$\le\int_{\Omega}\left[f^m(x)+h(x)+\phi(x)\cdot\frac{l(u^m)}{\bar a}\right]\cdot\exp(2\,G(|\,u^m\,|))\cdot T_1(u^m-T_R(u^m))$$
$$\times\,\eta_R(|\,x\,|)\,dx+\int_{\{\,\Omega:\,R\le|\,u^m\,|\le R+1\,\}}\phi(x)\cdot\exp(2\,G(|\,u^m\,|)\cdot\eta_R(|\,x\,|)\,dx.$$

Since $\eta_R(|\,x\,|)\ge 0,\ \exp(G(\pm\infty))\le\exp\left(2\,\frac{||\,l\,||_{L^1(\mathbb{R})}}{\bar a}\right)$, $f^m\in(L^1(\Omega))^N$, ϕ and $h\in L^1(\Omega)$. Then, $\forall\,\epsilon>0,\ \exists\,R(\epsilon)>0$ such as

$$\sum_{i=1}^{N}\int_{\{\,\Omega:\,|\,u^m\,|>R+1\,\}} B(|\,\nabla u^m\,|)\,dx\le\frac{\epsilon}{2}\quad\forall R>R(\epsilon).$$

Let $\mathring{V}(\Omega(m))$ be an arbitrary bounded subset for Ω, then, for any measurable set $E\subset\mathring{V}(\Omega(m))$ we have

$$\sum_{i=1}^{N}\int_{E}B_i(|\,\nabla u^m\,|)\;dx \le \sum_{i=1}^{N}\int_{E}B_i(|\,\nabla T_R(u^m)\,|)\;dx + \sum_{i=1}^{N}\int_{\{\,|\,u^m\,|>R+1\,\}}B_i(|\,\nabla u^m\,|)\;dx \tag{34}$$

we conclude that $\forall E\subset \mathring{V}(\Omega(m))$ with meas $(E)<\beta(\epsilon)$ and $T_R(u^m)\longrightarrow T_R(u)$ in $\mathring{W}^1_B(\Omega)$

$$\sum_{i=1}^{N}\int_{E}B_i(|\,\nabla T_R(u^m)\,|)\;dx \le \frac{\epsilon}{2}. \tag{35}$$

Finally, according to (34) and (35), we obtain

$$\sum_{i=1}^{N}\int_{E}B_i(|\,\nabla u^m\,|)\;dx \le \epsilon \quad \forall E\subset \mathring{V}(\Omega(m)) \;\text{ such as }\; \text{meas }(E)<\beta(\epsilon).$$

Which gives the results.

Step 6 Passing to the limit.

Let $\xi\in \mathring{W}^1_B(\Omega)\cap L^\infty(\Omega)$, using the following test function $v=\vartheta_k\,T_k(u^m-\xi)$ in the problem $(\mathcal{P}_m)$ with

$$\vartheta_k=\begin{cases}1 & \text{for }\Omega(m),\\ 0 & \text{for }\Omega(m+1)\backslash\Omega(m).\end{cases}$$

and $|\,u^m\,|-||\,\xi\,||_\infty<|\,u^m-\xi\,|\le j$. Then, $\{\,|\,u^m-\xi\,|\le j\,\}\subset\{\,|\,u^m\,|\le j+||\,\xi\,||_\infty\,\}$ we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_m(u^m),\nabla u^m)\cdot\vartheta_k\,\nabla T_k(u^m-\xi)\;dx&\\ +\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_m(u^m),\nabla u^m)\cdot T_k(u^m-\xi)\,\nabla\vartheta_k\;dx&\\ +\sum_{i=1}^{N}\int_{\Omega}b_i^m(x,u^m,\nabla u^m)\cdot\vartheta_k\,T_k(u^m-\xi)\;dx&\\ \le\int_{\Omega}f^m(x)\cdot\vartheta_k\,T_k(u^m-\xi)\;dx&\end{aligned} \tag{36}$$

which implies that

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega(m)} a_i(x,T_m(u^m),\nabla u^m)\cdot T_k(u^m-\xi)\ dx\\
&=\sum_{i=1}^{N}\int_{\Omega(m)} a_i(x,T_{j+||\xi||_\infty}(u^m),\nabla T_{j+||\xi||_\infty}(u^m))\cdot T_{j+||\xi||_\infty}(u^m-\xi)\cdot \chi_{\{|u^m-\xi|<j\}}dx\\
&=\sum_{i=1}^{N}\int_{\Omega(m)} \Big[a_i(x,T_{j+||\xi||_\infty}(u^m),\nabla T_{j+||\xi||_\infty}(u^m))-a_i(x,T_{j+||\xi||_\infty}(u^m),\nabla\xi)\Big]\\
&\quad\times \nabla T_{j+||\xi||_\infty}(u^m-\xi)\cdot \chi_{\{|u^m-\xi|<j\}}\ dx\\
&\quad+\sum_{i=1}^{N}\int_{\Omega(m)} a_i(x,T_{j+||\xi||_\infty}(u^m),\nabla\xi)\cdot\nabla T_{j+||\xi||_\infty}(u^m-\xi)\cdot \chi_{\{|u^m-\xi|<j\}}\ dx.
\end{aligned}
\tag{37}
$$

By Fatou's Lemma, we have

$$
\begin{aligned}
&\lim_{m\to\infty}\inf\sum_{i=1}^{N}\int_{\Omega(m)} a_i(x,T_m(u^m),\nabla u^m)\cdot \nabla T_k(u^m-\xi)\ dx\\
&\quad\geq \sum_{i=1}^{N}\int_{\Omega(m)} \Big[a_i(x,T_{j+||\xi||_\infty}(u^m),\nabla T_{j+||\xi||_\infty}(u^m))-a_i(x,T_{j+||\xi||_\infty}(u^m),\nabla\xi)\Big]\\
&\qquad\times \nabla T_{j+||\xi||_\infty}(u^m-\xi)\cdot \chi_{\{|u^m-\xi|<j\}}\ dx\\
&\quad+\lim_{m\to\infty}\sum_{i=1}^{N}\int_{\Omega(m)} a_i(x,T_{j+||\xi||_\infty}(u^m),\nabla\xi).\nabla T_{j+||\xi||_\infty}(u^m-\xi)\cdot \chi_{\{|u^m-\xi|<j\}}\ dx.
\end{aligned}
\tag{38}
$$

The second term on the right hand side of the previous inequality is equal to

$$
\int_{\Omega(m)} a_i(x,T_{j+||\xi||_\infty}(u),\nabla\xi)\cdot\nabla T_{j+||\xi||_\infty}(u-\xi)\cdot \chi_{\{|u-\xi|<j\}}\ dx.
$$

Then, since $T_k(u^m-\xi)\rightharpoonup T_k(u-\xi)$ weakly in $\mathring{W}^1_B(\Omega)$, and by (29), (33) we have

$$
\sum_{i=1}^{N}\int_{\Omega} b_i^m(x,u^m,\nabla u^m)\cdot\vartheta_k\, T_k(u^m-\xi)\ dx\longrightarrow \sum_{i=1}^{N}\int_{\Omega} b_i(x,u,\nabla u)\cdot\vartheta_k\, T_k(u-\xi)\ dx
\tag{39}
$$

and

$$
\int_{\Omega} f^m(x)\cdot\vartheta_k\, T_k(u^m-\xi)\ dx\longrightarrow \int_{\Omega} f(x)\cdot\vartheta_k\, T_k(u-\xi)\ dx.
\tag{40}
$$

Combining (36)–(40) and passing to the limit as $m\longrightarrow\infty$, we have the condition 3 in Definition 1.1. □

4 Uniqueness result in unbounded domain

In this section, we demonstrate the Theorem of uniqueness to the solution of problem $(\mathcal{P})$ in an unbounded domain; using the the fact given in [1, 11, 12] such as $b_i(x, u, \nabla u)$ are a contraction Lipschitz continuous functions.

Theorem 4.1 *Under assumptions* (20)–(23), and $b_i(x, u, \nabla u) : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ *for* $i = 1, \dots, N$ *contraction Lipschitz continuous functions do not satisfy any sign condition, and*

$$\sum_{i=1}^{N} \left[a_i(x, \xi, \nabla \xi) - a_i(x, \xi', \nabla \xi') \right] \cdot (\nabla \xi - \nabla \xi') > 0. \tag{41}$$

The problem $(\mathcal{P})$ *has a unique solution.*

Proof Let u^1 and u^2 be two solutions of problem $(\mathcal{P})$ with $u^1 \neq u^2$ then,

$$\sum_{i=1}^{N} \int_\Omega a_i(x, u^1, \nabla u^1) \cdot \nabla v \; dx + \sum_{i=1}^{N} \int_\Omega b_i(x, u^1, \nabla u^1) \cdot v \; dx = \int_\Omega f(x) \cdot v \; dx$$

and

$$\sum_{i=1}^{N} \int_\Omega a_i(x, u^2, \nabla u^2) \cdot \nabla v \; dx + \sum_{i=1}^{N} \int_\Omega b_i(x, u^2, \nabla u^2) \cdot v \; dx = \int_\Omega f(x) \cdot v \; dx$$

we subtract the previous inequality, we get

$$\sum_{i=1}^{N} \int_\Omega \left[a_i(x, u^1, \nabla u^1) - a_i(x, u^2, \nabla u^2) \right] \cdot \nabla v \; dx + \sum_{i=1}^{N} \int_\Omega \left[b_i(x, u^1, \nabla u^1) - b_i(x, u^2, \nabla u^2) \right] \cdot v \; dx = 0$$

we take $v = \eta(x) \cdot (u^1 - u^2)(x)$ with

$$\eta(x) = \begin{cases} 0 & \text{if } x \geq k, \\ k - \frac{|x|^2}{k} & \text{if } |x| < k, \\ 0 & \text{if } x \leq -k. \end{cases}$$

Combine to (41), we obtain

$$\sum_{i=1}^{N} \int_\Omega \left[a_i(x, u^1, \nabla u^1) - a_i(x, u^2, \nabla u^2) \right] \cdot (u^1 - u^2) \cdot \nabla \eta(x) \; dx + \sum_{i=1}^{N} \int_\Omega \left[b_i(x, u^1, \nabla u^1) - b_i(x, u^2, \nabla u^2) \right] \cdot (u^1 - u^2) \cdot \eta(x) \; dx \leq 0$$

according to (2) and the fact that $b_i(x, u, \nabla u)$ contraction Lipschitz functions for $i = 1, \dots, N$, we get

$$\sum_{i=1}^{N} \int_\Omega \bar{B}_i\big(a_i(x,u^1,\nabla u^1) - a_i(x,u^2,\nabla u^2)\big)\,dx + \sum_{i=1}^{N} \int_\Omega B_i(u^1 - u^2)\,\nabla \eta(x))\,dx$$
$$\leq \sum_{i=1}^{N} \int_\Omega \bar{B}_i\big(\, a_i(x,u^1,\nabla u^1) - a_i(x,u^2,\nabla u^2)\,\big)\ dx + 2\sum_{i=1}^{N} \int_\Omega B_i(u^1 - u^2)\ dx \tag{42}$$
$$\leq \alpha \sum_{i=1}^{N} \int_\Omega B_i(u^1 - u^2)\ dx + \alpha \sum_{i=1}^{N} \int_\Omega \bar{B}_i(\eta(x)\cdot(u^1 - u^2))\ dx$$

then

$$\sum_{i=1}^{N} \int_\Omega \bar{B}_i\big(\, a_i(x,u^1,\nabla u^1) - a_i(x,u^2,\nabla u^2)\,\big)\ dx$$
$$\leq (\alpha - 2)\sum_{i=1}^{N} \int_\Omega B_i(u^1 - u^2)\ dx + \alpha \sum_{i=1}^{N} \int_\Omega \bar{B}_i(\eta(x)\cdot(u^1 - u^2))\ dx. \tag{43}$$

Since,

$$\sum_{i=1}^{N} \int_\Omega \bar{B}_i(\eta(x)\cdot(u^1 - u^2))\ dx$$
$$\leq \sum_{i=1}^{N} \int_{\Omega\cap\{\,|x|\leq k\,\}} \bar{B}_i\left(\left(k - \frac{|x|^2}{k}\right)\cdot(u^1 - u^2)\right)\ dx$$
$$+ \sum_{i=1}^{N} \int_{\Omega\cap\{\,|x|>k\,\}} \bar{B}_i(\,\eta(x)\cdot(u^1 - u^2))\ dx$$
$$\longrightarrow 0 \ \text{ as } \ k \longrightarrow 0$$

and since the N-functions $\bar{B}_i$ verified the same conditions and properties of the B_i then, according to (6) and (20), we obtain

$$\sum_{i=1}^{N} \int_\Omega \bar{B}_i\big(\, a_i(x,u^1,\nabla u^1) - a_i(x,u^2,\nabla u^2)\,\big)\ dx$$
$$\leq \tilde{a}c \sum_{i=1}^{N} \int_\Omega B_i(\nabla(u^2 - u^2))\ dx$$
$$\leq \tilde{a}c\,||\,B(u^1 - u^2)\,||_{1,\Omega}.$$

Combine to (42) and (43), we deduce that

$$0 \leq (\tilde{a}c + 2 - \alpha)\,||\,B(u^1 - u^2)\,||_{1,\Omega} \leq 0.$$

Thus

$$||\,B(u^1 - u^2)\,||_{1,\Omega} = 0.$$

Hence, $u^1 = u^2$ a.e in Ω. □

Appendix

Let

$$A : \mathring{W}_B^1(\Omega) \longrightarrow (\mathring{W}_B^1(\Omega))'$$
$$v \quad \longmapsto < A(u), v >= \int_\Omega \sum_{i=1}^N \left(a_i(x,u,\nabla u) \cdot \frac{\partial v}{\partial x_i} + b_i(x,u,\nabla u) \cdot v \right) dx$$
$$- \int_\Omega f(x) \cdot v \, dx$$

and let denote $L_{\bar{B}}(\Omega) = \prod_{k=1}^N L_{\bar{B}_i}(\Omega)$ with the norm

$$|| \, v \, ||_{L_{\bar{B}}(\Omega)} = \sum_{i=1}^N || \, v_i \, ||_{\bar{B}_i,\Omega} \quad v = (v_1, \ldots, v_N) \in L_{\bar{B}}(\Omega).$$

Where $\bar{B}_i(t)$ are N-functions satisfying the Δ_2−conditions. Sobolev-space $\mathring{W}_B^1(\Omega)$ is the completions of the space $C_0^\infty(\Omega)$.

$$a(x,s,\xi) = \big(a_1(x,s,\xi), \ldots, a_N(x,s,\xi) \big)$$

and

$$b(x,s,\xi) = \big(b_1(x,s,\xi), \ldots, b_N(x,s,\xi) \big).$$

Let's show that operator A is bounded, so for $u \in \mathring{W}_B^1(\Omega)$, according to (9) and (20) we get

$$\begin{aligned} || \, a(x,u,\nabla u) \, ||_{L_{\bar{B}}(\Omega)} &= \sum_{i=1}^N || \, a_i(x,u,\nabla u) \, ||_{L_{\bar{B}_i}(\Omega)} \\ &\leq \sum_{i=1}^N \int_\Omega \bar{B}_i(\, a_i(x,u,\nabla u) \,) \; dx + N \\ &\leq \tilde{a}(\Omega) \cdot || \, B(u) \, ||_{1,\Omega} + || \, \varphi ||_{1,\Omega} + N. \end{aligned} \tag{44}$$

Further, for $a(x,u,\nabla u) \in L_{\bar{B}_i}(\Omega)$, $\; v \in \mathring{W}_B^1(\Omega)$ using Hölder's inequality we have

$$\begin{aligned} | \, < A(u), \, v >_\Omega \, | \leq \; & 2 \, || \, a(x,u,\nabla u) \, ||_{L_{\bar{B}}(\Omega)} \cdot || \, v \, ||_{\mathring{W}_B^1(\Omega)} \\ & + \; 2 \, || \, b(x,u,\nabla u) \, ||_{L_B(\Omega)} \cdot || \, v \, ||_{\mathring{W}_B^1(\Omega)} + c_0 \cdot || \, v \, ||_{\mathring{W}_B^1(\Omega)}. \end{aligned} \tag{45}$$

Thus, A is bounded. And that A is coercive, so for $u \in \mathring{W}_B^1(\Omega)$

$$< A(u), u >_\Omega = \sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u) \cdot \frac{\partial u}{\partial x_i}\, dx + \sum_{i=1}^{N} \int_\Omega b_i(x,u,\nabla u) \cdot u\, dx - \int_\Omega f(x) \cdot u\, dx.$$

Then,

$$\frac{< A(u),\, u >_\Omega}{|| u ||_{\mathring{W}_B^1(\Omega)}} \geq \frac{1}{|| u ||_{\mathring{W}_B^1(\Omega)}} \cdot \Bigg[\bar{a} \sum_{i=1}^{N} \int_\Omega B_i\Big(\Big| \frac{\partial u}{\partial x_i} \Big| \Big)\, dx - c_1 - c_0 - l(u) \cdot \sum_{i=1}^{N} \int_\Omega B_i\Big(\Big| \frac{\partial u}{\partial x_i} \Big| \Big)\, dx - \int_\Omega h(x)\, dx \Bigg]$$
$$\geq \frac{1}{|| u ||_{\mathring{W}_B^1(\Omega)}} \cdot \Bigg[(\, \bar{a}(\Omega) - c_2\,) \cdot \sum_{i=1}^{N} \int_\Omega B_i\Big(\Big| \frac{\partial u}{\partial x_i} \Big| \Big)\, dx - c_0 - c_1 - c_3 \Bigg]$$

According to (20), we have for all $k > 0$, $\exists\, \alpha_0 > 0$ such that

$$b_i(\,| u_{x_i} |\,) > k\, b_i\Bigg(\frac{| u_{x_i} |}{|| u_{x_i} ||_{B_i,\Omega}} \Bigg), \quad i = 1, \dots, N.$$

We take $|| u_{x_i} ||_{B_i,\Omega} > \alpha_0 \quad i = 1, \dots, N.$

Suppose that $|| u_{x_i} ||_{\mathring{W}_B^1(\Omega)} \longrightarrow 0$ as $j \to \infty$. We can assume that

$$|| u^j_{x_1} ||_{B_1,\Omega} + \cdots + || u^j_{x_N} ||_{B_N,\Omega} \geq N\, \alpha_0.$$

According to (9) for $c > 1$, we have

$$| u^j \,|\, b(\,| u^j |\,) < c\, B(u^j)$$

then, by (2.8) we obtain

$$\frac{< A(u^j),\, u^j >_\Omega}{|| u^j ||_{\mathring{W}_B^1(\Omega)}} \geq \frac{\bar{a}(\Omega) - c_2}{N\, \alpha_0} \cdot \sum_{i=1}^{N} \int_\Omega B_i\Big(\Big| \frac{\partial u}{\partial x_i} \Big| \Big)\, dx - \frac{c_4}{N\, \alpha_0}$$
$$\geq \frac{\bar{a}(\Omega) - c_2}{N\, \alpha_0} \cdot \sum_{i=1}^{N} \int_\Omega | u^j_{x_i} \,|\, b(\,| u^j_{x_i} |\,)\, dx - \frac{c_4}{N\, \alpha_0}$$
$$\geq \frac{(\, \bar{a}(\Omega) - c_2\,) \cdot k}{c\, N\, || u^j_{x_i} ||_{B_i}} \cdot \sum_{i=1}^{N} \int_\Omega | u^j_{x_i} \,|\, b_i\Bigg(\frac{| u^j_{x_i} |}{|| u^j_{x_i} ||_{B_i,\Omega}} \Bigg)\, dx - \frac{c_4}{N\, \alpha_0}$$
$$\geq \frac{(\, \bar{a}(\Omega) - c_2\,) \cdot k}{c\, N} \cdot \sum_{i=1}^{N} \int_\Omega B_i\Bigg(\frac{| u^j_{x_i} |}{|| u^j_{x_i} ||_{B_i,\Omega}} \Bigg)\, dx - \frac{c_4}{N\, \alpha_0}$$
$$\geq \frac{(\, \bar{a}(\Omega) - c_2\,) \cdot k}{c\, N} - \frac{c_4}{N\, \alpha_0}.$$

which shows that A is coercive, because k is arbitrary.

And for A pseudo-monotonic, we consider a sequence $\{\, u^m \,\}_{m=1}^{\infty}$ in the space $\mathring{W}_B^1(\Omega)$ such that

$$u^m \rightharpoonup u \ \text{ weakly in } \mathring{W}^1_B(\Omega) \quad m \to \infty. \tag{46}$$

$$\lim_{m\to\infty} \sup < A(u^m), u^m - u > \leq 0 \tag{47}$$

we demonstrate that

$$A(u^m) \rightharpoonup A(u) \ \text{ weakly in } (\mathring{W}^1_B(\Omega))', \ m \to \infty. \tag{48}$$

$$< A(u^m), u^m - u > \longrightarrow 0, \ m \to \infty. \tag{49}$$

Since $B(\theta)$ satisfy the Δ_2-condition, then by (9) we have

$$\int_\Omega B(\theta)\, dx \leq c_0 \,||\, \theta \,||_{B,\Omega}. \tag{50}$$

According to (46) we get

$$||\, u^m \,||_{\mathring{W}^1_B(\Omega)} \leq c_1 \quad m = 1, 2, \ldots \tag{51}$$

and

$$||\, B(\nabla u^m) \,||_1 \leq c_2 \quad m = 1, 2, \ldots. \tag{52}$$

Combining to (44) and (51) we obtain

$$||\, a^m(x, u, \nabla u) \,||_{\bar{B}} = \sum_{i=1}^{N} ||\, a_i^m(x, u^m, \nabla u^m) \,||_{\bar{B}_i} \leq c_3 \ m = 1, 2, \ldots. \tag{53}$$

And for $m \in \mathbb{N}^*$, $|\, b^m(x, u, \nabla) \,| = |\, T_m(b(x, u, \nabla u) \,| \leq m$. Then, by (23) and (51) we have

$$||\, b^m(x, u, \nabla u) \,||_B = \sum_{i=1}^{N} ||\, b_i^m(x, u^m, \nabla u^m) \,||_{B_i} \leq c_4 \ m = 1, 2, \ldots.$$

According again to proof of Lemmas 3.4 and 2.8, we have

$$\mathring{W}^1_B(\Omega(R+1)) \hookrightarrow L_{B_i}(\Omega(R+1)) \ \text{ for } \ R > 0 \ \text{ and } \ i = 1, \ldots, N.$$

We set

$$A^m(x) = \sum_{i=1}^{N} \left[a_i^m(x, u^m, \nabla u^m) - a_i^m(x, u, \nabla u) \right] (u^m - u)_{x_i}$$
$$+ \sum_{i=1}^{N} \left[b_i^m(x, u^m, \nabla u^m) - b_i^m(x, u, \nabla u) \right] (u^m - u), \ m = 1, \ldots.$$

then

$$< A(u^m) - A(u),\ u^m - u >= \int_\Omega A^m(x)\ dx \quad m = 1, \ldots .$$

By (46) and (47), we obtain

$$\lim_{m\to\infty} \sup \int_\Omega A^m(x)\ dx \leq 0.$$

So,

$$\begin{aligned} A^m(x) &= \sum_{i=1}^{N} \left[a_i^m(x, u^m, \nabla u^m) - a_i^m(x, u^m, \nabla u) \right] (u^m - u)_{x_i} \\ &\quad + \sum_{i=1}^{N} \left[a_i^m(x, u^m, \nabla u) - a_i^m(x, u, \nabla u) \right] (u^m - u)_{x_i} \\ &\quad + \sum_{i=1}^{N} \left[b_i^m(x, u^m, \nabla u^m) - b_i^m(x, u, \nabla u) \right] (u^m - u) \\ &= A_1^m(x) + A_2^m(x) + A_3^m(x) \quad m = 1, \ldots . \end{aligned} \tag{54}$$

We prove that

$$A_1^m(x) \longrightarrow 0 \text{ almost everywhere in } \Omega \quad m \to \infty. \tag{55}$$

$$A_2^m(x) \longrightarrow 0 \text{ almost everywhere in } \Omega \quad m \to \infty. \tag{56}$$

$$A_3^m(x) \longrightarrow 0 \text{ almost everywhere in } \Omega \quad m \to \infty. \tag{57}$$

$$\begin{aligned} A^m(x) &= \sum_{i=1}^{N} \left[a_i^m(x, u^m, \nabla u^m) - a_i^m(x, u^m, \nabla u) \right] (u^m - u)_{x_i} \\ &= \sum_{i=1}^{N} a_i^m(x, u^m, \nabla u^m) \cdot u_{x_i}^m - \sum_{i=1}^{N} a_i^m(x, u^m, \nabla u^m) \cdot u_{x_i} \\ &\quad - \sum_{i=1}^{N} a_i^m(x, u, \nabla u) \cdot u_{x_i}^m + \sum_{i=1}^{N} a_i^m(x, u, \nabla u) \cdot u_{x_i} \end{aligned}$$

applying (1), (22), (52) and (53) we obtain

$$A_1^m(x) \geq c(m) \longrightarrow 0 \text{ as } m \to \infty.$$

Hence, using the diagonal process, we conclude the convergence (55).

As in [32], let $A_i(u) = a_i(x, u, \nabla v)$ $i = 1, \ldots, N$ be Nemytsky operators for $v \in \mathring{W}_B^1(\Omega)$ fixed and $x \in \Omega(R)$, continuous in $L_{\bar{B}_i}(\Omega(R))$ for any $R > 0$.

Thus, according to (10), (27) and the diagonal process, we have for any $R > 0$

$$A_2^m(x) \longrightarrow 0 \text{ almost everywhere in } \Omega \quad m \to \infty.$$

Applying the inequality (10) we obtain

$$
\begin{aligned}
A_3^m(x) &\leq 2\sum_{i=1}^{N} ||\, b_i^m(x,u^m,\nabla u^m) - b_i^m(x,u,\nabla u)\,||_{B_i,\Omega(R)} \cdot ||\, u^m - u\,||_{\mathring{W}_B^1(\Omega)} \\
&\leq 2c(m)\cdot ||\, u^m - u\,||_{\mathring{W}_B^1(\Omega)}.
\end{aligned}
$$

Hence, combining to (27) and the diagonal process, we have for any $R > 0$

$$A_3^m(x) \longrightarrow 0 \ \text{ almost everywhere in } \ \Omega \quad m \to \infty.$$

Consequently, by (55), (56), (57) and the selective convergences we deduce that

$$A^m(x) \longrightarrow 0 \ \text{ almost everywhere in } \ \Omega \quad m \to \infty. \tag{58}$$

Let $\Omega' \subset \Omega$, meas $\Omega' =$ meas Ω, and the conditions (27), (58) are true, and (20)–(23) are satisfied.

We prove the convergence

$$u_{x_i}^m(x) \longrightarrow u_{x_i}(x) \ \text{ everywhere in } \ \Omega \ \text{ for } \ i = 1,\ldots,N\ , \ m \to \infty \tag{59}$$

By the absurd, suppose we do not have convergence at the point $x^* \in \Omega'$.

Let $u^m = u_{x_i}^m(x^*)$, $u = u_{x_i}(x^*)$, $i = 1,\ldots,N$, and $\hat{a} = \varphi_1(x^*)$, $\bar{a} = \varphi(x^*)$.

Suppose that the sequence $\sum_{i=1}^{N} B_i(u^m)$ $m = 1,\ldots,\infty$ is unbounded.

Let $\epsilon \in \left(0, \frac{\bar{a}}{1+\hat{a}}\right)$ is fixed, according to (2), (4) and the conditions (20), (22), we get

$$
\begin{aligned}
A^m(x^*) = &\sum_{i=1}^{N}\Big(a_i^m(x^*,u^m,\nabla u^m) - a_i^m(x^*,u,\nabla u)\Big)\nabla(u^m - u) \\
&+ \sum_{i=1}^{N}\Big(b_i^m(x^*,u^m,\nabla u^m) - b_i^m(x^*,u,\nabla u)\Big)(u^m - u) \\
= &\sum_{i=1}^{N} a_i^m(x^*,u^m,\nabla u^m)\,\nabla u^m - \sum_{i=1}^{N} a_i^m(x^*,u^m,\nabla u^m)\,\nabla u \\
&- \sum_{i=1}^{N} a_i^m(x^*,u,\nabla u)\,\nabla u^m + \sum_{i=1}^{N} a_i^m(x^*,u,\nabla u)\,\nabla u \\
&+ \sum_{i=1}^{N} b_i^m(x^*,u^m,\nabla u^m)\,u^j - \sum_{i=1}^{N} b_i^m(x^*,u^m,\nabla u^m)\,u \\
&- \sum_{i=1}^{N} b_i^m(x^*,u,\nabla u)\,u^m + \sum_{i=1}^{N} b_i^m(x^*,u,\nabla u)\,u.
\end{aligned}
$$

Applying the generalized Young inequality and (51), we obtain

$$
\begin{aligned}
A^m(x^*) \geq & \sum_{i=1}^N a_i^m(x^*, u, \nabla u) \cdot \nabla u + \sum_{i=1}^N a_i^m(x^*, u^m, \nabla u^m) \cdot \nabla u^m \\
& - \epsilon \sum_{i=1}^N \bar{B}_i(a_i^m(x^*, u^m, \nabla u^m)) \\
& - c_1(\epsilon) \sum_{i=1}^N B_i(\nabla u) - \epsilon \sum_{i=1}^N \bar{B}_i(a_i^m(x^*, u, \nabla u)) - c_2(\epsilon) \sum_{i=1}^N B_i(\nabla u^m) \\
& + \sum_{i=1}^N b_i^m(x^*, u^m, \nabla u^m) \cdot \nabla u^m + \sum_{i=1}^N b_i^m(x^*, u, \nabla u) \cdot \nabla u \\
& - \sum_{i=1}^N b_i^m(x^*, u^m, \nabla u^m) \cdot \nabla u \\
& - \sum_{i=1}^N b_i^m(x^*, u, \nabla u) \cdot \nabla u^m \\
\geq & \ \bar{a} \sum_{i=1}^N B_i(\nabla u) - \psi(x^*) + \sum_{i=1}^N B_i(\nabla u^m) - \psi(x^*) \\
& - \epsilon \, \hat{a} \sum_{i=1}^N B_i(\nabla u^m) - \epsilon \, \varphi(x^*) \\
& - c_1(\epsilon) \sum_{i=1}^N B_i(\nabla u) - \epsilon \, \hat{a} \sum_{i=1}^N B_i(\nabla u) - \epsilon \, \varphi(x^*) \\
& - c_2 \sum_{i=1}^N B_i(\nabla u^m) - 4h(x^*) \\
& - c_3 \, l(u) \sum_{i=1}^N B_i(\nabla u) - c_4 \, l(u^m) \sum_{i=1}^N B_i(\nabla u^m).
\end{aligned}
$$

So

$$
\begin{aligned}
A^j(x^*) \geq & \big[\bar{a} - c_1(\epsilon) - \epsilon \, \hat{a} \\
& - c_3 \, l(u)\big] \sum_{i=1}^N B_i(\nabla u) + \big[\bar{a} - \epsilon \, \hat{a} \, c_2 \\
& - c_4 \, l(u^m)\big] \sum_{i=1}^N B_i(\nabla u^m) - c_5(\epsilon).
\end{aligned}
$$

So we deduce that the sequence $A^m(x^*)$ is not bounded, which is absurd as far as what is in (58).

As a consequence, the sequences $u^m_{x_i}$, $i = 1, \ldots, N$, $m \to \infty$ are bounded.

Let $u^* = (u_1^*, u_2^*, \ldots, u_N^*)$ the limits of subsequence $u^m = (u_1^m, \ldots, u_N^m)$ with $m \to \infty$. Then, taking into account (27), we obtain

$$
u^m_{x_i} \longrightarrow u^*_{x_i} \quad , \quad i = 1, \ldots, N. \tag{60}
$$

As a result, from (58), (60) and the fact that $a_i^m(x^*, u, \nabla u)$ are continuous in u (because they are Carathéodory functions), we have

$$\sum_{i=1}^{N} \left(a_i^m(x^*, u^m, \nabla u^m) - a_i^m(x^*, u, \nabla u) \right) \cdot (u^m_{x_i} - u_{x_i}) = 0,$$

and from (21) we have, $u^*_{x_i} = u_{x_i}$. This contradicts the fact that there is no convergence at the point x^*.

And referring to (27), (60) and the fact that $a_i^m(x^*, u, \nabla u)$ are continuous u, so for $m \to \infty$ we get

$$a_i^m(x, u^m, \nabla u^m) \longrightarrow a_i^m(x, u, \nabla u), \;\; i = 1, \dots, N \;\; \text{almost everywhere in} \;\; \Omega.$$

Using Lemma 3.5 we find the weak convergences

$$a_i^m(x, u^m, \nabla u^m) \rightharpoonup a_i^m(x, u, \nabla u) \;\; \text{in} \;\; L_{\bar{B}_i(\Omega)}, \; i = 1, \dots, N. \tag{61}$$

The weak convergence (48) follows from (61).

Furthermore,to complete the proof, we note that (49) is implied from (46) and (58):

$$\begin{aligned} < A(u^m), u^m - u > =lt;A(u^m) - A(u), u^m - u > \\ + < A(u), u^m - u > \to 0, \; m \to \infty. \end{aligned}$$

We're ending this section by a suitable example, that checks all the above conditions and propositions,

Example 5.1 Let Ω be an unbounded domain of $\mathbb{R}^N$, $(N \geq 2)$. By Theorems 3.1 and 4.1 it exists a unique entropy solution based on the Definition 1.1 of the following anisotropic problem $(\mathcal{P}_1)$:

$$(\mathcal{P}_1) \begin{cases} \tilde{a} \displaystyle\sum_{i=1}^{N} \bar{B}_i^{-1} B_i(|\nabla u|) + l(u) \cdot \sum_{i=1}^{N} B_i(|\nabla u|) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

with $\tilde{a}$ is a positive constant, $l : \mathbb{R} \longrightarrow \mathbb{R}^+$ a positive continuous functions such as $l \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $f \in L^1(\Omega)$ and

$$B(z) = |z|^b (|ln|z|| + 1), \; b > 1$$

satisfying the Δ_2-condition.

References

1. Aberqi, A., Bennouna, J., Hammoumi, M.: Non-uniformly degenerated parabolic equations with L 1-data. AIP Conf. Proc. **2074**, 020002 (2019)
2. Aberqi, A., Bennouna, J., Elmassoudi, M., Hammoumi, M.: Existence and uniqueness of a renormalized solution of parabolic problems in Orlicz spaces. Monatshefte für Mathematik **189**, 195–219 (2019)

3. Aberqi, A., Bennouna, J., Mekkour, M., Redwane, H.: Nonlinear parabolic inequalities with lower order terms. Appl. Anal. **96**, 2102–2117 (2017)
4. Aharouch, L., Benkirane, A., Rhoudaf, M.: Existence results for some unilateral problems without sign condition with obstacle free in Orlicz spaces. Nonlinear Anal. Theory Methods Appl. **68**, 2362–2380 (2008)
5. Aharouch, L., Bennouna, J.: Existence and uniqueness of solutions of unilateral problems in Orlicz spaces. Nonlinear Anal. Theory Methods Appl. **72**, 3553–3565 (2010)
6. Aharrouch, B., Boukhrij, M., Bennouna, J.: Existence of solutions for a class of degenerate elliptic equations in $P(x)$-Sobolev spaces. Topol. Methods Nonlinear Anal. **51**, 389–411 (2018)
7. Barletta, G.: On a class of fully anisotropic elliptic equations. Nonlinear Anal. **197**, 111838 (2020)
8. Bendahmane, M., Karlsen, K.H.: Nonlinear anisotropic elliptic and parabolic equations in R N with advection and lower order terms and locally integrable data. Potential Anal. **22**, 207–227 (2005)
9. Benslimane, O., Aberqi, A., Bennouna, J.: Existence and Uniqueness of Weak solution of $p(x)$ -laplacian in Sobolev spaces with variable exponents in complete manifolds, arXiv preprint arXiv :2006.04763, (2020)
10. Benslimane, O., Aberqi, A., Bennouna, J.: The existence and uniqueness of an entropy solution to unilateral Orlicz anisotropic equations in an unbounded domain. Axioms **9**, 109 (2020)
11. Blanchard, D., Guibé, O., Redwane, H.: Existence and uniqueness of a solution for a class of parabolic equations with two unbounded nonlinearities. Commun. Pure Appl. Anal. **15**, 197–217 (2016)
12. Blanchard, D., Murat, F., Redwane, H.: Existence and uniqueness of a renormalized solution for a fairly general class of nonlinear parabolic problems. J. Differ. Equ. **177**, 331–374 (2001)
13. Boccardo, L., Gallouët, Th, Vazquez, J.L.: Nonlinear elliptic equations in $\mathbb{R}^N$ without growth restrictions on the data. J. Differ. Equ. **105**, 334–363 (1993)
14. Boccardo, L., Gallouët, Th: Nonlinear elliptic equations with right hand side measures. Commun. Partial Differ. Equ. **17**, 89–258 (1992)
15. Bonanno, G., Bisci, G.M., Rădulescu, V.: Quasilinear elliptic non-homogeneous Dirichlet problems through Orlicz–Sobolev spaces. Nonlinear Anal. Theory Methods Appl. **75**, 4441–4456 (2012)
16. Bonanno, G., Bisci, G.M., Rădulescu, V.: Arbitrarily small weak solutions for a nonlinear eigenvalue problem in Orlicz–Sobolev spaces. Monatshefte für Mathematik **165**, 305–318 (2012)
17. Bonanno, G., Bisci, G.M., Rădulescu, V.: Infinitely many solutions for a class of nonlinear eigenvalue problem in Orlicz–Sobolev spaces. Comptes Rendus Mathématique **349**, 263–268 (2011)
18. Bonanno, G., Bisci, G.M., Rădulescu, V.: Existence of three solutions for a non-homogeneous Neumann problem through Orlicz–Sobolev spaces. Nonlinear Anal. Theory Methods Appl. **74**, 4785–4795 (2011)
19. Brezis, H.: Semilinear equations in$\mathbb{R}^N$without condition at infinity. Appl. Math. Optim. **12**, 271–282 (1984)
20. Cammaroto, F., Vilasi, L.: On a perturbed p (x)-Laplacian problem in bounded and unbounded domain. J. Math. Anal. Appl. **402**, 71–83 (2013)
21. Chmara, M., Maksymiuk, J.: Anisotropic Orlicz–Sobolev spaces of vector valued functions and Lagrange equations. J. Math. Anal. Appl. **456**, 457–475 (2017)
22. Chmara, M., Maksymiuk, J.: Mountain pass type periodic solutions for Euler–Lagrange equations in anisotropic Orlicz–Sobolev space. J. Math. Anal. Appl. **470**, 584–598 (2019)
23. Cianchi, A.: A fully anisotropic Sobolev inequality. Pac. J. Math. **196**, 283–294 (2000)
24. Diaz, J.I., Oleinik, O.A.: Nonlinear elliptic boundary-value problems in unbounded domains and the asymptotic behaviour of its solutions. Comptes Rendus-Academie Des Sciences Paris Serie **1**(315), 787–787 (1992)
25. Dong, G., Fang, X.: Existence results for some nonlinear elliptic equations with measure data in Orlicz–Sobolev spaces. Bound. Value Probl. **2015**, 18 (2015)
26. Elmahi, A.: Sur certains problèmes elliptiques et paraboliques non linéaires dans les espaces d'Orlicz; Ph.D Thesis. Sidi Mohamed Ben Abdelah University: FEZ, Morocco, (1997)
27. Gossez, J.P.: Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients. Trans. Am. Math. Soc. **190**, 163–205 (1974)
28. Gushchin, A.K.: The Dirichlet problem for a second-order elliptic equation with an Lp boundary function. Sbornik Math. **203**, 1 (2012)
29. Korolev, A.G.: Embedding theorems for anisotropic Sobolev–Orlicz spaces, Vestnik Moskovskogo Universiteta Seriya 1 Matematika Mekhanika, pp. 32–37 (1983)
30. Kozhevnikova, L.M.: On the entropy solution to an elliptic problem in anisotropic Sobolev–Orlicz spaces. Comput. Math. Math. Phys. **57**, 434–452 (2017)
31. Kozhevnikova, L.M.: Existence of entropic solutions of an elliptic problem in anisotropic Sobolev–Orlicz spaces. J. Math. Sci. **241**, 258–284 (2019)

32. Krasnosel'skii, M.A., Rutickii, J.B.: Convex Functions and Orlicz Spaces. Fizmatgiz, Moscow (1958)
33. Laptev, G.I.: Existence of solutions of certain quasilinear elliptic equations in $\mathbb{R}^N$ without conditions at infinity. J. Math. Sci. **150**, 2384–2394 (2008)
34. Lions, J.L.: Quelques méthodes de résolution des problemes aux limites non linéaires. Dunod, Paris (1969)
35. Mihăilescu, M., Pucci, P., Rădulescu, V.: Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent. J. Math. Anal. Appl. **340**, 687–698 (2008)
36. Papageorgiou, N.S., Rădulescu, V., Repovš, D.D.: Ground state and nodal solutions for a class of double phase problems. Zeitschrift für angewandte Mathematik und Physik **71**, 1–15 (2020)
37. Rădulescu, V.D., Repovs, D.D.: Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis, vol. 9. CRC Press, Boca Raton (2015)
38. Ragusa, M.A., Tachikawa, A.: Regularity for minimizers for functionals of double phase with variable exponents. Adv. Nonlinear Anal. **9**, 710–728 (2019)
39. Yang, Sh, Dai, G.: Existence results for a variable exponent elliptic problem via topological method. Bound. Value Probl. **2012**, 99 (2012)