Rendiconti del Circolo Matematico di Palermo Series 2 (2020) 69:1193–1203
https://doi.org/10.1007/s12215-019-00465-6

On some rational contractions in $b_v(s)$-metric spaces

Zoran D. Mitrović[1,2] · Hassen Aydi[3,4] · Zoran Kadelburg[5] · Ghasem Soleimani Rad[6]

Received: 6 April 2019 / Accepted: 24 October 2019 / Published online: 2 November 2019

Abstract
In this paper, we prove versions of Khan type and Dass–Gupta type contraction principles in $b_v(s)$-metric spaces. The results which we obtain generalize many known results in fixed point theory. Examples show how these results can be applied in concrete situations.

Keywords Fixed point · b-metric space · Rectangular metric space · $b_v(s)$-metric space

Mathematics Subject Classification Primary 47H10 · Secondary 55M20

1 Introduction

A lot of generalizations of metric spaces exist, mostly introduced in order to obtain new types of fixed point results using various contractive conditions. Some of these results appear to be simple reformulations of the known results from the framework of metric spaces, with just slightly modified proofs, or even their direct consequences. However, the work in some of generalized spaces is essentially harder. We mention here two of such types of spaces.

✉ Ghasem Soleimani Rad
gh.soleimani2008@gmail.com

Zoran D. Mitrović
zoran.mitrovic@tdtu.edu.vn

Hassen Aydi
hassen.aydi@isima.rnu.tn

Zoran Kadelburg
kadelbur@matf.bg.ac.rs

1 Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam

2 Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

3 Institut Supérieur d'Informatique et des Techniques de Communication, Université de Sousse, 4000 H. Sousse, Tunisia

4 China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

5 Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Beograd, Serbia

6 Young Researchers and Elite club, West Tehran Branch, Islamic Azad University, Tehran, Iran

Bakhtin [5] and Czerwik [7] introduced *b-metric spaces*, modifying the triangle inequality to the following form

$$d(x, z) \leq s[d(x, y) + d(y, z)], \tag{1.1}$$

where $s \geq 1$ is a fixed real number. Going in this direction, Aydi and Czerwik [4] initiated the concept of generalized b-metric spaces, see also [16]. On the other hand, Branciari [6] substituted the triangle inequality by a *polygonal* inequality of the form

$$d(x, z) \leq d(x, y_1) + d(y_1, y_2) + \cdots + d(y_v, z), \tag{1.2}$$

for arbitrary x, z and for all distinct points $y_1, y_2, \ldots, y_v$, each of them different from x and z (in particular, for $v = 2$, the inequality (1.2) is called *rectangular*). Further, a lot of fixed point results for single and multi-valued mappings were obtained in both kind of spaces by various authors (see [3,11,14,15] and references contained therein).

George et al. [10], as well as Roshan et al. [21], independently introduced *b-rectangular metric spaces*, by combining inequalities (1.1) and (1.2) (in the case $v = 2$). Finally, Mitrović and Radenović defined in [18] the concept of $b_v(s)$-metric space for arbitrary positive integer v (see the definition in the next section), thus generalizing all the mentioned types of spaces. They obtained some fixed point results in this new framework. It should be noted that these spaces might not be Hausdorff, that a $b_v(s)$-metric need not be continuous and that a convergent sequence might not be a Cauchy one.

Rational expressions in contractive conditions were firstly used by Dass and Gupta [8], Khan [17] (corrected by Fisher [9]) and Jaggi [12]. Later on, there have been a lot of papers using several variants of such conditions in various contexts, see, e.g., [1,2,19–21].

In this paper, we use contractive conditions involving rational expressions of Khan type, as well as of Dass–Gupta type, to obtain some fixed point results in the framework of $b_v(s)$-metric spaces. Thus, we obtain generalizations of several known fixed point results from the literature. Examples are given to show how these results can be applied in concrete situations.

2 $\mathbf{b_v(s)}$-metric spaces

Definition 2.1 [18] Let X be a non-empty set, $s \geq 1$ be a real number, $v \in \mathbb{N}$ and let d be a function from $X \times X$ into $[0, \infty)$. Then (X, d) is said to be a $b_v(s)$-metric space if for all $x, y, z \in X$ and for all distinct points $y_1, y_2, \ldots, y_v \in X$, each of them different from x and z the following hold:

(B1) $d(x, y) = 0$ if and only if $x = y$;
(B2) $d(x, y) = d(y, x)$;
(B3) $d(x, z) \leq s[d(x, y_1) + d(y_1, y_2) + \cdots + d(y_v, z)]$.

Note that:

(1) $b_1(1)$-metric space is a usual metric space,
(2) $b_1(s)$-metric space is a b-metric space with coefficient s of [5] and [7],
(3) $b_2(1)$-metric space is a rectangular metric space of [6],
(4) $b_2(s)$-metric space is a rectangular b-metric space with coefficient s of [10] and [21],
(5) $b_v(1)$-metric space is a v-generalized metric space of [6].

Example Consider the set $X = \{\frac{1}{n} : n \in \mathbb{N},\ n \geq 2\}$. Define $d : X \times X \to [0, \infty)$ by

$$d\left(\frac{1}{k}, \frac{1}{m}\right) = \begin{cases} |k - m|, & \text{if } |k - m| \neq 1, \\ \frac{1}{2}, & \text{if } |k - m| = 1. \end{cases}$$

It is an easy task to verify that (X, d) is a $b_3(3)$-metric space.

The notions of a convergent sequence, a Cauchy sequence and completeness of a $b_v(s)$-metric space are introduced in the same way as in standard metric spaces.

We will make use of the following lemmas obtained in [18].

Lemma 2.2 *Let (X, d) be a $b_v(s)$-metric space, $T : X \to X$ and let $\{x_n\}$ be a sequence in X defined by $x_0 \in X$ and $x_{n+1} = Tx_n$ such that $x_n \neq x_{n+1}$, $(n \geq 0)$. Suppose there exists $\lambda \in [0, 1)$ such that*

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}) \text{ for all } n \in \mathbb{N}.$$

Then $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$.

Lemma 2.3 *Let (X, d) be a $b_v(s)$-metric space and let $\{x_n\}$ be a sequence in X such that the elements x_n are all different $(n \geq 0)$. Suppose there exist $\lambda \in [0, 1)$ and c_1, c_2 real nonnegative numbers such that*

$$d(x_m, x_n) \leq \lambda d(x_{m-1}, x_{n-1}) + c_1\lambda^m + c_2\lambda^n, \text{ for all } m, n \in \mathbb{N}.$$

Then $\{x_n\}$ is a Cauchy sequence.

3 A fixed point theorem of Khan type in $b_v(s)$-metric spaces

Let (X, d) be a $b_v(s)$-metric space and $T : X \to X$ be a mapping. We introduce the following function $k : X \times X \to [0, 1]$ by

$$k_{xy} = \begin{cases} \dfrac{d(x, Ty)}{\max\{d(x, Ty), d(y, Tx)\}}, & \text{if } \max\{d(x, Ty), d(y, Tx)\} \neq 0 \\ 1/2, & \text{if } \max\{d(x, Ty), d(y, Tx)\} = 0. \end{cases}$$

Theorem 3.1 *Let (X, d) be a complete $b_v(s)$-metric space and $T : X \to X$ be a mapping satisfying*

$$d(Tx, Ty) \leq \lambda \max\{d(x, y), k_{xy} d(x, Tx) + k_{yx} d(y, Ty)\}, \tag{3.1}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then T has a unique fixed point.

Proof Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. If for some n, $x_n = x_{n+1}$, then x_n is a fixed point of T and there is nothing to prove. Hence, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. From the condition (3.1), we obtain

$$d(x_{n+1}, x_n) \leq \lambda \max\{d(x_n, x_{n-1}), k_{x_n x_{n-1}} d(x_n, x_{n+1}) + k_{x_{n-1} x_n} d(x_{n-1}, x_n)\}. \tag{3.2}$$

We distinguish two cases.

1. For all $n \geq 1$, $d(x_{n-1}, x_{n+1}) \neq 0$.

In this case, we obtain

$$k_{x_n x_{n-1}} = \frac{d(x_n, x_n)}{\max\{d(x_n, x_n), d(x_{n-1}, x_{n+1})\}},$$

so, $k_{x_n x_{n-1}} = 0$. Now, from (3.2) we have

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}), \tag{3.3}$$

for all $n \in \mathbb{N}$.

2. For some $n \geq 1$, $d(x_{n-1}, x_{n+1}) = 0$.

Then we have that $k_{x_n x_{n-1}} = \frac{1}{2}$. It follows from (3.2) that

$$d(x_{n+1}, x_n) \leq \lambda \max\{d(x_n, x_{n-1}), \frac{1}{2}d(x_n, x_{n+1}) + \frac{1}{2}d(x_{n-1}, x_n)\}.$$

We get from the above inequality

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}), \tag{3.4}$$

or

$$d(x_{n+1}, x_n) \leq \frac{\lambda}{2-\lambda} d(x_n, x_{n-1}).$$

Since $\max\{\lambda, \frac{\lambda}{2-\lambda}\} = \lambda$, we have $d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1})$. We conclude from the two cases that (3.3) holds for all $n \in \mathbb{N}$. Then from Lemma 2.2 we obtain

$$x_n \neq x_m \text{ for all distinct } n, m \in \mathbb{N}.$$

By (3.3), it follows

$$d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0), \tag{3.5}$$

for all $n \in \mathbb{N}$. Let $m, n \in \mathbb{N}$ such that $m \neq n-1$ and $m \neq n+1$. Then $\max\{d(x_n, x_{m+1}), d(x_m, x_{n+1})\} \neq 0$, therefore

$$k_{x_n x_m} = \frac{d(x_n, x_{m+1})}{\max\{d(x_n, x_{m+1}), d(x_m, x_{n+1})\}} \leq 1 \tag{3.6}$$

and

$$k_{x_m x_n} = \frac{d(x_m, x_{n+1})}{\max\{d(x_n, x_{m+1}), d(x_m, x_{n+1})\}} \leq 1. \tag{3.7}$$

Let $m, n \in \mathbb{N}$ be such that $|m-n| \neq 1$ (if $|m-n| = 1$, (3.5) is used). Then from (3.1), (3.5), (3.6) and (3.7), we obtain

$$\begin{aligned} d(x_m, x_n) &\leq \lambda \max\{d(x_{m-1}, x_{n-1}), k_{x_{m-1}x_{n-1}} d(x_{m-1}, x_m) \\ &\quad + k_{x_{n-1}x_{m-1}} d(x_{n-1}, x_n)\} \\ &\leq \lambda \max\{d(x_{m-1}, x_{n-1}), \lambda^{m-1} d(x_0, x_1) + \lambda^{n-1} d(x_0, x_1)\} \\ &\leq \lambda d(x_{m-1}, x_{n-1}) + (\lambda^m + \lambda^n) d(x_0, x_1). \end{aligned}$$

Now, from Lemma 2.3, (by putting $c_1 = c_2 = d(x_0, x_1)$), we obtain that $\{x_n\}$ is a Cauchy sequence in X. By the completeness of (X, d), there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

We will prove that x^* is the unique fixed point of T.

If there exists a subsequence $\{x_{n_k}\}$ of sequence $\{x_n\}$ such that $x_{n_k} = x^*$ for all $k \in \mathbb{N}$, we obtain

$$d(x^*, Tx^*) = d(x_{n_k}, x_{n_k+1}) \leq \lambda^{n_k} d(x_1, x_0).$$

Letting k tend to ∞ yields that $x^* = Tx^*$. Similarly, if $x_{n_k} = Tx^*$ for all $k \in \mathbb{N}$, we obtain

$$d(Tx^*, T(Tx^*)) = d(x_{n_k}, x_{n_k+1}) \leq \lambda^{n_k} d(x_1, x_0),$$

and so again $x^* = Tx^*$.
Otherwise, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, $x_n \notin \{x^*, Tx^*\}$.

Let us consider the following two cases:

1. $\liminf\limits_{n\to\infty} d(x_n, Tx^*) = 0$.

In this case, there exists a subsequence $\{x_{n_k}\}_{k\geq 0}$ of $\{x_n\}$ having the property that $\lim_{k\to\infty} d(x_{n_k}, Tx^*) = 0$. Using (3.5), we have

$$\begin{aligned} d(x^*, Tx^*) &\leq s[d(x^*, x_{n_k-v+1}) + d(x_{n_k-v+1}, x_{n_k-v+2}) + \cdots + \\ &\qquad d(x_{n_k-2}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, Tx^*)] \\ &= s\left[d(x^*, x_{n_k-v+1}) + d(x_{n_k}, Tx^*) + \sum_{i=n_k-v+1}^{n_k-1} d(x_{i+1}, x_i)\right] \\ &\leq s\left[d(x^*, x_{n_k-v+1}) + d(x_{n_k}, Tx^*) + \sum_{i=n_k-v+1}^{n_k-1} \lambda^i d(x_1, x_0)\right] \\ &\leq s\left[d(x^*, x_{n_k-v+1}) + d(x_{n_k}, Tx^*) + \frac{\lambda^{n_k-v+1}}{1-\lambda} d(x_1, x_0)\right]. \end{aligned}$$

Since $\lambda \in [0, 1)$ and $\lim_{k\to\infty} d(x^*, x_{n_k-v+1}) = 0$, we get $d(Tx^*, x^*) = 0$, i.e., $Tx^* = x^*$.

2. $\liminf\limits_{n\to\infty} d(x_n, Tx^*) = c > 0$.

Then there exists a subsequence $\{x_{n_k}\}_{k\geq 0}$ of $\{x_n\}$ such that $\lim_{k\to\infty} d(x_{n_k}, Tx^*) = c$. Using again (3.5), we have

$$\begin{aligned} d(x^*, Tx^*) &\leq s[d(x^*, x_{n_k-v+2}) + d(x_{n_k-v+2}, x_{n_k-v+3}) + \cdots + \\ &\qquad d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, Tx^*)] \\ &= s\left[d(x^*, x_{n_k-v+2}) + d(x_{n_k+1}, Tx^*) + \sum_{i=n_k-v+2}^{n_k} d(x_{i+1}, x_i)\right] \\ &\leq s\left[d(x^*, x_{n_k-v+2}) + d(x_{n_k+1}, Tx^*) + \sum_{i=n_k-v+2}^{n_k} \lambda^i d(x_1, x_0)\right] \\ &\leq s\left[d(x^*, x_{n_k-v+2}) + d(x_{n_k+1}, Tx^*) + \frac{\lambda^{n_k-v+2}}{1-\lambda} d(x_1, x_0)\right]. \end{aligned}$$

From (3.1), we obtain

$$d(x_{n_k+1}, Tx^*) = \lambda \max\{d(x_{n_k}, x^*), k_{x_{n_k}x^*} d(x_{n_k}, x_{n_k+1}) + k_{x^*x_{n_k}} d(x^*.Tx^*)\}.$$

Since

$$k_{x_{n_k}x^*} = \frac{d(x_{n_k}, Tx^*)}{\max\{d(x_{n_k}, Tx^*), d(x^*, x_{n_k+1})\}} \to 1 \text{ as } k \to \infty,$$

and

$$k_{x^*x_{n_k}} = \frac{d(x^*, x_{n_k+1})}{\max\{d(x^*, x_{n_k+1}), d(x_{n_k}, Tx^*)\}} \to 0 \text{ as } k \to \infty,$$

we have $\lim_{k\to\infty} d(x_{n_k+1}, Tx^*) = 0$. We deduce that $d(x^*, Tx^*) = 0$, that is, $Tx^* = x^*$.

In order to prove uniqueness, let y^* be another fixed point of T. Then it follows from (3.1) that

$$\begin{aligned} 0 < d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\leq \lambda \max\{d(x^*, y^*), k_{x^*y^*} d(x^*, Tx^*) + k_{y^*x^*} d(y^*, Ty^*)\}, \\ &\leq \lambda d(x^*, y^*) < d(x^*, y^*), \end{aligned}$$

which is a contradiction. Therefore, we must have $d(x^*, y^*) = 0$, i.e., $x^* = y^*$. □

Example Let $X = \{0, 1, 2\}$ and define $d : X \times X \to [0, +\infty)$ as follows:

$$d(0, 2) = 2.2, \quad d(1, 2) = 1.1, \quad d(0, 1) = 1,$$
$$d(x, x) = 0 \text{ and } d(x, y) = d(y, x) \text{ for all } x, y \in X.$$

Then (X, d) is a b-metric space with $s = \frac{22}{21}$. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} 0, & \text{if } x \neq 2, \\ 1, & \text{if } x = 2. \end{cases}$$

We shall check that for all $x, y \in X$ the following contractive condition holds:

$$d(Tx, Ty) \leq \begin{cases} \gamma \max\left\{d(x, y), \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{\max\{d(x,Ty),d(Tx,y)\}}\right\}, \\ \quad \text{if } \max\{d(x, Ty), d(Tx, y)\} \neq 0, \\ 0, \quad \text{if } \max\{d(x, Ty), d(Tx, y)\} = 0. \end{cases} \tag{3.8}$$

We have the next three cases:

a) $x = 0, y = 1$. Then $d(T0, T1) = d(0, 0) = 0$. The condition (3.8) holds.

b) $x = 0, y = 2$. Then $d(T0, T2) = d(0, 1)$. Since

$$\max\{d(0, T2), d(T0, 2)\} = \max\{d(0, 1), d(0, 2)\} = 2.2 \neq 0,$$

we need

$$\begin{aligned} 1 &\leq \gamma \max\left\{d(0, 2), \frac{0 \cdot d(0, 1) + d(2, 1) \cdot d(2, 0)}{}\right\} \\ &= \gamma \max\{2.2, 1.1\} = \gamma \cdot 2.2. \end{aligned}$$

Hence, (3.8) holds if $\gamma \geq \frac{10}{22} = \frac{5}{11}$.

c) $x = 1, y = 2$, Then $d(T1, T2) = d(0, 1) = 1$. Again, since

$$\max\{d(1, T2), d(T1, 2)\} = \max\{d(0, 1), d(0, 2)\} = 2.2 \neq 0$$

and we need

$$\begin{aligned} 1 &\leq \gamma \max\left\{d(1, 2), \frac{d(1.T1) \cdot d(1, T2) + d(2, T2) \cdot d(2, T1)}{2.2}\right\} \\ &= \gamma \max\left\{1.1, \frac{0 + 1.1 \cdot 2.2}{2.2}\right\} = \gamma \cdot 1.1. \end{aligned}$$

Hence, (3.8) holds if $\gamma \geq \frac{10}{11}$.

We obtain that the contractive condition (3.8) holds for all $x, y \in X$ where $\gamma \in [\frac{10}{11}, 1)$.

So, by Theorem 3.1 in the context of b-metric spaces, T has a unique fixed point (which is $x^* = 0$).

Remark 3.2 1. It is clear that Theorem 3.1 generalizes Banach contraction principle in $b_v(s)$-metric spaces (see Theorem 2.1. in [18]).

2. Also, Theorem 3.1 generalizes the result of Piri et al. (see Theorem 2.1. in [20]).

4 Two fixed point theorems of Dass–Gupta type in $\mathbf{b_v(s)}$-metric spaces

Let (X, d) be a $b_v(s)$-metric space and $T : X \to X$. We will use the following expressions:

$$M(x, y) = \max\left\{d(x, y), \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}, \frac{d(x, Tx)[1 + d(y, Ty)]}{1 + d(Tx, Ty)}\right\},$$
$$m(x, y) = \max\left\{d(x, y), \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}\right\},$$
$$N(x, y) = \min\{d(x, Tx), d(y, Tx)\},$$

for $x, y \in X$.

Lemma 4.1 *Let (X, d) be a complete $b_v(s)$-metric space and $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \le \lambda M(x, y) + LN(x, y) \tag{4.1}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$ and $L \ge 0$. Then for any $x_0 \in X$, the sequence $\{T^n x_0\}$ converges.

Proof Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \ge 0$. We have

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\left\{d(x_n, x_{n+1}), \frac{d(x_{n+1}, x_{n+2})[1 + d(x_n, x_{n+1})]}{1 + d(x_{n+1}, x_n)},\right.\\ &\qquad \left.\frac{d(x_n, x_{n+1})[1 + d(x_{n+1}, x_{n+2})]}{1 + d(x_{n+1}, x_{n+2})}\right\}\\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} \end{aligned}$$

and

$$N(x_n, x_{n+1}) = \min\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+1})\} = 0.$$

From the condition (4.1), we have that

$$d(x_{n+1}, x_{n+2}) \le \lambda \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}.$$

Therefore,

$$d(x_{n+1}, x_{n+2}) \le \lambda d(x_n, x_{n+1}), \tag{4.2}$$

for all $n \ge 0$. It follows from (4.2) that

$$d(x_{n+1}, x_n) \le \lambda^n d(x_1, x_0) \text{ for all } n \ge 1. \tag{4.3}$$

If $x_n = x_{n+1}$ then x_n is a fixed point of T. So, suppose that $x_n \ne x_{n+1}$ for all $n \ge 0$. Then $\lambda \ne 0$. From the conditions (4.1) and (4.3) we obtain

$$d(x_m, x_n) \le \lambda \max\left\{d(x_{m-1}, x_{n-1}), \frac{d(x_{n-1}, x_n)[1 + d(x_{m-1}, x_m)]}{1 + d(x_{m-1}, x_{n-1})},\right.$$

$$
\begin{aligned}
&\left.\frac{d(x_{m-1},x_m)[1+d(x_{n-1},x_n)]}{1+d(x_m,x_n)}\right\}\\
&+L\min\{d(x_{m-1},x_m),d(x_{n-1},x_m)\}\\
&\le\lambda\max\{d(x_{m-1},x_{n-1}),d(x_{n-1},x_n)[1+d(x_{m-1},x_m)],\\
&\quad d(x_{m-1},x_m)[1+d(x_{n-1},x_n)]\}\\
&\quad +Ld(x_{m-1},x_m)\\
&\le\lambda d(x_{m-1},x_{n-1})+\lambda d(x_{n-1},x_n)[1+d(x_0,x_1)]\\
&\quad +d(x_{m-1},x_m)[\lambda(1+d(x_0,x_1))+L]\\
&\le\lambda d(x_{m-1},x_{n-1})+[\lambda^m(1+d(x_0,x_1))+L\lambda^{m-1}]\\
&\quad d(x_0,x_1)+\lambda^n[1+d(x_0,x_1)]d(x_0,x_1).
\end{aligned}
$$

Now, from Lemma 2.3, (by putting $c_1=[1+d(x_0,x_1)+L/\lambda]d(x_0,x_1)$, $c_2=[1+d(x_0,x_1)]d(x_0,x_1)$) we obtain that $\{x_n\}$ is a Cauchy sequence in X. By the completeness of (X,d) there exists $x^*\in X$ such that $\lim_{n\to\infty}x_n=x^*$. □

The following theorem is an analogue of Dass–Gupta contraction principle in $b_v(s)$-metric spaces.

Theorem 4.2 *Let (X,d) be a complete $b_v(s)$-metric space and $T:X\to X$ be a mapping satisfying:*

$$d(Tx,Ty)\le\lambda M(x,y)+LN(x,y)$$

for all $x,y\in X$, where $\lambda\in[0,1)$ and $L\ge0$. Then T has a unique fixed point x^* and for any $x_0\in X$ the sequence $\{T^nx_0\}$ converges to x^* if one of the following conditions is satisfied

(i) *T is continuous, or*
(ii) $\lambda s<1$.

Proof Let $x_0\in X$ be arbitrary. Define a sequence $\{x_n\}$ by $x_{n+1}=Tx_n$ for all $n\ge0$. From Lemma 4.1 we obtain that there exists $x^*\in X$ such that $\lim_{n\to\infty}x_n=x^*$.

(i) Let T be continuous. Then

$$x^*=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}Tx_n=T(\lim_{n\to\infty}x_n)=Tx^*.$$

(ii) $\lambda s<1$.
Without loss of generality, there exists $n_0\in\mathbb{N}$ such that for any $n\ge n_0$, $x_n\notin\{x^*,Tx^*\}$. Let us consider the following two cases:

1. $\liminf\limits_{n\to\infty}d(x_n,Tx^*)=0$.

In this case, there exists a subsequence $\{x_{n_k}\}_{k\ge0}$ of $\{x_n\}$ having the property that $\lim_{k\to\infty}d(x_{n_k},Tx^*)=0$. Proceeding similarly as the proof of Theorem 3.1, we get $d(Tx^*,x^*)=0$, i.e., $Tx^*=x^*$.

2. $\liminf\limits_{n\to\infty}d(x_n,Tx^*)=c>0$.

Then there exists a subsequence $\{x_{n_k}\}_{k\ge0}$ of $\{x_n\}$ such that $\lim_{k\to\infty}d(x_{n_k},Tx^*)=c$. Again, as in the proof of Theorem 3.1, we have

$$d(x^*,Tx^*)\le s[d(x^*,x_{n_k-v+2})+d(x_{n_k+1},Tx^*)+\frac{\lambda^{n_k-v+2}}{1-\lambda}d(x_1,x_0)].$$

From (4.1), we obtain

$$d(x_{n_k+1},Tx^*)=\lambda\max\left\{d(x_{n_k},x^*),\frac{d(x^*,Tx^*)[1+d(x_{n_k},x_{n_k+1})]}{1+d(x_{n_k},x^*)},\right.$$

$$\left.\frac{d(x_{n_k}, x_{n_k+1})[1 + d(x^*, Tx^*)]}{1 + d(x_{n_k+1}, Tx^*)}\right\}$$
$$+ L \min\{d(x_{n_k}, x_{n_k+1}), d(x^*, x_{n_k+1})\}$$
$$\to \lambda d(x^*, Tx^*) \text{ as } k \to \infty.$$

Therefore, $d(x^*, Tx^*) \leq s\lambda d(x^*, Tx^*)$. Since $s\lambda < 1$, we get $d(x^*, Tx^*) = 0$ and so $Tx^* = x^*$.

In order to prove uniqueness, let y^* be another fixed point of T. Then from (4.1), we have

$$\begin{aligned} 0 < d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\leq \lambda \max \left\{ d(x^*, y^*), \frac{d(y^*, Ty^*)[1 + d(x^*, Tx^*)]}{1 + d(x^*, y^*)}, \right. \\ &\quad \left. \frac{d(x^*, Tx^*)[1 + d(y^*, Ty^*)]}{1 + d(Tx^*, Ty^*)} \right\} \\ &\quad + L \min\{d(x^*, Tx^*), d(y^*, Tx^*)\} \\ &= \lambda d(x^*, y^*) < d(x^*, y^*), \end{aligned}$$

which is a contradiction. Therefore, $x^* = y^*$. □

Here, it is another version of Dass–Gupta type theorem.

Theorem 4.3 *Let (X, d) be a complete $b_v(s)$-metric space and $T : X \to X$ be a mapping satisfying*

$$d(Tx, Ty) \leq \lambda m(x, y) \tag{4.4}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then T has a unique fixed point x^* and for any $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to x^*.

Proof Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. Suppose that $x_n \neq x_{n+1}$ for all each $n \geq 0$ (otherwise, nothing is to prove). Since $m(x, y) \leq M(x, y)$ for all $x, y \in X$, from Lemma 4.1 we obtain that there exists x^* such that $\{T^n x_0\}$ converges to x^*. Without loss of generality, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, $x_n \notin \{x^*, Tx^*\}$. From inequality $(B3)$, we obtain

$$\begin{aligned} d(x^*, Tx^*) &\leq s[d(x^*, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + \\ &\quad d(x_{n+v-3}, x_{n+v-2}) + d(x_{n+v-2}, x_{n+v-1}) + d(x_{n+v}, Tx^*)] \\ &= s\left[d(x^*, x_{n+1}) + d(x_{n+v}, Tx^*) + \sum_{i=n+1}^{n+v-2} d(x_i, x_{i+1})\right] \\ &\leq s\left[d(x^*, x_{n+1}) + d(x_{n+v}, Tx^*) + \sum_{i=n+1}^{n+v-2} \lambda^i d(x_0, x_1)\right] \\ &\leq s\left[d(x^*, x_{n+1}) + d(x_{n+v}, Tx^*) + \frac{\lambda^{n+1}}{1-\lambda} d(x_0, x_1)\right]. \end{aligned}$$

From condition (4.4), we have

$$\begin{aligned} d(Tx^*, x_{n+v}) &= \lambda \max \left\{ d(x^*, x_{n+v-1}), \frac{d(x_{n+v-1}, x_{n+v})[1 + d(x^*, Tx^*)]}{1 + d(x^*, x_{n+v})} \right\}, \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

At the limit, we get $d(x^*, Tx^*) = 0$, that is, $Tx^* = x^*$.

To prove the uniqueness, let y^* be another fixed point of T. Then from (4.4) we have

$$\begin{aligned} 0 < d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\leq \lambda \max \left\{ d(x^*, y^*), \frac{d(y^*, Ty^*)[1 + d(x^*, Tx^*)]}{1 + d(x^*, y^*)} \right\} \\ &= \lambda d(x^*, y^*) < d(x^*, y^*), \end{aligned}$$

a contradiction. It follows that $x^* = y^*$. □

Remark 4.4 1. If $v = 1$ (resp. $v = 2$), from Theorems 3.1, 4.2, 4.3, we obtain results for b-metric spaces (resp. rectangular b-metric spaces).

2. Theorem 4.2 generalizes a result obtained in the paper [13].

3. From Theorem 4.3, Theorem 2.1. in [18] is obtained.

Example Let $X = \{a, b, c, \delta\}$, $d(x, y) = d(y, x)$, $d(x, x) = 0$ for all $x, y \in X$. Further, let $d(a, b) = \frac{1}{5}$, $d(\delta, c) = 5$, $d(a, c) = d(b, \delta) = d(b, c) = d(a, \delta) = 10$. Then (X, d) is a $b_1(\frac{11}{10})$-metric space (i.e., a b-metric space with the parameter $s = \frac{11}{10}$).

Define $T : X \to X$ by $Ta = Tb = T\delta = a$, $Tc = b$. We shall check that all conditions of Theorem 4.3 are satisfied.

Indeed, if $x = a, y = b$, or $x = a, y = \delta$ or $x = b, y = \delta$, the condition (4.4) trivially holds. Let $x = a, y = c$. Then $d(Ta, Tc) = d(a, b) = \frac{1}{5}$ and

$$\begin{aligned} m(a, c) &= \max \left\{ d(a, c), \frac{d(c, Tc)[1 + d(a, Ta)]}{1 + d(a, c)} \right\} \\ &= \max \left\{ 10, \frac{10[1 + 0]}{1 + 10} \right\} = 10. \end{aligned}$$

Hence, it is enough to have $\frac{1}{5} \leq \lambda \cdot 10$, i.e., $\lambda \in \left[\frac{1}{50}, 1\right)$.

Let $x = b, y = c$. Then $d(Tb, Tc) = d(a, b) = \frac{1}{5}$ and

$$\begin{aligned} m(b, c) &= \max \left\{ d(b, c), \frac{d(c, Tc)[1 + d(b, Tb)]}{1 + d(b, c)} \right\} \\ &= \max \left\{ 10, \frac{10\left[1 + \frac{1}{5}\right]}{1 + 10} \right\} = 10. \end{aligned}$$

Again, it is enough that $\frac{1}{5} \leq \lambda \cdot 10$, i.e., $\lambda \in \left[\frac{1}{50}, 1\right)$.

Let $x = c, y = \delta$. Then $d(Tc, T\delta) = d(b, a) = \frac{1}{5}$ and

$$\begin{aligned} m(c, \delta) &= \max \left\{ d(c, \delta), \frac{d(\delta, T\delta)[1 + d(c, Tc)]}{1 + d(c, \delta)} \right\} \\ &= \max \left\{ 5, \frac{10[1 + 10]}{1 + 5} \right\} = \frac{55}{3}. \end{aligned}$$

It follows that we need $\frac{1}{5} \leq \lambda \cdot \frac{55}{3} \cdot$, i.e., $\lambda \in \left[\frac{3}{275}, 1\right)$.

Hence, for $\lambda \in [\frac{1}{50}, 1)$, all conditions of Theorem 4.3 are satisfied and in this case T has a unique fixed point (which is $x^* = a$).

It can be checked in a similar way that the same conclusion can be derived from Theorem 4.2.

Example [10, Example 2.2] Let $X = A \cup B$, where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = [1, 2]$, be equipped with $d : X \times X \to [0, \infty)$ defined by $d(\frac{1}{2}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{5}) = 0.03$, $d(\frac{1}{2}, \frac{1}{5}) = d(\frac{1}{3}, \frac{1}{4}) = 0.02$, $d(\frac{1}{2}, \frac{1}{4}) = d(\frac{1}{3}, \frac{1}{5}) = 0.6$, and $d(x, y) = (x - y)^2$ in all other cases (with $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$). Then (X, d) is a $b_2(4)$-metric space. It is easy to check that the mapping

$$Tx = \begin{cases} \frac{1}{4}, & x \in A \\ \frac{1}{5}, & x \in B \end{cases}$$

satisfies the conditions of each of Theorems 3.1, 4.2 and 4.3 (for example, for Theorem 4.2, one can take $\lambda = \frac{3}{25}$). T has a unique fixed point $x^* = \frac{1}{4}$.

References

1. Ahmad, J., Arshad, M., Vetro, C.: On a theorem of Khan in a generalized metric space. Int. J. Anal. Article ID 852727, p 6 (2013)
2. Ansari, A.H., Aydi, H., Kumari, P.S., Yildirim, I.: New fixed point results via C-class functions in b-rectangular metric spaces. Commun. Math. Anal. **9**(2), 109–126 (2018)
3. Aydi, H., Chen, C.M., Karapinar, E.: Interpolative Ciric-Reich-Rus type contractions via the Branciari distance. Mathematics **7**(1), 84 (2019). https://doi.org/10.3390/math7010084
4. Aydi, H., Czerwik, S.: Fixed point theorems in generalized b-metric spaces. Modern Discrete Math. Anal. **131**, 1–9 (2018)
5. Bakhtin, I.A.: The contraction mapping principle in quasimetric spaces. Funct. Anal. Ulianowsk Gos. Ped. Inst. **30**, 26–37 (1989)
6. Branciari, A.: A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces. Publ. Math. Debr. **57**, 31–37 (2000)
7. Czerwik, S.: Contraction mappings in b-metric spaces. Acta Math. Inform. Univ. Ostrav. **1**, 5–11 (1993)
8. Dass, B.K., Gupta, S.: An extension of Banach contracion principle through rational expression. Indian J. Pure Appl. Math. **6**, 1455–1458 (1975)
9. Fisher, B.: A note on a theorem of Khan. Rend. Ist. Mat. Univ. Trieste **10**, 1–4 (1978)
10. George, R., Radenović, S., Reshma, K.P., Shukla, S.: Rectangular b-metric space and contraction principles. J. Nonlinear Sci. Appl. **8**, 1005–1013 (2015)
11. Gulyaz, S., Karapinar, E., Erhan, I.M.: Generalized α-Meir–Keeler contraction mappings on Branciari b-metric spaces. Filomat **31**(17), 5445–5456 (2017)
12. Jaggi, D.S.: Some unique fixed point theorems. Indian J. Pure. Appl. Math. **8**, 223–230 (1977)
13. Jovanović, M., Kadelburg, Z., Radenović, S.: Common fixed point results in metric-type spaces. Fixed Point Theory Appl. Article ID 978121, p 15 (2010)
14. Karapinar, E.: Some fixed points results on Branciari metric spaces via implicit functions. Carpathian J. Math. **31**(3), 339–348 (2015)
15. Karapinar, E., Pitea, A.: On α-ψ-Geraghty contraction type mappings on quasi-Branciari metric spaces. J. Nonlinear Convex Anal. **17**(7), 1291–1301 (2016)
16. Karapınar, E., Czerwik, S., Aydi, H.: (α, ψ)-Meir–Keeler contraction mappings in generalized b-metric spaces. J. Funct. Spaces. Article ID 3264620, p 4 (2018)
17. Khan, M.S.: A fixed point theorem for metric spaces. Rend. Inst. Math. Univ. Trieste **8**, 69–72 (1976)
18. Mitrović, Z.D., Radenović, S.: The Banach and Reich contractions in $b_v(s)$-metric spaces. J. Fixed Point Theory Appl. **19**, 3087–3095 (2017)
19. Mustafa, Z., Karapinar, E., Aydi, H.: A discussion on generalized almost contractions via rational expressions in partially ordered metric spaces. J. Inequal. Appl. **2014**, 219 (2014)
20. Piri, H., Rahrovi, S., Kumam, P.: Khan type fixed point theorems in a generalized metric space. J. Math. Comput. Sci. **16**, 211–217 (2016)
21. Roshan, J.R., Parvaneh, V., Kadelburg, Z., Hussain, N.: New fixed point results in b-rectangular metric spaces. Nonlinear Anal. Model. Control **21**(5), 614–634 (2016)