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On some classes of Dunford–Pettis-like operators

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Abstract
In this paper we give characterizations of weak Dunford–Pettis, weak* Dunford–Pettis, weak p-convergent, and weak* p-convergent operators.

Keywords Weak Dunford–Pettis operator · Weak* Dunford–Pettis operator · Weak p-convergent operator · Weak* p-convergent operator · The Dunford–Pettis property of order p

Mathematics Subject Classification 46B20 · 46B25 · 46B28

1 Introduction

In this paper Dunford–Pettis sets and limited sets are used to characterize the classes of weak Dunford–Pettis and weak* Dunford–Pettis operators. The classes of weak p-convergent and weak* p-convergent operators are also studied.

Our major results are Theorems 2, 6, 14, and 18. As consequences, we obtain equivalent characterizations of Banach spaces with the Dunford–Pettis property, DP^*-property, Dunford–Pettis property of order p, and DP^*-property of order p. We generalize some results in [1,5,11,15,18].

2 Definitions and notation

Throughout this paper, X and Y will denote real Banach spaces. The unit ball of X will be denoted by B_X and X^* will denote the continuous linear dual of X. An operator $T : X \to Y$ will be a continuous and linear function. The space of all operators from X to Y will be denoted by $L(X, Y)$.

The operator T is *completely continuous (or Dunford–Pettis)* if T maps weakly convergent sequences to norm convergent sequences.

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A Banach space X has the Dunford–Pettis property (DPP) if every weakly compact operator T with domain X is completely continuous. Equivalently, X has the DPP if and only if $x_n^*(x_n) \to 0$ for all weakly null sequences (x_n) in X and (x_n^*) in X^* [8, Theorem 1]. Schur spaces, $C(K)$ spaces, and $L_1(\mu)$ spaces have the DPP. The reader can check [7–9], and [2] for a guide to the extensive classical literature dealing with the DPP, equivalent formulations of the preceding definitions, and undefined notation and terminology.

A subset A of X is called a *Dunford–Pettis* (DP) subset (resp. *limited* subset) of X if each weakly null (resp. w^* -null) sequence (x_n^*) in X^* tends to 0 uniformly on A [2] (resp. [3,24]); i.e.

$$\sup_{x \in A} |x_n^*(x)| \to 0.$$

If A is a limited subset of X, then $T(A)$ is relatively compact for any operator $T : X \to c_0$ [3, p. 56], [24, p. 23]. The subset A of X is a DP subset of X if and only if $T(A)$ is relatively compact whenever $T : X \to Y$ is a weakly compact operator [2] if and only if $T(A)$ is relatively compact whenever $T : X \to Y$ is an operator with weakly precompact adjoint [20].

A bounded subset S of X is said to be *weakly precompact* provided that every sequence from S has a weakly Cauchy subsequence. Every DP subset of X is weakly precompact [2]. Since any limited set is a DP set, any limited set is weakly precompact. An operator $T : X \to Y$ is called *weakly precompact (or almost weakly compact)* if $T(B_X)$ is weakly precompact.

A Banach space X has the DP^*-property (DP^*P) if all weakly compact sets in X are limited [4,5,21]. The space X has the DP^*P if and only if $x_n^*(x_n) \to 0$ for all weakly null sequences (x_n) in X and w^*-null sequences (x_n^*) in X^* [16]. If X has the DP^*P, then it has the DPP. If X is a Schur space or if X has the DPP and the Grothendieck property (weak and weak* convergence of sequences in X^* coincide), then X has the DP^*P.

3 Weak Dunford–Pettis operators and weak* Dunford–Pettis operators

An operator $T : X \to Y$ is called *weak Dunford–Pettis* [1, p. 349] if $\langle T(x_n), y_n^* \rangle \to 0$, whenever (x_n) is a weakly null sequence in X and (y_n^*) is a weakly null sequence in Y^*. An operator $T : X \to Y$ is called weak* *Dunford–Pettis* [11] if $\langle T(x_n), y_n^* \rangle \to 0$, whenever (x_n) is a weakly null sequence in X and (y_n^*) is a w^*-null sequence in Y^*.

In this section we give some characterizations of weak Dunford–Pettis and weak* Dunford–Pettis operators.

Observation 1 If $T : X \to Y$ is an operator, then $T(B_X)$ is a DP (resp. limited) subset of Y if and only if $T^* : Y^* \to X^*$ is completely continuous (resp. T^* is w^*-norm sequentially continuous).

To see this, note that $T(B_X)$ is a DP (resp. limited) subset of Y if and only if

$$0 = \lim_n \sup\{|\langle y_n^*, T(x)\rangle| : x \in B_X\} = \lim_n \sup\{|\langle T^*(y_n^*), x\rangle| : x \in B_X\} = \lim_n \|T^*(y_n^*)\|$$

for each weakly null (resp. w^*-null) sequence (y_n^*) in Y^*; that is, $T^* : Y^* \to X^*$ is completely continuous (resp. T^* is w^*-norm sequentially continuous).

Theorem 1 [1, Theorem 5.99, p. 351] *Let $T : X \to Y$ be an operator. The following statements are equivalent:*

(1) *T is a weak Dunford–Pettis operator.*

(2) *T carries weakly compact subsets of X to Dunford–Pettis subsets of Y.*
(3) *If $S : Y \to Z$ is a weakly compact operator, then $ST : X \to Z$ is completely continuous, for any Banach space Z.*

We are now giving our first major result. It gives characterizations of weak Dunford–Pettis operators and generalizes [1, Theorem 5.99, p. 351].

Theorem 2 *Let X and Y be Banach spaces, and let $T : X \to Y$ be an operator. The following statements are equivalent.*

(1) *T is a weak Dunford–Pettis operator.*
(2) *T carries weakly precompact subsets of X to Dunford–Pettis subsets of Y.*
(3) *For all Banach spaces Z, if $S : Y \to Z$ has a weakly precompact adjoint, then $ST : X \to Z$ is completely continuous.*
(4) *If $S : Y \to c_0$ has a weakly precompact adjoint, then $ST : X \to c_0$ is completely continuous.*
(5) *If (x_n) is a weakly null sequence in X and (y_n^*) is a weakly Cauchy sequence in Y^*, then $\langle y_n^*, T(x_n)\rangle \to 0$.*

Proof (1) $\Rightarrow$ (2) Let A be a weakly precompact subset of X. Suppose by contradiction that $T(A)$ is not a Dunford–Pettis subset of Y. Suppose that (y_n^*) is a weakly null sequence in Y^*, (x_n) is a sequence in A, and $\epsilon > 0$ such that $|\langle y_n^*, T(x_n)\rangle| > \epsilon$ for all n. Without loss of generality assume that (x_n) is weakly Cauchy.

Let $n_1 = 1$ and choose $n_2 > n_1$ so that $|\langle y_{n_2}^*, T(x_{n_1})\rangle| < \epsilon/2$. We can do this since (y_n^*) is w^*-null. Continue inductively. Choose $n_k > n_{k-1}$ so that $|\langle y_{n_k}^*, T(x_{n_{k-1}})\rangle| < \epsilon/2$. Since T is a weak Dunford–Pettis operator, $\langle y_{n_k}^*, T(x_{n_k} - x_{n_{k-1}})\rangle \to 0$. However,

$$|\langle y_{n_k}^*, T(x_{n_k} - x_{n_{k-1}})\rangle| \geq |\langle y_{n_k}^*, T(x_{n_k})\rangle| - |\langle y_{n_k}^*, T(x_{n_{k-1}})\rangle| > \epsilon/2,$$

a contradiction.

(2) $\Rightarrow$ (3) Let $S : Y \to Z$ be an operator such that $S^* : Z^* \to Y^*$ is weakly precompact. Suppose (x_n) is a weakly null sequence in X. Since $\{T(x_n) : n \in \mathbb{N}\}$ is a Dunford–Pettis set in Y and S^* is weakly precompact, $\{ST(x_n) : n \in \mathbb{N}\}$ is relatively compact [20, Corollary 4]. Then $\|ST(x_n)\| \to 0$, and thus $ST : X \to Z$ is completely continuous.

(3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (1) are obvious.

(4) $\Rightarrow$ (1) Suppose (x_n) is weakly null in X and (y_n^*) is weakly null in Y^*. Define $S : Y \to c_0$ by $S(y) = (y_i^*(y))$. Then $S^* : \ell_1 \to Y^*$, $S^*(b) = \sum b_i y_i^*$. Note that S^* maps B_{ℓ_1} into the closed and absolutely convex hull of $\{y_i^* : i \in \mathbb{N}\}$, which is relatively weakly compact [9, p. 51]. Then S^* is weakly compact. Hence $ST : X \to c_0$ is completely continuous. Therefore $\langle T(x_n), y_n^*\rangle \leq \|ST(x_n)\| = \sup_i |\langle y_i^*, T(x_n)\rangle| \to 0$, and T is weak Dunford–Pettis.

(1) $\Rightarrow$ (5) Suppose that (x_n) is a weakly null sequence in X, (y_n^*) is a weakly Cauchy sequence in Y^*, and $\langle y_n^*, T(x_n)\rangle \not\to 0$. Without loss of generality suppose that $|\langle y_n^*, T(x_n)\rangle| > \epsilon$ for each $n \in \mathbb{N}$, for some $\epsilon > 0$.

Let $n_1 = 1$ and choose $n_2 > n_1$ so that $|\langle y_{n_1}^*, T(x_{n_2})\rangle| < \epsilon/2$. We can do this since $(T(x_n))$ is weakly null. Continue inductively. Choose $n_{k+1} > n_k$ so that $|\langle y_{n_k}^*, T(x_{n_{k+1}})\rangle| < \epsilon/2$. Since T is weak Dunford–Pettis, $\langle y_{n_{k+1}}^* - y_{n_k}^*, T(x_{n_{k+1}})\rangle \to 0$. Since

$$|\langle y_{n_{k+1}}^* - y_{n_k}^*, T(x_{n_{k+1}})\rangle| \geq |\langle y_{n_{k+1}}^*, T(x_{n_{k+1}})\rangle| - |\langle y_{n_k}^*, T(x_{n_{k+1}})\rangle| > \epsilon/2,$$

we have a contradiction. □

Corollary 3 *Let X and Y be Banach spaces, and let $T : X \to Y$ be an operator. The following statements are equivalent.*

(i) *T is a weak Dunford–Pettis operator.*
(ii) *For all Banach spaces Z, if $S : Z \to X$ is a weakly precompact operator, then $TS : Z \to Y$ has a completely continuous adjoint.*
(iii) *If $S : \ell_1 \to X$ is a weakly precompact operator, then $TS : \ell_1 \to Y$ has a completely continuous adjoint.*
(iv) *If (x_n) is a weakly Cauchy sequence in X and (y_n^*) is a weakly null sequence in Y^*, then $\langle y_n^*, T(x_n)\rangle \to 0$.*

Proof (i) $\Rightarrow$ (ii) Let $S : Z \to X$ be a weakly precompact operator. Then $TS(B_Z)$ is a Dunford–Pettis set. Thus $TS : Z \to Y$ has a completely continuous adjoint.

(iii) $\Rightarrow$ (iv) Suppose (x_n) is a weakly Cauchy sequence in X and (y_n^*) is a weakly null sequence in Y^*. Define $S : \ell_1 \to X$ by

$$S(b) = \sum b_n x_n,$$

where $b = (b_n) \in \ell_1$. Since $S(B_{\ell_1})$ is contained in the closed and absolutely convex hull of $\{x_n : n \in \mathbb{N}\}$, which is weakly precompact [24, p. 27], S is weakly precompact.

By assumption, $(TS)^* = S^*T^*$ is completely continuous. Note that $S^*(x^*) = (\langle x^*, x_i\rangle)_i$, $x^* \in X^*$, and $S^*T^*(y_n^*) = (\langle T^*(y_n^*), x_i\rangle)_i$. Hence

$$\langle y_n^*, T(x_n)\rangle = \langle T^*(y_n^*), x_n\rangle \le \|S^*T^*(y_n^*)\| = \sup_i |\langle T^*(y_n^*), x_i\rangle| \to 0.$$

(ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (i) are obvious. □

The following two corollaries provide equivalent characterizations of spaces with the Dunford–Pettis property.

Corollary 4 *Let X be a Banach space. Then the following statements are equivalent:*

(i) *X has the DPP.*
(ii) *The identity operator $i : X \to X$ is a weak Dunford–Pettis operator; that is, every weakly precompact subset of X is a Dunford–Pettis set.*
(iii) [8] *Every operator $S : X \to Z$ with weakly precompact adjoint is completely continuous, for any Banach space Z.*
(iv) [8] *Every operator $S : X \to c_0$ with weakly precompact adjoint is completely continuous.*
(v) [8] *If (x_n) is a weakly null sequence in X and (x_n^*) is a weakly Cauchy sequence in X^*, then $x_n^*(x_n) \to 0$.*

Proof Apply Theorem 2 to the identity operator $i : X \to X$. □

We note that X has the DPP if and only if weakly precompact sets and DP sets coincide (since every DP set is weakly precompact [2]).

Corollary 5 *Let X be a Banach space. Then the following statements are equivalent:*

(i) *X has the DPP.*
(ii) [8] *For all Banach spaces Z, every weakly precompact operator $S : Z \to X$ has a completely continuous adjoint.*
(iii) *Every weakly precompact operator $S : \ell_1 \to X$ has a completely continuous adjoint.*

(iv) [8] *If* (x_n) *is a weakly Cauchy sequence in* X *and* (x_n^*) *is a weakly null sequence in* X^*, *then* $x_n^*(x_n) \to 0$.

Proof Apply Corollary 3 to the identity operator $i : X \to X$. □

An operator $T : X \to Y$ is called *limited* if $T(B_X)$ is limited in Y. The operator $T : X \to Y$ is limited if and only if $T^* : Y^* \to X^*$ is w^*-norm sequentially continuous (by Observation 1).

We are now giving our second major result. It gives a characterization of weak* Dunford–Pettis operators and generalizes [11, Theorem 3.2].

Theorem 6 *Let* X *and* Y *be Banach spaces and let* $T : X \to Y$ *be an operator. The following statements are equivalent.*

(1) T *is a weak** *Dunford–Pettis operator.*
(2) T *carries weakly precompact subsets of* X *to limited subsets of* Y.
(3) *If* $S : Z \to X$ *is a weakly precompact operator, then* $TS : Z \to Y$ *is limited, for any Banach space* Z.
(4) *If* $S : \ell_1 \to X$ *is a weakly precompact operator, then* $TS : \ell_1 \to Y$ *is limited.*
(5) *If* (x_n) *is a weakly null sequence in* X *and* (y_n^*) *is a* w^**-Cauchy sequence in* Y^*, *then* $\langle y_n^*, T(x_n)\rangle \to 0$.

Proof (1) ⇒ (2) is similar to the proof of (1) ⇒ (2) in Theorem 2.

(2) ⇒ (3) Suppose $S : Z \to X$ is weakly precompact. Then $TS(B_Z)$ is limited, and thus TS is limited.

(3) ⇒ (4) and (5) ⇒ (1) are obvious.

(4) ⇒ (1) Let (x_n) be a weakly null sequence in X and (y_n^*) be a w^*-null sequence in Y^*. Define $S : \ell_1 \to X$ by

$$S(b) = \sum b_n x_n,$$

where $b = (b_n) \in \ell_1$. Since $S(B_{\ell_1})$ is contained in the closed and absolutely convex hull of $\{x_n : n \in \mathbb{N}\}$, which is relatively weakly compact [9, p. 51], S is weakly compact. By assumption, TS is limited. Suppose (e_n^*) denotes the unit vector basis of ℓ_1. Then

$$\langle y_n^*, T(x_n)\rangle = \langle y_n^*, TS(e_n^*)\rangle \to 0.$$

(1) ⇒ (5) is similar to the proof of (1) ⇒ (5) in Theorem 2. □

Corollary 7 *Let* X *and* Y *be Banach spaces and let* $T : X \to Y$ *be an operator. The following statements are equivalent.*

(i) T *is a weak** *Dunford–Pettis operator.*
(ii) *If* (x_n) *is a weakly Cauchy sequence in* X *and* (y_n^*) *is a* w^**-null sequence in* Y^*, *then* $\langle y_n^*, T(x_n)\rangle \to 0$.
(iii) *If* $S : Y \to Z$ *is an operator such that* $S^*(B_{Z^*})$ *is* w^**-sequentially compact, then* $ST : X \to Z$ *is completely continuous.*
(iv) *If* $S : Y \to c_0$ *is an operator, then* $ST : X \to c_0$ *is completely continuous.*

Proof (i) ⇒ (ii) Suppose that (x_n) is a weakly Cauchy sequence in X and (y_n^*) is a w^*-null sequence in Y^*. Since $(T(x_n))$ is limited in Y, $\langle y_n^*, T(x_n)\rangle \to 0$.

(ii) ⇒ (iii) Let $S : Y \to Z$ be an operator such that $S^*(B_{Z^*})$ is w^*-sequentially compact, but $ST : X \to Z$ is not completely continuous. Let (x_n) be weakly null in X so that

$\|ST(x_n)\| > \epsilon$, for some $\epsilon > 0$. Choose (z_n^*) in B_{Z^*} so that $\langle z_n^*, ST(x_n)\rangle > \epsilon$. Without loss of generality $(S^*(z_n^*))$ is w^*-convergent. Hence $\langle S^*(z_n^*), T(x_n)\rangle = \langle z_n^*, ST(x_n)\rangle \to 0$, a contradiction.

(iii) $\Rightarrow$ (iv) Let $S : Y \to c_0$ be an operator. Note that B_{ℓ_1}, and thus $S^*(B_{\ell_1})$ is w^*-sequentially compact (since c_0 is separable). Then $ST : X \to c_0$ is completely continuous.

(iv) $\Rightarrow$ (i) Suppose (x_n) is a weakly null sequence in X and (y_n^*) is a w^*-null sequence in Y^*. Define $S : Y \to c_0$ by $S(y) = (y_i^*(y))$. Since $ST : X \to c_0$ is completely continuous, $\langle y_n^*, T(x_n)\rangle \leq \|ST(x_n)\| \to 0$. □

The following corollary provides a characterization of spaces with the DP^*P and generalizes [11, Corollary 3.3].

Corollary 8 *Let X be a Banach space. Then the following statements are equivalent:*

(i) *X has the DP^*P.*
(ii) [16] *The identity operator $i : X \to X$ is a weak* Dunford–Pettis operator ; that is, every weakly precompact subset of X is a limited set.*
(iii) [16] *Every weakly precompact operator $S : Z \to X$ is limited, for any Banach space Z.*
(iv) *Every weakly precompact operator $S : \ell_1 \to X$ is limited.*
(v) [16] *If (x_n) is a weakly null sequence in X and (x_n^*) is a w^*-Cauchy sequence in X^*, then $x_n^*(x_n) \to 0$.*

Proof Apply Theorem 6 to the identity operator $i : X \to X$. □

We note that X has the DP^*P if and only if weakly precompact sets and limited sets coincide (since every limited set is weakly precompact [3]).

Corollary 9 *Let X be a Banach space. Then the following statements are equivalent:*

(i) *X has the DP^*P.*
(ii) [16] *If (x_n) is a weakly Cauchy sequence in X and (x_n^*) is a w^*-null sequence in Y^*, then $x_n^*(x_n) \to 0$.*
(iii) [16] *If $S : X \to Z$ is an operator such that $S^*(B_{Z^*})$ is w^*-sequentially compact, then S is completely continuous.*
(iv) [5] *Every operator $S : X \to c_0$ is completely continuous.*

Proof Apply Corollary 7 to the identity operator $i : X \to X$. □

Corollary 10 (i) *If Y^* does not contain a copy of ℓ_1, then every weak Dunford–Pettis operator $T : X \to Y$ is completely continuous.*
(ii) *If B_{Y^*} is w^*-sequentially compact (in particular if Y is separable), then every weak* Dunford–Pettis operator $T : X \to Y$ is completely continuous.*
(iii) *If X or Y has the DPP, then every operator $T : X \to Y$ is weak Dunford–Pettis.*
(iv) *If X or Y has the DP^*P, then every operator $T : X \to Y$ is weak* Dunford–Pettis.*

Proof (i) Let $i : Y \to Y$ be the identity operator on Y. Suppose that $T : X \to Y$ is a weak Dunford–Pettis operator. Since Y^* does not contain a copy of ℓ_1, i^* is weakly precompact (by Rosenthal's ℓ_1 theorem). Then $T = iT$ is completely continuous by Theorem 2.
(ii) Let $i : Y \to Y$ be the identity operator on Y. Suppose $T : X \to Y$ is a weak* Dunford–Pettis operator. Since $i^*(B_{Y^*})$ is w^*-sequentially compact, $T = iT$ is completely continuous by Corollary 7.

(iii) Let $T : X \to Y$ be an operator. If Y has the DPP, then the identity operator $i : Y \to Y$ is weak Dunford–Pettis. Hence $T = iT$ is weak Dunford–Pettis. If X has the DPP, then the identity operator $i : X \to X$ is weak Dunford–Pettis. Hence $T = Ti$ is weak Dunford–Pettis.

(iv) The proof is similar to that of (iii).

□

Clearly each completely continuous operator $T : X \to Y$ is weak* Dunford–Pettis and each weak* Dunford–Pettis operator is weak Dunford–Pettis. By Corollary 10, we obtain the following result.

Corollary 11 *If Y^* does not contain a copy of ℓ_1, then the families of completely continuous operators, weak* Dunford–Pettis operators, and weak Dunford–Pettis operators $T : X \to Y$ coincide.*

Examples (a) Note that ℓ_∞ has the DP^*P (since it has the DPP and the Grothendieck property [5]). Then the identity operator $i : \ell_\infty \to \ell_\infty$ is weak* Dunford–Pettis and is not completely continuous.

(b) A space X has the DP^*P if and only if every operator $T : X \to c_0$ is completetely continuous [5]. Since the identity operator $i : c_0 \to c_0$ is not completetely continuous, c_0 does not have the DP^*P. Thus $i : c_0 \to c_0$ is weak Dunford–Pettis (since c_0 has the DPP) and not weak* Dunford–Pettis.

Corollary 12 (i) *Suppose that Y has the DPP. If $T : X \to Y$ is an operator such that T^* is not completely continuous, then T fixes a copy of ℓ_1.*

(ii) *Suppose that Y has the DP^*P. If $T : X \to Y$ is a non-limited operator, then T fixes a copy of ℓ_1.*

Proof (i) Suppose that T^* is not completely continuous. Let (y_n^*) be weakly null in Y^* so that $\|T^*(y_n^*)\| \not\to 0$. Suppose (x_n) is a sequence in B_X such that $|\langle y_n^*, T(x_n)\rangle| > \epsilon$ for some $\epsilon > 0$. We claim that (x_n) has no weakly Cauchy subsequence. If the claim is false, suppose without loss of generality that (x_n) is weakly Cauchy. Since Y has the DPP, T is weak Dunford–Pettis. Then $\langle y_n^*, T(x_n)\rangle \to 0$ by Corollary 3. This contradiction shows that (x_n) has no weakly Cauchy subsequence. By Rosenthal's ℓ_1 theorem, (x_n) has a subsequence equivalent to the ℓ_1 basis. Suppose without loss of generality that (x_n) is equivalent to (e_n^*), where (e_n^*) denotes the basis of ℓ_1.
Now, since $|\langle y_n^*, T(x_n)\rangle| > \epsilon$ and Y has the DPP, $(T(x_n))$ has no weakly Cauchy subsequence (by Corollary 5). By Rosenthal's ℓ_1 theorem, $(T(x_n))$ has a subsequence equivalent to (e_n^*). Suppose without loss of generality that $(T(x_n))$ is equivalent to (e_n^*). Hence T fixes a copy of ℓ_1.

(ii) The proof is similar to that of (i).

□

Corollary 12 (ii) generalizes [5, Theorem 2.3] (which states that if X and Y have the the DP^*P and $T : X \to Y$ is a non-limited operator, then T fixes a copy of ℓ_1).

A Banach space X has the *Dunford–Pettis relatively compact property (DPrcP)* if every DP subset of X is relatively compact [13]. Schur spaces have the $DPrcP$. The space X does not contain a copy of ℓ_1 if and only if X^* has the $DPrcP$ [12,13]. We note that if X^* does not contain a copy of ℓ_1, then X^{**}, thus X, has the $DPrcP$ [12,13].

The space X has the *Gelfand–Phillips (GP)* property (or X is a *Gelfand–Phillips* space) if every limited subset of X is relatively compact. Schur spaces and separable spaces have the Gelfand–Phillips property [3].

An operator $T : X \to Y$ is called *Dunford–Pettis completely continuous (DPcc)* if T carries weakly null and DP sequences to norm null ones [22]. An operator $T : X \to Y$ is called *limited completely continuous (lcc)* if T maps weakly null limited sequences to norm null sequences [23].

The sets of all limited completely continuous, Dunford–Pettis completely continuous operators, weak Dunford Pettis, and weak* Dunford Pettis operators from X to Y will be respectively denoted by $LCC(X, Y)$, $DPCC(X, Y)$, $WDP(X, Y)$, and $W^*DP(X, Y)$.

In the following result, we characterize Banach spaces X on which every weak (resp. weak*) Dunford–Pettis operator is a DPcc (resp. lcc) operator.

Corollary 13 (i) *A Banach space X has the DPrcP if and only if $DPCC(X, \ell_\infty) = WDP(X, \ell_\infty)$.*

(ii) *A Banach space X has the GP property if and only if $LCC(X, \ell_\infty) = W^*DP(X, \ell_\infty)$.*

Proof (i) A Banach space X has the $DPrcP$ if and only if $DPCC(X, \ell_\infty) = L(X, \ell_\infty)$ [22]. Since ℓ_∞ has the DPP, $L(X, \ell_\infty) = WDP(X, \ell_\infty)$.

(ii) A Banach space X has the GP property if and only if $LCC(X, \ell_\infty) = L(X, \ell_\infty)$ [23]. Since ℓ_∞ has the DP^*P, $L(X, \ell_\infty) = W^*DP(X, \ell_\infty)$. □

If X has the $DPrcP$, then X has the GP property (since any limited set is a DP set). Thus, if X has the $DPrcP$, then $L(X, \ell_\infty) = LCC(X, \ell_\infty) = DPCC(X, \ell_\infty) = WDP(X, \ell_\infty) = W^*DP(X, \ell_\infty)$.

Example We note that the identity operator $i : \ell_\infty \to \ell_\infty$ is weak* Dunford–Pettis and not lcc (since ℓ_∞ does not have the GP property). Further, $i : \ell_\infty \to \ell_\infty$ is weak Dunford–Pettis (since ℓ_∞ has the DPP) and not DPcc (since ℓ_∞ does not have the $DPrcP$).

4 Weak *p*-convergent operators and weak* *p*-convergent operators

For $1 \le p < \infty$, p^* denotes the conjugate of p. If $p = 1$, we take c_0 instead of ℓ_{p^*}. The unit vector basis of ℓ_p will be denoted by (e_n).

Let $1 \le p < \infty$. A sequence (x_n) in X is called *weakly p-summable* if $(x^*(x_n)) \in \ell_p$ for each $x^* \in X^*$ [10, p. 32]. Let $\ell_p^w(X)$ denote the set of all weakly p-summable sequences in X. The space $\ell_p^w(X)$ is a Banach space with the norm

$$\|(x_n)\|_p^w = \sup\left\{\left(\sum_{n=1}^{\infty} |\langle x^*, x_n\rangle|^p\right)^{1/p} : x^* \in B_{X^*}\right\}$$

We recall the following isometries: $L(\ell_{p^*}, X) \simeq \ell_p^w(X)$ for $1 < p < \infty$; $L(c_0, X) \simeq \ell_p^w(X)$ for $p = 1$; that are obtained via the isometry $T \to (T(e_n))$ [10, Proposition 2.2, p. 36].

A series $\sum x_n$ in X is said to be *weakly unconditionally convergent (wuc)* if for every $x^* \in X^*$, the series $\sum |x^*(x_n)|$ is convergent. An operator $T : X \to Y$ is *unconditionally converging* if it maps weakly unconditionally convergent series to unconditionally convergent ones.

Let $1 \le p \le \infty$. An operator $T : X \to Y$ is called *p-convergent* if T maps weakly p-summable sequences into norm null sequences. The set of all p-convergent operators from X to Y is denoted by $C_p(X, Y)$ [6].

The 1-convergent operators are precisely the unconditionally converging operators and the ∞-convergent operators are precisely the completely continuous operators. If $p < q$, then $C_q(X, Y) \subseteq C_p(X, Y)$.

A sequence (x_n) in X is called *weakly p-convergent* to $x \in X$ if the sequence $(x_n - x)$ is weakly p-summable [6]. Weakly ∞-convergent sequences are precisely the weakly convergent sequences.

Let $1 \leq p \leq \infty$. A bounded subset K of X is *relatively weakly p-compact* if every sequence in K has a weakly p-convergent subsequence. An operator $T : X \to Y$ is *weakly p-compact* if $T(B_X)$ is relatively weakly p-compact [6].

The set of weakly p-compact operators $T : X \to Y$ is denoted by $W_p(X, Y)$. If $p < q$, then $W_p(X, Y) \subseteq W_q(X, Y)$. A Banach space $X \in C_p$ (resp. $X \in W_p$) if $id(X) \in C_p(X, X)$ (resp. $id(X) \in W_p(X, X)$) [6], where $id(X)$ is the identity map on X.

A sequence (x_n) in X is called *weakly p-Cauchy* if $(x_{n_k} - x_{m_k})$ is weakly p-summable for any increasing sequences (n_k) and (m_k) of positive integers.

Every weakly p-convergent sequence is weakly p-Cauchy, and the weakly ∞-Cauchy sequences are precisely the weakly Cauchy sequences.

Let $1 \leq p \leq \infty$. A subset S of X is called *weakly p-precompact* if every sequence from S has a weakly p-Cauchy subsequence [18]. An operator $T : X \to Y$ is called *weakly p-precompact* if $T(B_X)$ is weakly p-precompact.

Let $1 \leq p \leq \infty$. A Banach space X has the *Dunford–Pettis property of order p* (DPP_p) ($1 \leq p \leq \infty$) if every weakly compact operator $T : X \to Y$ is p-convergent, for any Banach space Y [6]. Equivalently, X has the DPP_p if and only if $x_n^*(x_n) \to 0$ whenever (x_n) is weakly p-summable in X and (x_n^*) is weakly null in X^* [6, Proposition 3.2].

If X has the DPP_p, then it has the DPP_q, if $q < p$. Also, the DPP_∞ is precisely the DPP, and every Banach space has the DPP_1. $C(K)$ spaces and L_1 have the DPP, and thus the DPP_p for all p. If $1 < r < \infty$, then ℓ_r has the DPP_p for $p < r^*$. If $1 < r < \infty$, then $L_r(\mu)$ has the DPP_p for $p < min(2, r^*)$. Tsirelson's space T has the DPP_p for all $p < \infty$. Since T is reflexive, it does not have the DPP. Tsirelson's dual space T^* does not have the DPP_p, if $p > 1$ [6].

Let $1 \leq p \leq \infty$. A Banach space X has the *DP^*-property of order p* (DP^*P_p) if all weakly p-compact sets in X are limited [14]. Equivalently, X has the DP^*P_p if and only if $x_n^*(x_n) \to 0$ whenever (x_n) is weakly p-summable in X and (x_n^*) is weakly null in X^* [14].

If X has the DP^*P_q, then it has the DP^*P_p, if $q > p$. Further, the DP^*P_∞ is precisely the DP^*P and every Banach space has the DP^*P_1. If X has the DP^*P, then X has the DP^*P_p, $1 \leq p \leq \infty$. If X has the DP^*P_p, then X has the DPP_p.

Let $1 \leq p < \infty$. An operator $T : X \to Y$ is called *weak p-convergent* if $\langle y_n^*, T(x_n)\rangle \to 0$ whenever (x_n) is weakly p-summable in X and (y_n^*) is weakly null in Y^* [15]. An operator $T : X \to Y$ is called *weak* p-convergent* if $\langle y_n^*, T(x_n)\rangle \to 0$ whenever (x_n) is weakly p-summable in X and (y_n^*) is w^*-null in Y^* [15].

In the following we study weak p-convergent and weak* p-convergent operators. The following result generalizes [18, Theorem 8].

Theorem 14 *Let X and Y be Banach spaces, and let $1 < p < \infty$. The following statements are equivalent about an operator $T : X \to Y$.*

(1) *T is weak p-convergent.*
(2) *T takes weakly p-precompact subsets of X to DP subsets of Y.*
(3) *For any Banach space Z, if $S : Y \to Z$ has a weakly precompact adjoint, then $ST : X \to Z$ is p-convergent.*
(4) *If $S : Y \to c_0$ has a weakly precompact adjoint, then $ST : X \to c_0$ is p-convergent.*

(5) *If* (x_n) *is a weakly* p*-summable sequence in* X *and* (y_n^*) *is a weakly Cauchy sequence in* Y^**, then* $\langle y_n^*, T(x_n)\rangle \to 0$.

Proof (1) $\Rightarrow$ (2) Let A be a weakly p-precompact subset of X. Suppose by contradiction that $T(A)$ is not a Dunford–Pettis subset of Y. Let (y_n^*) be a weakly null sequence in Y^*, and let (x_n) be a sequence in A such that $|\langle y_n^*, T(x_n)\rangle| > \epsilon$ for all n, for some $\epsilon > 0$. By passing to a subsequence, we can assume that (x_n) is weakly p-Cauchy.

Let $n_1 = 1$ and choose $n_2 > n_1$ so that $|\langle y_{n_2}^*, T(x_{n_1})\rangle| < \epsilon/2$. We can do this since (y_n^*) is w^*-null. Continue inductively. Choose $n_k > n_{k-1}$ so that $|\langle y_{n_k}^*, T(x_{n_{k-1}})\rangle| < \epsilon/2$. Since T is weak p-convergent, $\langle y_{n_k}^*, T(x_{n_k} - x_{n_{k-1}})\rangle \to 0$. However,

$$|\langle y_{n_k}^*, T(x_{n_k} - x_{n_{k-1}})\rangle| \geq |\langle y_{n_k}^*, T(x_{n_k})\rangle| - |\langle y_{n_k}^*, T(x_{n_{k-1}})\rangle| > \epsilon/2,$$

a contradiction.

(2) $\Rightarrow$ (3) Suppose $S : Y \to Z$ is an operator with weakly precompact adjoint. Let (x_n) be a weakly p-summable sequence in X. By (2), $(T(x_n))$ is a DP subset of Y. Therefore $(ST(x_n))$ is relatively compact [20, Corollary 4]. Hence $\|ST(x_n)\| \to 0$, and thus ST is p-convergent.

(3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (1) are obvious.

(4) $\Rightarrow$ (1) Let (x_n) be a weakly p-summable sequence in X and (y_n^*) be a weakly null sequence in Y^*. Define $S : Y \to c_0$ by $S(y) = (y_i^*(y))$. Then $S^* : \ell_1 \to Y^*$, $S^*(b) = \sum b_i y_i^*$. Note that S^* maps B_{ℓ_1} into the closed and absolutely convex hull of $\{y_i^* : i \in \mathbb{N}\}$, which is relatively weakly compact [9, p. 51]. Then S^* is weakly compact. Hence $ST : X \to c_0$ is p-convergent. Therefore $\langle T(x_n), y_n^*\rangle \leq \|ST(x_n)\| = \sup_i |\langle y_i^*, T(x_n)\rangle| \to 0$, and T is weak p-convergent.

(1) $\Rightarrow$ (5) Let (x_n) be a weakly p-summable sequence in X and (y_n^*) be a weakly Cauchy sequence in Y^*. Suppose $\langle y_n^*, T(x_n)\rangle \not\to 0$. Without loss of generality suppose that $|\langle y_n^*, T(x_n)\rangle| > \epsilon$ for each $n \in \mathbb{N}$, for some $\epsilon > 0$.

Let $n_1 = 1$ and choose $n_2 > n_1$ so that $|\langle y_{n_1}^*, T(x_{n_2})\rangle| < \epsilon/2$. We can do this since $(T(x_n))$ is weakly null. Continue inductively. Choose $n_{k+1} > n_k$ so that $|\langle y_{n_k}^*, T(x_{n_{k+1}})\rangle| < \epsilon/2$. Since T is a weak p-convergent operator, $|\langle y_{n_{k+1}}^* - y_{n_k}^*, T(x_{n_{k+1}})\rangle| \to 0$. However,

$$|\langle y_{n_{k+1}}^* - y_{n_k}^*, T(x_{n_{k+1}})\rangle| \geq |\langle y_{n_{k+1}}^*, T(x_{n_{k+1}})\rangle| - |\langle y_{n_k}^*, T(x_{n_{k+1}})\rangle| > \epsilon/2,$$

and we have a contradiction. □

Corollary 15 *Let* X *and* Y *be Banach spaces, and let* $1 < p < \infty$*. The following statements are equivalent about an operator* $T : X \to Y$*.*

(i) T *is weak* p*-convergent.*
(ii) *For every Banach space* Z*, if* $S : Z \to X$ *is a weakly* p*-precompact operator, then* $TS : Z \to Y$ *has a completely continuous adjoint.*
(iii) [18] *If* $S : \ell_{p^*} \to X$ *is an operator, then* $TS : \ell_{p^*} \to Y$ *has a completely continuous adjoint.*
(iv) *If* (x_n) *is a weakly* p*-Cauchy sequence in* X *and* (y_n^*) *is a weakly null sequence in* Y^**, then* $\langle y_n^*, T(x_n)\rangle \to 0$.

Proof (i) $\Rightarrow$ (ii) Let $S : Z \to X$ be a weakly p-precompact operator. Then $TS(B_Z)$ is a DP set in Y. Hence $(TS)^*$ is completely continuous.

(ii) $\Rightarrow$ (iii) Let $S : \ell_{p^*} \to X$ be an operator. Since $1 < p < \infty$, $\ell_{p^*} \in W_p$ [6]. Hence S is weakly p-compact, and thus $(TS)^*$ is completely continuous.

(iii) $\Rightarrow$ (i) Let (x_n) be weakly p-summable in X and let (y_n^*) be weakly null in Y^*. Define $S : \ell_{p^*} \to X$ by $S(b) = \sum b_i x_i$ [10, Proposition 2.2, p. 36]. Since $TS(B_{\ell_{p^*}})$ is a DP set in Y, $\langle y_n^*, TS(e_n)\rangle = \langle y_n^*, T(x_n)\rangle \to 0$.

(i) $\Rightarrow$ (iv) Let (x_n) be weakly p-Cauchy in X and let (y_n^*) be weakly null in Y^*. Since $(T(x_n))$ is a DP set in Y, $\langle y_n^*, T(x_n)\rangle \to 0$.

(iv) $\Rightarrow$ (i) is obvious. □

As a consequence of the previous two results we obtain the following characterizations of Banach spaces with the DPP_p.

Corollary 16 [19, Theorem 1] *Let $1 < p < \infty$. The following statements are equivalent about a Banach space X.*

(1) *X has the DPP_p.*
(2) *The identity operator $i : X \to X$ is weak p-convergent; that is, every weakly p-precompact subset of X is a Dunford–Pettis set.*
(3) *Every operator $S : X \to Z$ with weakly precompact adjoint is p-convergent, for any Banach space Z.*
(4) *Every operator $S : X \to c_0$ with weakly precompact adjoint is p-convergent.*
(5) *If (x_n) is a weakly p-summable sequence in X and (x_n^*) is a weakly Cauchy sequence in X^*, then $x_n^*(x_n) \to 0$.*

Proof Apply Theorem 14 to the identity operator $i : X \to X$. □

Corollary 17 [19, Theorem 1] *Let $1 < p < \infty$. The following statements are equivalent about a Banach space X.*

(i) *X has the DPP_p.*
(ii) *For all Banach spaces Z, every weakly p-precompact operator $S : Z \to X$ has a completely continuous adjoint.*
(iii) *Every operator $S : \ell_{p^*} \to X$ has a completely continuous adjoint.*
(iv) *If (x_n) is a weakly p-Cauchy sequence in X and (x_n^*) is a weakly null sequence in X^*, then $x_n^*(x_n) \to 0$.*

Proof Apply Corollary 15 to the identity map $i : X \to X$. □

The following result generalizes [15, Theorem 2.11].

Theorem 18 *Let X and Y be Banach spaces and $T : X \to Y$ be an operator. Let $1 < p < \infty$. The following statements are equivalent.*

(1) *T is weak* p-convergent.*
(2) *T carries weakly p-precompact subsets of X to limited subsets of Y.*
(3) *If $S : Z \to X$ is a weakly p-precompact operator, then $TS : Z \to Y$ is limited, for any Banach space Z.*
(4) *If $S : \ell_{p^*} \to X$ is an operator, then $TS : \ell_{p^*} \to Y$ is limited.*
(5) *If (x_n) is a weakly p-summable sequence in X and (y_n^*) is a w^*-Cauchy sequence in Y^*, then $\langle y_n^*, T(x_n)\rangle \to 0$.*

Proof (1) $\Rightarrow$ (2) is similar to the proof of (1) $\Rightarrow$ (2) in Theorem 14.

(2) $\Rightarrow$ (3) Let $S : Z \to X$ be a weakly p-precompact operator. Then $TS(B_Z)$ is limited, and thus $TS : Z \to Y$ is limited.

(3) $\Rightarrow$ (4) Let $S : \ell_{p^*} \to X$ be an operator. Since $1 < p < \infty$, $\ell_{p^*} \in W_p$ [6]. Hence S is weakly p-compact, and thus TS limited.

(4) $\Rightarrow$ (1) Suppose (x_n) is weakly p-summable in X and (y_n^*) is w^*-null in Y^*. Define $S : \ell_{p^*} \to X$ by $S(b) = \sum b_i x_i$ [10, Proposition 2.2, p. 36]. Since $TS(B_{\ell_{p^*}})$ is a limited set in Y, $\langle y_n^*, TS(e_n)\rangle = \langle y_n^*, T(x_n)\rangle \to 0$.

(1) $\Rightarrow$ (5) is similar to the proof of (1) $\Rightarrow$ (5) in Theorem 14. □

Corollary 19 *Let X and Y be Banach spaces and $T : X \to Y$ be an operator. Let $1 < p < \infty$. The following statements are equivalent.*

(i) *T is weak* p-convergent.*
(ii) *If (x_n) is a weakly p-Cauchy sequence in X and (y_n^*) is a w^*-null sequence in Y^*, then $\langle y_n^*, T(x_n)\rangle \to 0$.*
(iii) *If $S : Y \to Z$ is an operator such that $S^*(B_{Z^*})$ is w^*-sequentially compact, then $ST : X \to Z$ is p-convergent.*
(iv) *If $S : Y \to c_0$ is an operator, then $ST : X \to c_0$ is p-convergent.*

Proof (i) $\Rightarrow$ (ii) Suppose that (x_n) is a weakly p-Cauchy sequence in X and (y_n^*) is a w^*-null sequence in Y^*. Since $(T(x_n))$ is limited in Y, $\langle y_n^*, T(x_n)\rangle \to 0$.

(ii) $\Rightarrow$ (iii) Let $S : Y \to Z$ be an operator such that $S^*(B_{Z^*})$ is w^*-sequentially compact, but $ST : X \to Z$ is not p-convergent. Let (x_n) be weakly p-summable in X so that $\|ST(x_n)\| > \epsilon$, for some $\epsilon > 0$. Choose (z_n^*) in B_{Z^*} so that $\langle z_n^*, ST(x_n)\rangle > \epsilon$. Without loss of generality, $(S^*(z_n^*))$ is w^*-convergent. Then $\langle S^*(z_n^*), T(x_n)\rangle = \langle z_n^*, ST(x_n)\rangle \to 0$, a contradiction.

(iii) $\Rightarrow$ (iv) Let $S : Y \to c_0$ be an operator. Note that B_{ℓ_1}, and thus $S^*(B_{\ell_1})$ is w^*-sequentially compact. Then $ST : X \to c_0$ is p-convergent.

(iv) $\Rightarrow$ (i) Let (x_n) be a weakly p-summable sequence in X and let (y_n^*) be a w^*-null sequence in Y^*. Define $S : Y \to c_0$ by $S(y) = (y_i^*(y))$. Since ST is p-convergent, $\langle y_n^*, T(x_n)\rangle \le \|ST(x_n)\| \to 0$. □

The following two corollaries provide equivalent characterizations of spaces with the DP^*P_p.

Corollary 20 *Let $1 < p < \infty$. The following statements are equivalent about a Banach space X.*

(i) *X has the DP^*P_p.*
(ii) [15] *The identity operator $i : X \to X$ is weak* p-convergent; that is, every weakly p-precompact subset of X is a limited set.*
(iii) [18] *Every weakly p-precompact operator $S : Z \to X$ is limited, for any Banach space Z.*
(iv) [15] *Every operator $S : \ell_{p^*} \to X$ is limited.*
(v) [18] *If (x_n) is a weakly p-summable sequence in X and (x_n^*) is a w^*-Cauchy sequence in X^*, then $x_n^*(x_n) \to 0$.*

Proof Apply Theorem 18 to the identity operator $i : X \to X$. □

Corollary 21 *Let $1 < p < \infty$. The following statements are equivalent about a Banach space X.*

(i) *X has the DP^*P_p.*
(ii) [18] *If (x_n) is a weakly p-Cauchy sequence in X and (x_n^*) is a w^*-null sequence in X^*, then $x_n^*(x_n) \to 0$.*

(iii) [18] *If* $S : X \to Z$ *is an operator such that* $S^*(B_{Z^*})$ *is* w^**-sequentially compact, then* S *is* p*-convergent.*
(iv) [15] *Every operator* $S : X \to c_0$ *is* p*-convergent.*

Proof Apply Corollary 19 to the identity operator $i : X \to X$. □

We note that an operator $T : X \to Y$ is p-convergent if and only if T takes weakly p-precompact subsets of X into norm compact subsets of Y.

Corollary 22 *Let* $1 < p < \infty$.

(i) *Suppose* $S : X \to Y$ *is weakly* p*-precompact and* $T : Y \to Z$ *is an operator with weakly precompact adjoint. If* Y *has the* DPP_p*, then* TS *is compact.*
(ii) *Suppose* $S : X \to Y$ *is weakly* p*-precompact and* $T : Y \to Z$ *is an operator such that* $T^*(B_{Z^*})$ *is* w^**-sequentially compact. If* Y *has the* DP^*P_p*, then* TS *is compact.*

Proof (i) Suppose $S : X \to Y$ is weakly p-precompact and $T : Y \to Z$ is an operator such that T^* is weakly precompact. Since Y has the DPP_p, T is p-convergent by Corollary 16. Then $TS(B_X)$ is relatively compact, and thus TS is compact.
(ii) The proof is similar to that of (i).

□

Corollary 23 *Let* $1 < p < \infty$.

(i) *If* Y^* *does not contain a copy of* ℓ_1*, then every weak* p*-convergent operator* $T : X \to Y$ *is* p*-convergent.*
(ii) *If* B_{Y^*} *is* w^**-sequentially compact (in particular if* Y *is separable), then every weak** p*-convergent operator* $T : X \to Y$ *is* p*-convergent.*
(iii) *If* X *or* Y *has the* DPP_p*, then every operator* $T : X \to Y$ *is weak* p*-convergent.*
(iv) *If* X *or* Y *has the* DP^*P_p*, then every operator* $T : X \to Y$ *is weak** p*-convergent.*

Proof (i) Let $i : Y \to Y$ be the identity operator on Y. Suppose $T : X \to Y$ is a weak p-convergent operator. By Rosenthal's ℓ_1 theorem, i^* is weakly precompact. Then $T = iT$ is p-convergent by Theorem 14.
(ii) The proof is similar to that of (ii).
(iii) Let $T : X \to Y$ be an operator. If Y has the DPP_p, then the identity operator $i : Y \to Y$ is weak p-convergent. Hence $T = iT$ is weak p-convergent. If X has the DPP_p, then the identity operator $i : X \to X$ is weak p-convergent. Hence $T = Ti$ is weak p-convergent.
(iv) The proof is similar to that of (iii).

□

Clearly each p-convergent operator $T : X \to Y$ is weak* p-convergent and each weak* p-convergent operator is weak p-convergent. By Corollay 23, we obtain the following result. It generalizes [15, Proposition 2.5].

Corollary 24 *If* Y^* *does not contain a copy of* ℓ_1*, then the families of* p*-convergent operators, weak** p*-convergent operators, and weak* p*-convergent operators* $T : X \to Y$ *coincide.*

Let $1 \leq p < \infty$. A Banach space X has the *p-Gelfand–Phillips* (p-GP) *property* (or is a *p-Gelfand–Phillips space*) if every limited weakly p-summable sequence in X is norm null [15].

If X has the GP property, then X has the p-GP property for any $1 \leq p < \infty$. The space ℓ_∞ does not have the p-GP property for any $1 \leq p < \infty$ [15].

Let $1 \leq p < \infty$. A space X has the *p-Dunford Pettis relatively compact property* (p-$DPrcP$) if every DP weakly p-summable sequence (x_n) in X is norm null [17].

If X has the $DPrcP$ property, then X has the p-$DPrcP$ property for any $1 \leq p < \infty$.

Corollary 25 *Let $1 \leq p < \infty$. If X has the p-GP (resp. the p-$DPrcP$) property, then the following are equivalent.*

(i) *X has the DP^*P_p (resp. the DPP_p).*
(ii) *$X \in C_p$.*

Proof $(i) \Rightarrow (ii)$ We only prove the result when X has the the p-GP and the DP^*P_p. The other case is similar.

Let (x_n) be weakly p-summable in X. Then (x_n) is limited by Corollary 20. Therefore $\|x_n\| \to 0$, and thus $X \in C_p$. □

Let $1 \leq p < \infty$. An operator $T : X \to Y$ is called *limited p-convergent* if it carries limited weakly p-summable sequences in X to norm null ones in Y [15]. An operator $T : X \to Y$ is called *DP p-convergent* if it takes DP weakly p-summable sequences to norm null sequences [17].

The sets of all limited p-convergent, DP p-convergent, weak p-convergent, and weak* p-convergent operators from X to Y will be respectively denoted by $LC_p(X, Y)$, $DPC_p(X, Y)$, $WC_p(X, Y)$, and $W^*C_p(X, Y)$.

Corollary 26 *Let $1 \leq p < \infty$. Let X be a Banach space. The following statements hold.*

(i) *X has the p-$DPrcP$ if and only if $WC_p(X, \ell_\infty) = DPC_p(X, \ell_\infty)$.*
(ii) *X has the p-GP property if and only if $W^*C_p(X, \ell_\infty) = LC_p(X, \ell_\infty)$.*

Proof (i) A Banach space X has the p-$DPrcP$ if and only if $DPC_p(X, \ell_\infty) = L(X, \ell_\infty)$ [17]. Since ℓ_∞ has the DPP_p, $L(X, \ell_\infty) = WC_p(X, \ell_\infty)$.

(ii) A Banach space X has the p-GP if and only if $LC_p(X, \ell_\infty) = L(X, \ell_\infty)$ [17]. Since ℓ_∞ has the DP^*P_p, $L(X, \ell_\infty) = W^*C_p(X, \ell_\infty)$.

□

Since any limited set is a DP set, any limited weakly p-summable sequence is also DP weakly p-summable. Hence if X has the p-$DPrcP$, then X has the p-GP property. Thus, if X has the p-$DPrcP$, then $L(X, \ell_\infty) = LC_p(X, \ell_\infty) = DPC_p(X, \ell_\infty) = WC_p(X, \ell_\infty) = W^*C_p(X, \ell_\infty)$.

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