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Well-posedness of the Cauchy problem of Ostrovsky equation in analytic Gevrey spaces and time regularity

Aissa Boukarou[1] · Khaled Zennir[2] · Kaddour Guerbati[1] · Svetlin G. Georgiev[3]

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Abstract
We study the Cauchy problem of the Ostrovsky equation (Ost) $\partial_t u + \partial_x^3 u - \partial_x^{-1} u + u\partial_x u = 0$, where the data in analytic Gevrey spaces on the line and the circle is considered and its local well-posedness in these spaces is proved. The proof is based on bilinear estimates in Bourgain type spaces. Also, Gevrey regularity of the solution in time variable is provided.

Keywords Ostrovsky equation · Well-posedness · Analytic Gevrey spaces · Bourgain spaces · Bilinear estimates · Time regularity

Mathematics Subject Classification 35E15 · 35Q53 · 35B65 · 35C07

1 Introduction

In this paper we investigate a nonlinear model of long waves which describes the propagation of surface waves in the ocean

$$\begin{cases} \partial_t u + \partial_x^3 u - \partial_x^{-1} u + u\partial_x u = 0, x \in \mathbb{R} \; or \; \mathbb{T}, \; t \in \mathbb{R} \\ u(x,0) = \varphi(x), \end{cases} \tag{1.1}$$

✉ Aissa Boukarou
boukarouaissa@gmail.com

Khaled Zennir
k.zennir@qu.edu.sa

Kaddour Guerbati
guerbati_k@yahoo.com

Svetlin G. Georgiev
svetlingeorgiev1@gmail.com

[1] Laboratoire de Mathématiques et Sciences appliquées, Université de Ghardaia, Ghardaïa, Algerie

[2] Department of Mathematics, College of Sciences and Arts, Al-Ras, Qassim University, Al-Mulida, Kingdom of Saudi Arabia

[3] Department of Differential Equations, Faculty of Mathematics and Informatics, University of Sofia, Sofia, Bulgaria

where the operator ∂_x^{-1} denotes a certain antiderivative with respect to the variable x defined for 0-mean value periodic function by the Fourier transform $\widehat{(\partial_x^{-1} f)} = \frac{\widehat{f}(\xi)}{i\xi}$. The model (1.1) was introduced by Ostrovsky [1]. This type of problem comes from the Korteweg-de Vries-Burgers equation

$$\partial_t u + \partial_x^3 u - \partial_{xx} u + u\partial_x u = 0. \tag{1.2}$$

The Eq. (1.2) appears in the literature as a dissipative version of the Korteweg-de Vries equation

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0. \tag{1.3}$$

In some typical situations, due to the effects of viscosity, it is impossible to neglect dissipative effects, and which can lead to the KdV-Burgers equation [2]. Therefore this is a model for the propagation of waves in a non-linear medium that is both dispersive and dissipative. In [3], it was shown that the equations of Kadomtsev–Petviashvili–Burgers

$$\partial_x(\partial_t u + \partial_x^3 u - \partial_x^2 u + u\partial_x u) + \varepsilon\partial_y^2 u = 0, \ \varepsilon = \pm 1. \tag{1.4}$$

model the propagation of electromagnetic waves in a saturated ferromagnetic medium. We can consider these equations as models for the propagation of two-dimensional waves taking into account damping effects. These equation is also dissipative versions of the Kadomtsev–Petviashvili equation

$$\partial_x(\partial_t u + \partial_x^3 u + u\partial_x u) + \varepsilon\partial_y^2 u = 0, \ \varepsilon \pm 1. \tag{1.5}$$

In the context of waves, (KP) equations are universal models for non-linear, nearly unidirectional dispersive waves with weak transverse effects. The sign $\varepsilon = +1$ corresponds to the equation of (KP-II), while the sign $\varepsilon = -1$ corresponds to the equation of (KP-I). The KP-II equation models long waves with small surface tension effects, whereas the KP-I equation models the flow in the presence of strong surface tension effects. These equations are two-dimensional extensions of the Korteweg-de Vries (KdV) equation. By disturbing the Korteweg Vries equation (KdV) with a non-local term, we can obtain the Ostrovsky equation $(1.1)_1$. Several papers have been published and many results have been obtained in classical Sobolev spaces $H^s(\mathbb{R})$ for dynamical system generated by nonlinear partial differentiel equations (see [4–10], and references therein).

Our main goal here is to show, where data in analytic Gevrey spaces on the line and the circle, that the considered problem admits a local well-posedness in analytic Gevrey Bourgain-spaces. The proof is based on bilinear estimates in Bourgain type spaces. Also, Gevrey regularity of the solution in time variable is provided. There is few results about this subject so far.

We are working mainly on the integral equivalent formulation of (1.1) given as follows

$$u(t) = S(t)\varphi - \frac{1}{2}\int_0^t S(t-t')(\partial_x u^2(t'))dt', \tag{1.6}$$

where the unit operator related to the corresponding linear equation is

$$(S(t)\varphi)(x) = \int_{\mathbb{R}} e^{i(x\xi + t\phi(\xi))}\widehat{\varphi}(\xi)d\xi. \tag{1.7}$$

With ϕ we denote the phase function as follows

$$\phi(\xi) = \xi^3 - \xi^{-1}. \tag{1.8}$$

We define the needed spaces begining by the spaces of analytic Gevrey functions that contain our initial data. For $s \in \mathbb{R}$, $\sigma \geq 1$, $a \geq 0$ and $\delta > 0$, denote

$$G^{\sigma,\delta,s,a}(\mathbb{R}) = \left\{\varphi \in L^2(\mathbb{R}); \|\varphi\|_{G^{\sigma,\delta,s,a}}(\mathbb{R}) < \infty\right\}, \tag{1.9}$$

where

$$\|\varphi\|^2_{G^{\sigma,\delta,s,a}(\mathbb{R})} = \int_{\mathbb{R}} e^{2\delta|\xi|^{\frac{1}{\sigma}}} \langle\xi\rangle^{2(s+a)} |\xi|^{-2a} |\hat{\varphi}(\xi)|^2 d\xi,$$

and in the periodic case we define

$$G^{\sigma,\delta,s,a}(\mathbb{T}) = \left\{\varphi \in L^2(\mathbb{T}); \|\varphi\|_{G^{\sigma,\delta,s,a}(\mathbb{T})} < \infty\right\}, \tag{1.10}$$

for

$$\|\varphi\|^2_{G^{\sigma,\delta,s,a}(\mathbb{T})} = \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} e^{2\delta|k|^{\frac{1}{\sigma}}} \langle k\rangle^{2(s+a)} |k|^{-2a} |\hat{\varphi}(k)|^2.$$

Here $\langle\cdot\rangle$ stands for $(1 + |\cdot|^2)^{\frac{1}{2}}$.

The completion of the Schwartz class $S(\mathbb{R}^2)$ is given by $X^a_{\sigma,\delta,s,b}(\mathbb{R}^2)$, subjected to the norm

$$\|u\|_{X^a_{\sigma,\delta,s,b}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} e^{2\delta|\xi|^{\frac{1}{\sigma}}} \langle\xi\rangle^{2(s+a)} |\xi|^{-2a} \langle\tau - \phi(\xi)\rangle^{2b} \mid \hat{u}(\xi,\tau) \mid^2 d\xi d\tau\right)^{\frac{1}{2}}. \tag{1.11}$$

For the periodic case, it is defined as the completion of the space of the functions defined on $\mathbb{T} \times \mathbb{R}$ that are in the Schwartz class in t and are supposed smooth in x, with the norm

$$\|u\|_{X^a_{\sigma,\delta,s,b}(\mathbb{T}\times\mathbb{R})} = \left(\sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \int e^{2\delta|k|^{\frac{1}{\sigma}}} \langle k\rangle^{2(s+a)} |k|^{-2a} \langle\tau - \phi(k)\rangle^{2b} \mid \hat{u}(k,\tau) \mid^2 d\tau\right)^{\frac{1}{2}}. \tag{1.12}$$

For a given interval $I = [-T, T], T > 0$, with $X^a_{\sigma,\delta,s,b}(I \times \mathbb{R})$ we denote the restriction of $X^a_{\sigma,\delta,s,b}(\mathbb{R}^2)$ on $I \times \mathbb{R}$ with the following norm

$$\|u\|_{X^{T,a}_{\sigma,\delta,s,b}} := \|u\|_{X^a_{\sigma,\delta,s,b}(I\times\mathbb{R})} = \inf\left\{\| \, U \, \|_{X^a_{\sigma,\delta,s,b}(\mathbb{R}^2)} : U \mid_{I\times\mathbb{R}} = u\right\}.$$

The paper is organized as follows. In Sect. 2, our main results regarding the well-posedness and regularity in the analytic Gevrey–Bourgain spaces for (1.1) are stated. In Sect. 3, time regularity is proved in details.

2 Main results and proofs

For a $b \in \mathbb{R}$ with $b\pm$ we denote $b \pm \epsilon$ for $\epsilon > 0$ small enough

Theorem 2.1 *Let $s \geq s_0 = -\frac{5}{8}+, b > \frac{1}{2}, \sigma \geq 1$ and $\delta > 0$. Then for any $\varphi \in G^{\sigma,\delta,s,a}$, where $a = \frac{1}{2}-$, there exists $T = T(\| \varphi \|_{G^{\sigma,\delta,s_0,a}})$ such that (1.1) with the initial condition $u \mid_{t=0} = \varphi$ has a solution u, satisfying*

$$u \in X^{T,a}_{\sigma,\delta,s,b} \subseteq C\big([-T,T], G^{\sigma,\delta,s,a}\big).$$

Moreover, this solution is unique in the class of $X^{T,a}_{\sigma,\delta,s,b}$, and the mapping

$$F : G^{\sigma,\delta,s,a} \longrightarrow X^{a,T}_{\sigma,\delta,s,b}, \quad \varphi \mapsto u,$$

is Lipschitz continuous.

Our next goal is to study Gevrey's temporal regularity of the unique solution obtained in Theorem . A periodic and non-periodic function $f(x)$ is the Gevrey class of order σ, if there exists a constant $C > 0$ such that

$$\sup_{x \in \mathbb{R} or \mathbb{T}} \mid \partial_x^l f(x) \mid \leq C^{l+1}(l!)^\sigma \quad l = 0,1,2,\ldots. \tag{2.1}$$

Here we will show that for $x \in \mathbb{R}$ or $\mathbb{T}$, for every $t \in (-T,T)$ and $j,l \in \{0,1,2,\ldots\}$, there exist $C > 0$ such that,

$$\sup_{\substack{t \in (-T,T) \\ x \in \mathbb{R} or \mathbb{T}}} \mid \partial_t^j \partial_x^l u(x,t) \mid \leq C^{j+l+1}(j!)^{3\sigma}(l!)^\sigma. \tag{2.2}$$

i.e, $u(\cdot,t) \in G^\sigma$ in spacial variable and $u(x,\cdot) \in G^{3\sigma}$ in time variable .

Theorem 2.2 *Let $s > -\frac{5}{8}+, \sigma \geq 1, a = \frac{1}{2}-$ and $\delta > 0$. If $\varphi \in G^{\sigma,\delta,s,a}$, then the solution $u \in C\big([-T,T], G^{\sigma,\delta,s,a}\big)$, given by Theorem 2.1, belongs to the Gevrey class $G^{3\sigma}$ in time variable.*

To prove our main results we have a need of some bilinear estimates in the analytic Bourgain spaces. Note that the spaces $X^a_{\sigma,\delta,s,b}$ are continuously embedded in $C\big([-T,T], G^{\sigma,\delta,s,a}(\mathbb{R})\big)$, provided $b > 1/2$. We start with the following useful lemma.

Lemma 2.3 *Let $b > \dfrac{1}{2}, s \in \mathbb{R}, \sigma \geq 1, \delta > 0$ and $a \geq 0$. Then, for all $T > 0$ we have*

$$X^a_{\sigma,\delta,s,b}(\mathbb{R}^2) \hookrightarrow C\big(\mathbb{R}, G^{\sigma,\delta,s,a}(\mathbb{R})\big),$$

and

$$X^{T,a}_{\sigma,\delta,s,b} \hookrightarrow C\big([-T,T], G^{\sigma,\delta,s,a}(\mathbb{R})\big).$$

Proof Observe that the operator A, defined by

$$\widehat{Au}^x(\xi,t) = e^{\delta|\xi|^{\frac{1}{\sigma}}}\widehat{u}^x(\xi,t), \tag{2.3}$$

satisfies the relations

$$\|u\|_{X^a_{\sigma,\delta,s,b}} = \|Au\|_{X_{s,a,b}},$$

and

$$\|u\|_{G^{\sigma,\delta,s,a}} = \|Au\|_{H^{s,a}},$$

where $X_{s,a,b}$ is the space defined in [11]. From Lemma 1.1 of [11], we have that Au belongs to $C(\mathbb{R}, H^{s,a})$ and there exists $C_0 > 0$ such that

$$\| Au \|_{C(\mathbb{R},H^{s,a})} \le C_0 \| Au \|_{X_{s,a,b}} .$$

Hence, it follows that u belongs to $C\big([-T,T], G^{\sigma,\delta,s,a}\big)$ and

$$\| u(x,t) \|_{C([-T,T],G^{\sigma,\delta,s,a})} \le C_0 \| u \|_{X^a_{\sigma,\delta,s,b}} .$$

This completes the proof. □

2.1 Existence of solution

We take Fourier transform with respect to x and y of the Cauchy problems (1.1) and we get

$$u(t) = S(t)\varphi - \frac{1}{2}\int_0^t S(t-t')\big(\partial_x u^2(t')\big)dt',$$

Now we take a cut-off function $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi = 1$ in $[-1,1]$ and supp$\psi \subset [-2,2]$. We consider the operator Φu, given by

$$\Phi(u) = \psi(t)S(t)\varphi - \frac{\psi_T(t)}{2}\int_0^t S(t-t')\big(\partial_x u^2(t')\big)dt', \tag{2.4}$$

where $\psi_T(t) = \psi(\frac{t}{T})$. Now we will estimate the fist part in the RHS of (2.4).

Lemma 2.4 *Let $s \in \mathbb{R}, b \ge 0, 0 < a < 1, \delta > 0$ and $\sigma \ge 1$. For any $T > 0$, there is a constant $C > 0$, depending only on ψ and b, such that*

$$\|\psi_T(t)S(t)\varphi\|_{X^a_{\sigma,\delta,s,b}} \le C\, T^{\frac{1}{2}-b}\|\varphi\|_{G^{\sigma,\delta,s,a}}, \tag{2.5}$$

for all $\varphi \in G^{\sigma,\delta,s,a}$.

Proof We have

$$\begin{aligned}\psi_T(t)S(t)\varphi &= C\psi_T(t)\int_{-\infty}^{+\infty} e^{i(x\xi+t\phi(\xi))}\widehat{\varphi}(\xi)d\xi\\ &= CT\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(x\xi+t\tau)}\widehat{\psi}(T(\tau-\phi(\xi)))\widehat{\varphi}(\xi)d\xi d\tau.\end{aligned}$$

Then

$$\begin{aligned}&\| \psi_T(t)S(t)\varphi \|^2_{X^a_{\sigma,\delta,s,b}}\\ &= CT^2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{2\delta|\xi|^{\frac{1}{\sigma}}}\left(1+\mid\xi\mid^2\right)^{s+a}\mid\xi\mid^{-2a}\left(1+\mid\tau-\phi(\xi)\mid^2\right)^b\mid\widehat{\psi}(T(\tau-\phi(\xi)))\mid^2\mid\widehat{\varphi}(\xi)\mid^2 d\xi d\tau\\ &= C\int_{-\infty}^{+\infty} e^{2\delta|\xi|^{\frac{1}{\sigma}}}\left(1+\mid\xi\mid^2\right)^{s+a}\mid\xi\mid^{-2a}\mid\widehat{\varphi}(\xi)\mid^2\left(T^2\int_{-\infty}^{+\infty}\mid\widehat{\psi}(T(\tau-\phi(\xi)))\mid^2\left(1+\mid\tau-\phi(\xi)\mid^2\right)^b d\tau\right)d\xi.\end{aligned}$$

For the inner integral, using that $b > 1/2$and $0 < T < 1$, we obtain

$$\begin{aligned}&T^2\int_{-\infty}^{+\infty}\mid\widehat{\psi}(T(\tau-\phi(\xi)))\mid^2\left(1+\mid\tau-\phi(\xi)\mid^2\right)^b d\tau\\ &\le CT^2\int_{-\infty}^{+\infty}\mid\widehat{\psi}(T(\tau-\phi(\xi)))\mid^2 d\tau + CT^2\int_{-\infty}^{+\infty}\mid\widehat{\psi}(T(\tau-\phi(\xi)))\mid^2\mid\tau-\phi(\xi)\mid^{2b} d\tau\\ &\le CT + CT^{(1-2b)} \le CT^{(1-2b)}.\end{aligned}$$

This completes the proof. □

Now we will estimate the second part in RHS of (2.4).

Lemma 2.5 *Let* $s\in\mathbb{R}$, $0<a<1$, $\delta>0$, $\sigma\ge 1$ *and* $-\frac{1}{2}<b'\le 0\le b\le b'+1$, *for any* $T>0$. *We have*

$$\|\psi_T(t)\int_0^t S(t-t')F(x,t')\mathrm{d}t'\|_{X^a_{\sigma,\delta,s,b}} \le CT^{1+b'-b}\|F\|_{X^a_{\sigma,\delta,s,b'}}. \tag{2.6}$$

Proof Define $U(x,t)=\psi_T(t)\int_0^t S(t-t')F(x,t')\mathrm{d}t'$. For the operator A, given by 2.3, we have

$$\begin{aligned}\widehat{AU}^x(\xi,t) &= \psi_T(t)\int_0^t \left(e^{i(t-t')\phi(\xi)}\right)e^{\delta|\xi|^{\frac{1}{\sigma}}}\widehat{F}^x(\xi,t')dt'\\ &= \psi_T(t)\int_0^t [S(t-\widehat{t')(A}F)]^x(\xi,t')\mathrm{d}t'.\end{aligned}$$

Thus,

$$\| U\|_{X^a_{\sigma,\delta,s,b}} = \| AU\|_{X_{s,a,b}} = \|\psi_T(t)\int_0^t S(t-t')AF(x,t')\mathrm{d}t'\|_{X_{s,a,b}}.$$

Now, applying Lemma 2.3-(ii) from [11], we obtain

$$\|\psi_T(t)\int_0^t S(t-t')AF(x,t')\mathrm{d}t'\|_{X_{s,a,b}} \le CT^{1+b'-b}\|AF\|_{X_{s,a,b'}}.$$

This completes the proof. □

Lemma 2.6 *Let $s \geq s_0 = -\frac{5}{8}+, \sigma \geq 1, \delta > 0$. Then for $a = \frac{1}{2}-, b' = -\frac{1}{2}+$ and $b > \frac{1}{2}$, one has*

$$\| \partial_x(u_1u_2) \|_{X^a_{\sigma,\delta,s,b'}} \leq C\Big(\| u_1 \|_{X^a_{\sigma,\delta,s,b}} \| u_2 \|_{X^a_{\sigma,\delta,s_0,b}} + \| u_1 \|_{X^a_{\sigma,\delta,s_0,b}} \| u_2 \|_{X^a_{\sigma,\delta,s,b}}\Big). \tag{2.7}$$

Proof Firstly, observe that

$$\begin{aligned} e^{\delta|\xi|^{\frac{1}{\sigma}}} \widehat{u_1u_2} &= (2\pi)^{-2} e^{\delta|\xi|^{\frac{1}{\sigma}}} \widehat{u_1} * \widehat{u_2} \\ &\leq (2\pi)^{-2} \int\int e^{\delta|\xi-\eta|^{\frac{1}{\sigma}}} \widehat{u_1}(\xi-\eta, \tau-\rho) e^{\delta|\eta|^{\frac{1}{\sigma}}} \widehat{u_2}(\eta,\rho) d\eta d\rho \\ &= \widehat{Au_1Au_2}. \end{aligned} \tag{2.8}$$

Since $e^{\delta|\xi|^{\frac{1}{\sigma}}} \leq e^{\delta|\xi-\eta|^{\frac{1}{\sigma}}+\delta|\eta|^{\frac{1}{\sigma}}}$, for all $\sigma \geq 1$, we have

$$\begin{aligned} \| \partial_x(u_1u_2) \|_{X^a_{\sigma,\delta,s,b'}} &= \| e^{\delta|\xi|^{\frac{1}{\sigma}}} \langle\xi\rangle^{(s+a)} | \xi |^{-a} \langle \tau - \phi(\xi)\rangle^b \widehat{\partial_x(u_1u_2)}(\xi,\tau) \|_{L^2_{\xi,\tau}} \\ &= \| \langle\xi\rangle^{(s+a)} | \xi |^{-a} \langle \tau - \phi(\xi)\rangle^b e^{\delta|\xi|^{\frac{1}{\sigma}}} \xi\widehat{u_1u_2}(\xi,\tau) \|_{L^2_{\xi,\tau}} \\ &\leq \| \langle\xi\rangle^{(s+a)} | \xi |^{-a} \langle \tau - \phi(\xi)\rangle^b (\widehat{\partial_x Au_1Au_2})(\xi,\tau) \|_{L^2_{\xi,\tau}} \\ &= \| \partial_x(Au_1Au_2) \|_{X_{s,a,b'}} . \end{aligned}$$

Now, applying Theorem 3.1 of [11], we get that there exists a constant $C > 0$ such that

$$\begin{aligned} \| \partial_x(Au_1Au_2) \|_{X_{s,a,b'}} &\leq C\Big(\| Au_1 \|_{X_{s,a,b}} \| Au_2 \|_{X_{s_0,a,b}} + \| Au_1 \|_{X_{s_0,a,b}} \| Au_2 \|_{X_{s,a,b}}\Big) \\ &= C\Big(\| u_1 \|_{X^a_{\sigma,\delta,s,b}} \| u_2 \|_{X^a_{\sigma,\delta,s_0,b}} + \| u_1 \|_{X^a_{\sigma,\delta,s_0,b}} \| u_2 \|_{X^a_{\sigma,\delta,s,b}}\Big). \end{aligned}$$

This completes the proof. □

Now, we are ready to estimate all terms in (2.4) by using the bilinear estimates in the above Lemmas.

Lemma 2.7 *Let $s \geq s_0 = -\frac{5}{8}+, a = \frac{1}{2}-, \sigma \geq 1, \delta > 0$, $b' = \frac{1}{2}-$ and $b = \frac{1}{2} + \frac{\epsilon}{2}$. Then for $\varphi \in G^{\sigma,\delta,s,a}$ and $0 < T \leq 1$, with some constant $C > 0$, we have*

$$\| \Phi(u) \|_{X^a_{\sigma,\delta,s,b}} \leq C\Big(\| \varphi \|_{G^{\sigma,\delta,s,a}} + T^{1+b'-b} \| u \|^2_{X^a_{\sigma,\delta,s,b}}\Big), \text{ for all } u \in X^a_{\sigma,\delta,s,b}, \tag{2.9}$$

and

$$\| \Phi(u) - \Phi(v) \|_{X^a_{\sigma,\delta,s,b}} \leq CT^{1+b'-b} \| u - v \|_{X^a_{\sigma,\delta,s,b}} \| u + v \|_{X^a_{\sigma,\delta,s,b}}, \text{ for all } u, v \in X^a_{\sigma,\delta,s,b}. \tag{2.10}$$

Proof In order to prove (2.9), we use that

$$\begin{aligned}\| \Phi(u) \|_{X^a_{\sigma,\delta,s,b}} &\leq \|\psi_T(t)S(t)\varphi\|_{X^a_{\sigma,\delta,s,b}} + \|\psi_T(t)\int_0^t S(t-t')\big(\partial_x u^2\big)(t')dt'\|_{X^a_{\sigma,\delta,s,b}}\\ &\leq C \| \varphi \|_{G^{\sigma,\delta,s,a}} + CT^{1+b'-b} \| \partial_x u^2 \|_{X^a_{\sigma,\delta,s,b'}}\\ &\leq C \| \varphi \|_{G^{\sigma,\delta,s,a}} + CT^{1+b'-b} \| u \|^2_{X^a_{\sigma,\delta,s,b}} .\end{aligned}$$

For the estimate (2.10), we observe that

$$\Phi(u) - \Phi(v) = \psi_T(t) \int_0^t S(t-t')\big(\partial_x u^2 - \partial_x v^2\big)(x,t')dt',$$

where $\omega = \partial_x u^2 - \partial_x v^2$ is now given by

$$\omega = \partial_x(u^2 - v^2) = \partial_x[(u+v)(u-v)].$$

Thus, from the previous results, we obtain (2.10). This completes the proof. □

We shall exhibit that the map Φ is a contraction on the ball $\mathbb{B}(0,r)$ to $\mathbb{B}(0,r)$ where,

$$\mathbb{B}(0,r) = \{u \in X^a_{\sigma,\delta,s,b}; \|u\|_{X^a_{\sigma,\delta,s,b}} \leq r\},$$

with $r = 2C\|\varphi\|_{G^{\sigma,\delta,s,a}}$.

Lemma 2.8 *Let $s \geq s_0 = -\frac{5}{8}+$, $a = \frac{1}{2}-$, $\sigma \geq 1$, $\delta > 0$ and $b = \frac{1}{2} + \frac{\epsilon}{2}$. Then for $\varphi \in G^{\sigma,\delta,s,a}$, there exist $c_0 \leq 1$ and $\beta > 1$ such that for*

$$T = \frac{c_0}{(1 + \| \varphi \|_{G^{\sigma,\delta,s,a}})^{\beta}}, \tag{2.11}$$

the map $\Phi : \mathbb{B}(0,r) \to \mathbb{B}(0,r)$ is a contraction. Here $\mathbb{B}(0,r)$ is given by

$$\mathbb{B}(0,r) = \{u \in X^a_{\sigma,\delta,s,b}; \|u\|_{X^a_{\sigma,\delta,s,b}} \leq r\},$$

with $r = 2C\|\varphi\|_{G^{\sigma,\delta,s,a}}$.

Proof From Lemma 2.7, for any $u \in \mathbb{B}(0,r)$, we have

$$\| \Phi(u) \|_{X^a_{\sigma,\delta,s,b}} \leq C\Big(\| \varphi \|_{G^{\sigma,\delta,s,a}} + T^{1+b'-b} \| u \|^2_{X^a_{\sigma,\delta,s,b}}\Big) \leq \frac{r}{2} + CT^{1+b'-b} r^2.$$

If we take $\beta = \frac{1}{1+b'-b}$ and $c_0 = (8C^2)^{-\frac{1}{1+b'-b}}$, then for T given by (2.11), we have that $T^{1+b'-b} \leq \frac{1}{4Cr}$. Hence,

$$\| \Phi(u) \|_{X^a_{\sigma,\delta,s,b}} \leq r, \quad \forall u \in \mathbb{B}(0,r).$$

Then, Φ maps $\mathbb{B}(0,r)$ into $\mathbb{B}(0,r)$, which is a contraction, because

$$\begin{aligned} \| \Phi(u) - \Phi(v) \|_{X^a_{\sigma,\delta,s,b}} &\leq CT^{1+b'-b} \| u - v \|_{X^a_{\sigma,\delta,s,b}} \| u + v \|_{X^a_{\sigma,\delta,s,b}} \\ &\leq CT^{1+b'-b} 2r \| u - v \|_{X^a_{\sigma,\delta,s,b}} \\ &\leq \tfrac{1}{2} \| u - v \|_{X^a_{\sigma,\delta,s,b}}, \quad \forall u, v \in \mathbb{B}(0, r). \end{aligned}$$

This completes the proof. □

2.2 The uniqueness

Note that the following embedding

$$X^a_{\sigma,\delta,s,b} \subseteq C\big(\mathbb{R}, G^{\sigma,\delta,s,a}\big),$$

is a key for the persistence property. Let $u, v \in X^{T,a}_{\sigma,\delta,s_0,b}$ be two solutions of (1.1) with extensions $\tilde{u}, \tilde{v} \in X^a_{\sigma,\delta,s_0,b}$ such that

$$T' = \sup\{t \in [0, T] : u(t) = v(t)\} < T.$$

Define $u^*(t) = \tilde{u}(t + T'), v^*(t) = \tilde{u}(t + T')$ for $T' \leq t \leq T - T'$. Since u and v are two solutions of (1.1), we have

$$u^*(t) - v^*(t) = \psi_T(t) \int_0^t S(t-t')\big(\partial_x u^{*2}\big)(x, t')dt' - \psi_T(t) \int_0^t S(t-t')\big(\partial_x v^{*2}\big)(x, t')dt',$$

for $T' \leq t \leq T - T'$. Therefore, for a small $\lambda > 0$, we get

$$\|\psi_\lambda(t)(u^* - v^*)\|_{X^a_{\sigma,\delta,s_0,b}} \leq C\lambda^{1+b'-b} \|\psi_\lambda(t)(u^* - v^*)\|_{X^a_{\sigma,\delta,s_0,b}} \Big(\|u^*\|_{X^a_{\sigma,\delta,s_0,b}} + \|v^*\|_{X^a_{\sigma,\delta,s_0,b}} \Big).$$

Choosing λ small enough, one can conclude that $u^*(t) = v^*(t)$ for $|t| \leq \lambda$. This implies that $u(t + T') = v(t + T')$, for $|t| \leq \lambda$, which contradicts with the definition of T'. If u, v did not coincide on $[-T, 0]$, we would obtain a similar contradiction.

2.3 Continuous dependence on the initial data

We need to prove the following Lemma.

Lemma 2.9 *Let $s \geq s_0 = -\frac{5}{8}+$, $a = \frac{1}{2}-$, $\sigma \geq 1$, $\delta > 0$ and $b = \frac{1}{2} + \frac{\epsilon}{2}$. Then for $\varphi \in G^{\sigma,\delta,s,a}$, and $T = T(\|\varphi\|_{G^{\sigma,\delta,s,a}})$ be given as in (2.11). Suppose that the solution $u \in X^{T,a}_{\sigma,\delta,s,b} \subseteq C\big([-T, T], G^{\sigma,\delta,s,a}\big)$ of (1.1) is unique. Then, for a given $T' \in (0, T)$ there exists $R = R(T') > 0$ such that*

$$\Gamma : W \longrightarrow X^{T',a}_{\sigma,\delta,s,b}$$

$$\tilde{u}_0 \mapsto \tilde{u},$$

is a Lipschitz map. Here W is defined by

$$W = \left\{ \tilde{u}_0 \in G^{\sigma,\delta,s,a}; \|\tilde{u}_0 - \varphi\|_{G^{\sigma,\delta,s,a}} < R \right\}.$$

Proof Since $T' \in (0, T)$, there exists $R > 0$ so that

$$T' < \tilde{T}, \forall \tilde{u}_0 \in W, \tag{2.12}$$

which is equivalent to

$$1 + \|\tilde{u}_0\|_{G^{\sigma,\delta,s,a}} < \left(\frac{c_0}{T'} \right)^{1/\beta}. \tag{2.13}$$

Also, if $\tilde{u}_0 \in W$, we have

$$\|\tilde{u}_0\|_{G^{\sigma,\delta,s,a}} < R + \|\varphi\|_{G^{\sigma,\delta,s,a}}.$$

To obtain (2.13), we choose R so that

$$0 < R < \left(\frac{c_0}{T'} \right)^{1/\beta} - \left(1 + \|\tilde{u}_0\|_{G^{\sigma,\delta,s,a}} \right). \tag{2.14}$$

Since $T' < T$, this involves that the RHS of (2.14) is positive. If $\tilde{u}_0, u_0^* \in W$, with $\Gamma(\tilde{u}_0) = \tilde{u}$ and $\Gamma(u_0^*) = u^*$, then using Lemma 2.3, we obtain

$$\begin{aligned} &\|\Gamma(\tilde{u}_0) - \Gamma(u_0^*)\|_{C\left([-T',T'],G^{\sigma,\delta,s,a}\right)} \\ &= \|\tilde{u}_0 - u_0^*\|_{C\left([-T',T'],G^{\sigma,\delta,s,a}\right)} \\ &\leq C_0 \|\tilde{u}_0 - u_0^*\|_{X^a_{\sigma,\delta,s,b}}. \end{aligned}$$

As $\tilde{u}$ is a fixed point of $\Phi_{\tilde{T}}$ and u^* is a fixed point of Φ_{T^*}, the inequality (2.12) implies that $\psi_{\tilde{T}} = \psi_{T^*}$ on $[-T', T']$. Therefore

$$\begin{aligned} \| \tilde{u} - u^* \|_{X^a_{\sigma,\delta,s,b}} &\leq \|\psi(t) S(t)(\tilde{u}_0 - u_0^*)\|_{X^a_{\sigma,\delta,s,b}} \\ &+ \|\psi_{T'}(t) \int_0^t S(t-t')\left(\partial_x \tilde{u}^2 - \partial_x u^{*2}\right)(x,t')dt'\|_{X^a_{\sigma,\delta,s,b}} \\ &\leq C \| \tilde{u}_0 - u_0^* \|_{G^{\sigma,\delta,s,a}} + C T'^{(1+b'-b)} (\tilde{r} + r^*) \| \tilde{u} - u^* \|_{X^a_{\sigma,\delta,s,b}}, \end{aligned}$$

because $\tilde{u} \in \mathbb{B}(0, \tilde{r})$ and $u^* \in \mathbb{B}(0, r^*)$, where

$$\tilde{r} = 2C \|\tilde{u}_0\|_{G^{\sigma,\delta,s,a}}$$

and

$$r^* = 2C \|u_0^*\|_{G^{\sigma,\delta,s,a}}.$$

Since $c_0 = \left(8C^2\right)^{-\beta}$ and $\beta = \frac{1}{(1+b'-b)}$, we have

$$T'^{(1+b'-b)} \leq \tilde{T}^{(1+b'-b)} = (4C\tilde{r})^{-1},$$

and

$$T'^{(1+b'-b)} \leq T^{*(1+b'-b)} = (4Cr^*)^{-1}.$$

From here, $CT'^{(1+b'-b)}(\tilde{r}+r^*) \leq \frac{1}{2}$ and

$$\|\Gamma(\tilde{u}_0)-\Gamma(u_0^*)\|_{C\left([-T',T'],G^{\sigma,\delta,s,a}\right)} \leq 2CC_0 \parallel \tilde{u}-u^* \parallel_{G^{\sigma,\delta,s,a}} .$$

This completes the proof. □

3 Time regularity

In this section, we shall prove the time regularity of the solution as stated in Theorem 1.2 on the circle, the proof on the line is analogous.

We begin by proving that solution $u \in G^\sigma$ in spacial variable, i.e

$$\sup_{x\in\mathbb{T}} |\partial_x^l u(x,t)| \leq C^{l+1}(l!)^\sigma, \quad l=0,1,2,\dots \tag{3.1}$$

Proposition 3.1 *Let $s>-\frac{5}{8}, \delta>0, \sigma\geq 1, a=\frac{1}{2}-$ and let $u\in C\big([-T,T];G^{\sigma,\delta,s,a}\big)$ be the solution to the Cauchy problem (1.1). Then u belong to G^σ in x variable, for all $t\in[-T,T]$ and there exists a constant $C>0$ for which*

$$|\partial_x^l u(x,t)| \leq C^{l+1}(l!)^\sigma, \tag{3.2}$$

for all $x\in\mathbb{R}$ or $\mathbb{T}$, $|t|\leq T$, for all $l\in\{0,1,2,\dots\}$.

Proof For any $t\in[-T,T]$, we get

$$\begin{aligned}\|\partial_x^l u(\cdot,t)\|_{H^{s,a}}^2 &= \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} |k|^{2l}|k|^{-2a}\langle k\rangle^{2(s+a)}|\widehat{u}(k,t)|^2\\ &= \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} |k|^{2l}e^{-2\delta|k|^{\frac{1}{\sigma}}}\langle k\rangle^{2(s+a)}|k|^{-2a}e^{2\delta|k|^{\frac{1}{\sigma}}}|\widehat{u}(k,t)|^2.\end{aligned}$$

Observe that

$$e^{\frac{2\delta}{\sigma}|k|^{\frac{1}{\sigma}}} = \sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{2\delta}{\sigma}|k|^{\frac{1}{\sigma}}\right)^j \geq \frac{1}{(2l)!}\left(\frac{2\delta}{\sigma}\right)^{2l}|k|^{\frac{2l}{\sigma}}, \quad \forall l\in\{0,1,\dots\}, k\in\mathbb{Z}.$$

This implies that

$$|k|^{2l}e^{-2\delta|k|^{\frac{1}{\sigma}}} \leq C_{\delta,\sigma}^{2l}(2l)!^\sigma.$$

Thus,

$$\begin{aligned}\|\partial_x^l u(\cdot,t)\|^2_{H^{s,a}} \leq & C^{2l}_{\delta,\sigma}(2l)!^\sigma \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} e^{2\delta|k|^{\frac{1}{\sigma}}}\langle k\rangle^{2(s+a)}|k|^{-2a}|\widehat{u}(k,t)|^2\\ =& C^{2l}_{\delta,\sigma}(2l)!^\sigma\|u(\cdot,t)\|^2_{G^{\sigma,\delta,s,a}}.\end{aligned}$$

Since $(2l)! \leq A_1^{2l}(l!)^2$, for some $A_1 > 0$, if $s \geq 0$, then

$$\|\partial_x^l u(\cdot,t)\|_{L^2} \leq \|\partial_x^l u(\cdot,t)\|_{H^{s,a}} \leq C_0 C_1^l (l)!^\sigma \quad \forall\, t\in[-T,T].$$

Here $C_0 = \|u\|_{G^{\sigma,\delta,s,a}}$ and $C_1 = A_1 C_{\sigma,\delta}$. This implies that u is Gevrey of order σ in x, for $s \geq 0$.

Now, for $-\frac{5}{8} < s < 0$, we have

$$\begin{aligned}\|u(\cdot,t)\|_{H^{0,a}} &= \sum_{k\in\mathbb{Z}}\langle k\rangle^{2a}|k|^{-2a}|\widehat{u}(k,t)|^2\\ &\leq C\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\frac{1}{\langle k\rangle^{-2s}}\langle k\rangle^{2a}|k|^{-2a}|\widehat{u}(k,t)|^2\\ &= C\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\langle k\rangle^{2(s+a)}|k|^{-2a}|\widehat{u}(k,t)|^2\\ &= C\|u(\cdot,t)\|_{H^{s,a}}.\end{aligned} \tag{3.3}$$

We note that for any $0 < \epsilon < \delta$ and by (3.3) there exists a positive constant $C_2, C = C_{s,\epsilon} > 0$ such that

$$\|\partial_x^l u(\cdot,t)\|_{H^{0,a}} \leq C_2^l (l)!^\sigma \|u\|_{G^{\sigma,\delta-\epsilon,0,a}} \quad \forall\, t\in[-T,T].$$

and

$$\begin{aligned}\|u(\cdot,t)\|_{G^{\sigma,\delta-\epsilon,0,a}} &= \sum_{k\in\mathbb{Z}} e^{2(\delta-\epsilon)|k|^{\frac{1}{\sigma}}}\langle k\rangle^{2a}|k|^{-2a}|\widehat{u}(k,t)|^2\\ &\leq C\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\frac{e^{2\epsilon|k|^{\frac{1}{\sigma}}}}{\langle k\rangle^{-2s}}\langle k\rangle^{2a}|k|^{-2a}e^{2(\delta-\epsilon)|k|^{\frac{1}{\sigma}}}|\widehat{u}(k,t)|^2\\ &= C\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} e^{2\delta|k|^{\frac{1}{\sigma}}}\langle k\rangle^{2(s+a)}|k|^{-2a}|\widehat{u}(k,t)|^2\\ &= \|u(\cdot,t)\|_{G^{\sigma,\delta,s,a}}.\end{aligned}$$

This implies that if $u \in C\big([-T,T];G^{\sigma,\delta,s,a}\big)$ and $s<0$, then $u \in C\big([-T,T];G^{\sigma,\delta-\epsilon,0,a}\big)$, which allows us to conclude that u is in G^{σ} in x, for all $s>-\frac{5}{8}$. This completes the proof. □

We follow now the strategy adopted by Petronilho et al. [12]. We start by introducing some notations. For $\epsilon>0$, consider the sequences

$$m_q = \frac{c(q!)^{\sigma}}{(q+1)^2}, (q=0,1,2,\dots), \tag{3.4}$$

and

$$M_q = \epsilon^{1-q} m_q, \epsilon>0 \;\; and \;\; (q=1,2,3,\dots), \tag{3.5}$$

where c is chosen (see [13]) such that the following inequality

$$\sum_{0\le l\le k} \binom{k}{l} m_l m_{k-l} \le m_k, \tag{3.6}$$

holds. Removing the endpoints 0 and k in the left-hand side of (3.6) and using the sequence M_q, we obtain

$$\sum_{0<l<k} \binom{k}{l} M_l M_{k-l} \le M_k, \; for \; any \; \epsilon>0. \tag{3.7}$$

Next, one can check that for any $\epsilon>0$ the sequence M_q satisfies the following inequality

$$M_j \le \epsilon M_{j+1}, \; for \; j\ge 2. \tag{3.8}$$

Also, one can check that for a given $C>1$, there exists $\epsilon_0>0$ such that for any $0<\epsilon\le\epsilon_0$, we have

$$C^{j+1}(j!)^{\sigma} \le M_j, \; for \; j\ge 2. \tag{3.9}$$

For $j=1$, it follows from the definition of M_1 and M_2 that

$$M_1 = a\epsilon M_2, \; where \; a=\frac{9}{4(2!)^{\sigma}},$$

for some $C>0$. Also, define the following constants

$$M_0 = \frac{c}{8} \; and \; M = max\{3, \frac{8C}{c}, \frac{4C^2}{c}\}.$$

Lemma 3.2 *Let $u(x,t)$ be the solution to the Cauchy problem (1.1). If $u(x,t)$ satisfies the inequality (3.2), then there exists $\epsilon_0>0$ such that for any $0<\epsilon\le\epsilon_0$ we have*

$$|\partial_t^j \partial_x^l u(x,t)| \le M^{j+1} M_{l+3j}, \tag{3.10}$$

for all $x\in\mathbb{R}$ or $\mathbb{T}, t\in[-T,T]$, $j\in\{0,1,2,\dots\}, l\in\{0,1,2,\dots\}$.

Lemma 3.3 *For given $n,k\in\{0,1,2,\dots\}$, we have*

$$\sum_{p=0}^{n}\sum_{q=0}^{k}\binom{n}{p}\binom{k}{q}L_{(n-p)+3(k-q)}L_{p+3q}\leq\sum_{r=1}^{m}\binom{m}{r}L_rL_{m-r},\tag{3.11}$$

where $L_j, j=0,1,\dots,m$ *are positive real numbers with* $m=n+3k$.

Proof We will prove (3.10) using induction. Let $j=0$. For $l=0$, it follows from (3.2) and the definition of M, that

$$|u(x,t)|\leq C\leq MM_0,\ \ \forall x\in\mathbb{T},\ |t|\leq T.$$

Similarly, for $l=1$, we have

$$|\partial_x u(x,t)|\leq C^2\leq MM_1,\ \ \forall x\in\mathbb{T},\ |t|\leq T.$$

For $l\geq 2$, it follows from (3.2) and (3.9), that there exists $\epsilon_0>0$ such that for any $0<\epsilon\leq\epsilon_0$ we have

$$|\partial_x^l u(x,t)|\leq\leq C^{l+1}(l!)^{\sigma}\leq M_l\leq MM_l,\ \ \forall x\in\mathbb{T},\ |t|\leq T.$$

This completes the proof of (3.10) for $j=0$ and $l\in\{0,1,\dots\}$.

Next, we will assume that (3.10) is true for $0\leq q\leq j$ and $l\in\{0,1,\dots\}$ and we will prove it for $q=j+1$ and $l\in\{0,1,\dots\}$. We begin by noticing that

$$|\partial_t^{j+1}\partial_x^l u|=|\partial_t^j\partial_x^l(\partial_t u)|\ \leq|\partial_t^j\partial_x^{l+3}u|+|\partial_t^j\partial_x^{l-1}u|+|\partial_t^j\partial_x^{l+1}(u^2)|.$$

Using the induction hypotheses and the condition $M>2$ we estimate the term $\partial_t^j\partial_x^{l+3}u$ and $\partial_t^j\partial_x^{l-1}u$ in the following

$$\begin{aligned}|\partial_t^j\partial_x^{l+3}u|&\leq M^{j+1}M_{l+3+3j}=M^{-1}M^{(j+1)+1}M_{l+3(j+1)}\\&\leq\frac{1}{3}M^{(j+1)+1}M_{l+3(j+1)},\end{aligned}\tag{3.12}$$

$$|\partial_t^j\partial_x^{l-1}u|\ \leq M^{j+1}M_{l-1+3j}\leq\frac{\epsilon^4}{3}M^{(j+1)+1}M_{l+3(j+1)}.\tag{3.13}$$

Note that in the last inequality we have used the fact that $l+3j-1\geq 2$.

For the nonlinear term, applying Leibniz's rule twice and using the induction hypothesis, we obtain

$$\begin{aligned}|\partial_t^j\partial_x^{l+1}(u^2)|&\leq\sum_{p=0}^{l+1}\sum_{q=0}^{j}\binom{l+1}{p}\binom{j}{p}|\partial_t^{j-q}\partial_x^{l+1-p}u||\partial_t^q\partial_x^p u|\\&\leq\sum_{p=0}^{l+1}\sum_{q=0}^{j}\binom{l+1}{p}\binom{j}{p}M^{(j-q)+1}M_{l+1-p+3(j-q)}M^{q+1}M_{p+3q}\\&=M^{(j+1)+1}\sum_{p=0}^{l+1}\sum_{q=0}^{j}\binom{l+1}{p}\binom{j}{p}M_{l+1-p+3(j-q)}M_{p+3q}.\end{aligned}$$

Next, using Lemma 3.3 with $n = l + 1, k = j, L_j = M_j, m = l + 1 + 3j$, we obtain

$$\begin{aligned}
&\sum_{p=0}^{l+1}\sum_{q=0}^{j}\binom{l+1}{p}\binom{j}{p}M_{l+1-p+3(j-q)}M_{p+3q} \\
&\leq \sum_{r=1}^{m}\binom{m}{r}L_rL_{m-r} \leq (M_0+\epsilon)M_m \\
&= (M_0+\epsilon)M_{l+3j+1}.
\end{aligned} \tag{3.14}$$

Thus,

$$\begin{aligned}
|\partial_t^j\partial_x^{l+1}(u^2)| &\leq M^{(j+1)+1}(M_0+\epsilon)M_{l+3j+1} \\
&\leq M^{(j+1)+1}\epsilon^2(M_0+\epsilon)M_{l+3(j+1)} \\
&\leq \epsilon^2(M_0+\epsilon)M^{(j+1)+1}M_{l+3(j+1)}.
\end{aligned}$$

Note that in the last inequality we have used the fact that $l + 3j + 1 \geq 2$. Now, we choose $\epsilon \leq \epsilon_0 = \left(\frac{1}{3(M_0+\epsilon)}\right)^{\frac{1}{2}} < 1$ and we obtain that

$$\epsilon^2(M_0+\epsilon) \leq \epsilon^2(M_0+1) \leq (M_0+1)\left(\frac{1}{3(M_0+1)}\right) = \frac{1}{3}.$$

Then

$$|\partial_t^j\partial_x^{l+1}(u^2)| \leq \frac{1}{3}M^{(j+1)+1}M_{l+3(j+1)}.$$

This completes the proof. □

Proof of Theorem 1.2 We have

$$|\partial_t^j\partial_x^l u(x,t)| \leq M^{j+1}M_{l+3j},\ j \in \{0,1,2,\dots\}, \quad l \in \{0,1,2,\dots\},$$

where

$$M_q = \epsilon^{1-q}\frac{c(q!)^\sigma}{(q+1)^2},\ q = 1,2,\dots$$

Applying this inequality for $j \in \{1, 2, \dots\}$ and $l = 0$, we obtain

$$\begin{aligned}
|\partial_t^j u(x,t)| &\leq M^{j+1}M_{3j} = MM^j\epsilon^{1-3j}\frac{c((3j)!)^\sigma}{(3j+1)^2} \\
&\leq M\epsilon c\left(\frac{M}{\epsilon^3}\right)^j((3j)!)^\sigma \\
&\leq L_0L^j((3j)!)^\sigma \\
&\leq L_0L^jA^{3\sigma j}(j!)^{3\sigma} \\
&\leq A_0^{j+1}(j!)^{3\sigma},
\end{aligned} \tag{3.15}$$

where $L_0 = M\epsilon c, L = \frac{M}{\epsilon^3}$ since $(3j)! \leq A^{3j}(j!)^3$ for $A > 0$ and $A_0 = max\{L_0, LA^{3\sigma}\}$. We also have from (3.10) for $j = 0,\ l = 0$ that,

$$|u(x,t)| \leq MM_0 = M\frac{c}{8}, \tag{3.16}$$

for all $(x,t) \in \mathbb{T} \times [-T, T]$. Setting $C = max\{M\frac{c}{8}, A_0\}$, it follows from (3.15) and (3.16), for $j \in \{0, 1, 2, \dots\}$, that we have

$$|\partial_t^j u(x,t)| \leq C^{j+1}(j!)^{3\sigma},$$

for all $(x,t) \in \mathbb{T} \times [-T, T]$. Hence, $u \in G^{3\sigma}$ in the time variable. This completes the proof. □

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