Rendiconti del Circolo Matematico di Palermo Series 2 (2020) 69:813–822
https://doi.org/10.1007/s12215-019-00437-w

# Quasi-complete intersections in $\mathbb{P}^2$ and syzygies

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Received: 9 March 2019 / Accepted: 23 July 2019 / Published online: 1 August 2019


**Abstract**

Let $C \subset \mathbb{P}^2$ be a reduced, singular curve of degree $d$ and equation $f = 0$. Let $\Sigma$ denote the jacobian subscheme of $C$. We have $0 \to E \to 3.\mathcal{O} \to \mathcal{I}_\Sigma(d-1) \to 0$ (the surjection is given by the partials of $f$). We study the relationships between the Betti numbers of the module $H^0_*(E)$ and the integers, $d, \tau$, where $\tau = \deg(\Sigma)$. We observe that our results apply to any quasi-complete intersection of type $(s, s, s)$.

**Keywords** Quasi complete intersections · Vector bundle · Syzygies · Global Tjurina number · Plane curves

**Mathematics Subject Classification** Primary 14H50; Secondary 14M06 , 14M07 , 13D02

## 1 Introduction

Let $C \subset \mathbb{P}^2$ be a reduced, singular curve, of degree $d$, of equation $f = 0$. The partials of $f$ determine a morphism: $3.\mathcal{O} \stackrel{\partial f}{\to} \mathcal{O}(d-1)$, whose image is $\mathcal{I}_\Sigma(d-1)$, where according to our assumptions, $\Sigma \subset \mathbb{P}^2$, is a closed subscheme of codimension two. The subscheme $\Sigma$, whose support is the singular locus of $C$, is called the *jacobian subscheme* of $C$. We denote by $\tau$ its degree, it is the *global Tjurina number* of the plane curve $C$.

We have:

$$0 \to E \to 3.\mathcal{O} \to \mathcal{I}_\Sigma(d-1) \to 0 \tag{1}$$

where $E$ is a rank two vector bundle with Chern classes $c_1 = 1-d$, $c_2 = (d-1)^2 - \tau$ (see for instance [11] and references therein). The bundle $E$ is the sheaf of logarithmic vector fields along $C$, also denoted $Der(-\log C)$ [5,14,15]. A particular case of this situation is when $C$ is an *arrangement of lines* [8,13,17]. This is a very active field of research with a huge literature.

In [9], using techniques of the theory of singularities, du Plessis and Wall gave sharp bounds on $\tau$ in function of $d$ and $d_1$, the least twist of $E$ having a section. Observe that

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$H^0_*(E)$ is the module of syzygies between the partials. This result has been extended (see [11]) to the case of quasi-complete intersections (q.c.i.), using vector bundles techniques.

In this note, inspired by [6], instead of considering only $d_1$, the minimal degree of a generator of $H^0_*(E)$, we consider the full minimal resolution of this module. So we will assume that $H^0_*(E)$ is minimally generated by $m$ elements of degree $d_1 \leq d_2 \leq \cdots \leq d_m$. The $m$-uple $(d_1, \ldots, d_m)$ is the *exponent* of $C$. We have $m \geq 2$, with equality if and only if $E$ splits. In this case one say that $C$ is a *free divisor* [1,14] or, equivalently, that $\Sigma$ is an almost complete intersection. The case $m = 3$ is handled in [6]. Here we deal with the general case $m \geq 3$.

Starting from the minimal free resolution of $H^0_*(E)$ we show how to get a free (non necessarily minimal) resolution of $\mathcal{I}_\Sigma$. With this we show (Corollary 6) that if $\Sigma$ is a complete intersection, then $m \leq 4$. Then (Theorem 8) we prove that $2d - 4 \geq d_i, \forall i$ and that the inequality is sharp if and only if $\Sigma$ is a point ($\tau = 1$). Finally we prove: $d_m = d - 1$ or $2d - m \geq d_m$.

Then (Theorem 13), shows that $d_3 \leq d - 1$ and characterizes the q.c.i. realizing the lower bound, $(d-1)(d-1-d_1) = \tau$, in du Plessis–Wall theorem: this happens if and only if $\Sigma$ is a complete intersection $(d-1, d-1-d_1)$. We also describe what happens in the next degree.

Finally, in the setting of q.c.i., we answer to a conjecture raised in [7] (Proposition 15) and describe the sub-maximal case (see Proposition 17).

The exact sequence (1) presents $\Sigma$ as a quasi-complete intersections (q.c.i.) of type $(d-1, d-1, d-1)$. In our proofs we will *never* use the fact that the three curves giving the q.c.i. are the partials of a polynomial $f$ (!). *So setting $s = d - 1$ , all our results are true for q.c.i. of type $(s, s, s)$.* Actually, after appropriate changes in notations (see [11]) they should hold for all q.c.i. (i.e. of any type $(a, b, c)$). Observe that to determine the minimal free resolution (m.f.r.) of $H^0_*(E)$ amounts to determine the m.f.r. of the (non saturated if $m > 2$) q.c.i. ideal $J = (F_1, F_2, F_3)$. For a purely algebraic approach to q.c.i. see for example [16].

As the first version of this paper was finished I received the preprint [7] containing some overlaps. This obliged me to revisit my text. This version contains some improvements (so thank you to the authors of [7] !), but overlaps are still present. However, since the methods are different, it could be useful to see how geometric techniques apply in this context.

I thank Alexandru Dimca for useful discussions, in particular about (i) of Theorem 13.

## 2 Setting, notations

Following [6] we have:

**Definition 1** We will say that $C$ is a $m$-syzygy curve if $H^0_*(E)$ is minimally generated by $m$ elements of degree $d_1 \leq d_2 \leq \cdots \leq d_m$. The $m$-uple $(d_1, \ldots, d_m)$ is the exponent of $C$.

***Remark 2*** We have $m \geq 2$. Moreover $m = 2$ if and only if $E$ is the direct sum of two line bundles.

In the sequel we will always assume $m \geq 3$.

For any $i$, $E(d_i)$ has a section vanishing in codimension two.

Besides the exact sequence (1) we will also consider the following ones:

$$0 \to \bigoplus_{j=1}^{m-2} \mathcal{O}(-b_j) \to \bigoplus_{i=1}^{m} \mathcal{O}(-d_i) \to E \to 0 \tag{2}$$

The minimal presentation of $H^0_*(E)$ yields $\bigoplus_{i=1}^m \mathcal{O}(-d_i) \to E \to 0$, the kernel; $K$, is locally free of rank $m-2$ with $H^1_*(K)=0$, hence $K$ is a direct sum of line bundles.

$$0 \to \mathcal{O} \to E(d_1) \to \mathcal{I}_Z(2d_1+1-d) \to 0 \tag{3}$$

Here $Z \subset \mathbb{P}^2$ is a locally complete intersection (l.c.i.), zero-dimensional subscheme of degree

$$\deg(Z) = c_2(E(d_1)) = d_1(1-d) + (d-1)^2 - \tau + d_1^2 \tag{4}$$

## 3 Resolutions

Starting from (2) we can get the minimal free resolution of $H^1_*(E)$ and $H^0_*(\mathcal{I}_Z)$, more precisely:

**Lemma 3** *Let $E$ be a rank two vector bundle on $\mathbb{P}^2$ and let $Z=(s)_0$, $s \in H^0(E(d_1))$, where $d_1 = min\{k \mid h^0(E(k)) \neq 0\}$.*

(i) *The following are equivalent:*

  (a) *$H^0_*(E)$ is minimally generated by $m$ elements*
  (b) *$H^1_*(E)$ is minimally generated by $m-2$ elements*
  (c) *$H^0(\mathcal{I}_Z)$ is minimally generated by $m-1$ elements.*

*Assume the minimal free resolution of $H^0_*(E)$ is given by (2) and that $c_1(E)=1-d$, then:*

(ii) *The minimal free resolution of $H^1_*(E)$ is*

$$0 \to \bigoplus_{j=1}^{m-2} S(-b_j) \to \bigoplus_{i=1}^{m} S(-d_i) \to \bigoplus_{i=1}^{m} S(d_i+1-d) \to \bigoplus_{j=1}^{m-2} S(b_j+1-d) \to H^1_*(E) \to 0 \tag{5}$$

(iii) *The minimal free resolution of $H^0_*(\mathcal{I}_Z)$ is:*

$$0 \to \bigoplus_{j=1}^{m-2} \mathcal{O}(-b_j+d-1-d_1) \to \bigoplus_{i=2}^{m} \mathcal{O}(-d_i+d-1-d_1) \to \mathcal{I}_Z \to 0 \tag{6}$$

***Proof*** Let $E$ be a rank two vector bundle on $\mathbb{P}^2$ and assume that $H^0_*(E)$ is minimally generated by $m$ elements. We have $\mathcal{G}_1 \to E \to 0$, with $\mathcal{G}_1 = \bigoplus_1^m \mathcal{O}(-d_i)$. As explained before the kernel, $\mathcal{G}_2$, is a direct sum of line bundles: $\mathcal{G}_2 = \bigoplus \mathcal{O}(-b_j)$. By dualizing the exact sequence: $0 \to \mathcal{G}_2 \to \mathcal{G}_1 \to E \to 0$, we get: $0 \to E^* \to \mathcal{G}_1^* \to \mathcal{G}_2^* \to 0$. Taking into account that $E^* \simeq E(-c_1)$ ($c_1 = c_1(E)$) because $E$ has rank two, we get: $0 \to E \to \mathcal{G}_1^*(c_1) \to \mathcal{G}_2^*(c_1) \to 0$. Taking cohomology this yields: $0 \to H^0_*(E) \to G_1^*(c_1) \to G_2^*(c_1) \to H^1_*(E) \to 0$. This is the beginning of a minimal free resolution of $H^1_*(E)$. We conclude with (2). This proves (ii) and also (a) $\Rightarrow$ (b) in (i). By uniqueness of the minimal free resolution this also proves (b) $\Rightarrow$ (a) in (i).

We have:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & \mathcal{O} & = & \mathcal{O} & \\
 & & & \downarrow & & \downarrow & \\
0 \to & \bigoplus_{j=1}^{m-2} \mathcal{O}(-b_j + d_1) & \to & \bigoplus_{i=2}^{m} \mathcal{O}(-d_i + d_1) \oplus \mathcal{O} & \to & E(d_1) & \to 0 \\
 & || & & \downarrow & & \downarrow & \\
0 \to & \bigoplus_{j=1}^{m-2} \mathcal{O}(-b_j + d_1) & \to & \bigoplus_{i=2}^{m} \mathcal{O}(-d_i + d_1) & \to & \mathcal{I}_Z(-d + 1 + 2d_1) & \to 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array}$$

which proves (iii) and also (a) $\Leftrightarrow$ (c) in (i) (observe that we have $0 \to S \xrightarrow{f} H^0_*(E(d_1)) \to H^0_*(\mathcal{I}_Z(2d_1 - d + 1)) \to 0$, where, by assumption, the image of $f$ yields a minimal generator of $H^0_*(E(d_1))$. □

## 4 Resolution of $H^0_*(\mathcal{I}_\Sigma)$

Starting from the resolution of $H^0_*(E)$ it is also possible to get a resolution of $H^0_*(\mathcal{I}_\Sigma)$ but this resolution is not necessarily minimal:

**Proposition 4** *We have the following free resolution*

$$0 \to \bigoplus_{i=1}^{m} \mathcal{O}(d_i - 2d + 2) \to \bigoplus_{j=1}^{m-2} \mathcal{O}(b_j - 2d + 2) \oplus 3.\mathcal{O}(1 - d) \to \mathcal{I}_\Sigma \to 0 \quad (7)$$

*This resolution is minimal up to cancellation of $\mathcal{O}(1 - d)$ terms with some $\mathcal{O}(d_i - 2d + 2)$ (in this case $d_i = d - 1$).*

***Proof*** Since $\mathcal{I}_\Sigma(d - 1)$ is generated by global sections we can link $\Sigma$ to a zero-dimensional subscheme $T$ by a complete intersection of type $(d - 1, d - 1)$. From the exact sequence (1), by mapping cone, we get that $T$ is a section of $E(d - 1)$. So we have an exact sequence: $0 \to \mathcal{O}(1 - d) \to E \to \mathcal{I}_T \to 0$. From (2) we get a surjection: $\bigoplus_1^m \mathcal{O}(-d_i) \to \mathcal{I}_T \to 0$. Using (2) we can build a commutative diagram and by the snake lemma we get:

$$0 \to \bigoplus_{1}^{m-2} \mathcal{O}(-b_j) \oplus \mathcal{O}(1 - d) \to \bigoplus_{1}^{m} \mathcal{O}(-d_i) \to \mathcal{I}_T \to 0$$

This resolution is minimal unless the section of $E(d - 1)$ yielding $T$ is a minimal generator of $H^0_*(E)$. From the above resolution, by mapping cone, we get the desired resolution of $\mathcal{I}_\Sigma$. Again this resolution is minimal unless one curve (resp. both curves) of the complete intersection $(d - 1, d - 1)$ linking $T$ to $\Sigma$ is a minimal generator (resp. both curves are minimal generators) of $\mathcal{I}_T$.

On the other hand, by minimality of the resolution (2) no term $\mathcal{O}(b_j - 2d + 2)$ can cancel. □

***Remark 5*** Cancellations can occur. Let $C = X \cup L$, where $X$ is a smooth curve of degree $d-1$, $d \geq 3$, and where $L$ is a line intersecting $X$ transversally. Clearly $\Sigma$ is a set of $d - 1$ points on the line $L$. The minimal free resolution of $\mathcal{I}_\Sigma$ is: $0 \to \mathcal{O}(-d) \to \mathcal{O}(-1) \oplus \mathcal{O}(1 - d) \to$

$\mathcal{I}_\Sigma \to 0$. Comparing with (4) we see that $m = 3$ and that two terms $\mathcal{O}(1-d)$ did cancel. So we have $d_1 = d-2$, $d_2 = d_3 = d-1$.

See Remark 9 for another example.

**Corollary 6** *If $m \geq 5$, $\Sigma$ can't be a complete intersection.*

***Proof*** Indeed $\Sigma$ is a complete intersection if and only if the minimal free resolution of $\mathcal{I}_\Sigma$ starts with two generators. According to Proposition 4 we have certainly $m-2$ minimal generators of degrees $2d-2-b_j$ in the minimal free resolution of $\mathcal{I}_\Sigma$. □

Before to go on we recall a basic fact about zero-dimensional subscheme of $\mathbb{P}^2$:

**Lemma 7** *Let $X \subset \mathbb{P}^2$ be a zero-dimensional subscheme with minimal free resolution:*

$$0 \to \bigoplus_1^t \mathcal{O}(-b_j) \xrightarrow{M} \bigoplus_1^{t+1} \mathcal{O}(-a_i) \to \mathcal{I}_X \to 0 \tag{8}$$

*Then $a_i \geq t, \forall i$.*

*In particular if $h^0(\mathcal{I}_X(n)) \neq 0$, then $H^0_*(\mathcal{I}_X)$ can be generated by $n+1$ elements.*

***Proof*** This should be well known (see for example [10], Corollary 3.9), but for the convenience of the reader we give a proof. We work by induction on $t$. The case $t = 1$ is clear. Assume the statement for $t-1$. Let $a_1 \leq \cdots \leq a_{t+1}$. Since $\mathcal{I}_X(a_{t+1})$ is generated by global sections we can always perform a liaison of type $(a_1, a_{t+1})$. By mapping-cone the linked scheme, $T$, has the following resolution:

$$0 \to \bigoplus_2^t \mathcal{O}(a_i - a_1 - a_{t+1}) \to \bigoplus_1^t \mathcal{O}(b_j - a_1 - a_{t+1}) \to \mathcal{I}_T \to 0$$

This resolution is minimal and by the inductive assumption we get: $a_1 + a_{t+1} - b_j \geq t-1$, hence $a_1 \geq b_j - a_{t+1} + t - 1$. We have $b_j - a_{t+1} \geq 0, \forall j$ (they are the degrees of the elements of the last row of the matrix $M$). If $b_j - a_{t+1} = 0, \forall j$, then, by minimality, the last row of $M$ is zero. By the Hilbert–Buch Theorem (see [10], Theorem 3.2) the maximal minors of $M$ yield a minimal set of generators of the ideal $I(X) := H^0_*(\mathcal{I}_X)$. If $M$ has a row of zeroes, we get only one non-zero generator, this is impossible. It follows that $a_1 \geq t$. □

**Theorem 8** (i) *With notations as in Sect. 2, if $d \geq 3$, then $2d-4 \geq d_i, \forall i$.*
(ii) *Moreover, if $d > 3$, we have equality (i.e. $d_m = 2d-4$) if and only if $\tau = 1$.*
(iii) *We have $d_m = d-1$ (hence $d_i \leq d-1, \forall i$) or $d_i \leq 2d-m, \forall i$.*

***Proof*** (i) This is clear if $d_i = d-1$, so we may assume that the term $\mathcal{O}(d_i - 2d + 2)$ really appears in (7) even after possible cancellations. This implies $2d-2-d_i \geq 2$.

(ii) We have $min\{2d - d_i - 2\} = 2d - d_m - 2$. Assume $2d - d_m - 2 = 2$. For $d > 3$, the term $\mathcal{O}(d_m - 2d + 2) \simeq \mathcal{O}(-2)$ really appears in the minimal free resolution of $\mathcal{I}_\Sigma$. This implies that there are two generators of degree one, hence $\Sigma$ is a point.

Conversely if $\Sigma$ is a point, let $T$ be linked to $\Sigma$ by a complete intersection $(d-1, d-1)$. Then using the minimal free resolution of $\mathcal{I}_\Sigma$, by mapping-cone, we have: $0 \to 2.\mathcal{O}(-2d+3) \to 2.\mathcal{O}(1-d) \oplus \mathcal{O}(-2d+4) \to \mathcal{I}_T \to 0$. But using instead the resolution (1) we see that $T$ is a section of $E(d-1)$, so we have $0 \to \mathcal{O}(1-d) \to E \to \mathcal{I}_T \to 0$. Using the above resolution of $\mathcal{I}_T$, we get after some diagram-chasing: $0 \to 2.\mathcal{O}(-2d+3) \to 3.\mathcal{O}(1-d) \oplus \mathcal{O}(-2d+4) \to E \to 0$. This resolution is clearly minimal. It follows that $m = 4$ and $d_m = 2d-4$.

(iii) Assume $d_m \neq d-1$, then, according to Proposition 4, the term $\mathcal{O}(d_m - 2d + 2)$ appears in the minimal free resolution of $\mathcal{I}_\Sigma$. Let $2d - 4 - u = d_m$. We have $u \geq 0$ by (i). Since there is a relation of degree $u + 2$, there are at least two minimal generators of degree $\leq u + 1$ in the minimal free resolution of $\mathcal{I}_\Sigma$. So $h^0(\mathcal{I}_\Sigma(u+1)) \neq 0$ and $\mathcal{I}_\Sigma$ can be generated by $u + 2$ elements (Lemma 7). This implies (see 7) that $m - 3 \leq u + 1$, hence $d_m \leq 2d - m$. □

***Remark 9*** (i) Point (i) was known by different methods (see [4,7]).

(ii) The proof of (iii) above shows the following: if $d \neq 4$ and if $d_m = 2d - 5$, then $\tau \leq 4$ or $h^0(\mathcal{I}_\Sigma(1)) = 0$ but $\Sigma$ contains a subscheme of length $\tau - 1$ lying on a line.

(iii) If $\Sigma = \{p\}$, then for any $d \geq 3$ we can present $\Sigma$ as a q.c.i. of type $(d-1, d-1, d-1)$ and, clearly, the term $3.\mathcal{O}(1-d)$ will cancel in (7).

***Example 10*** We can have $m = 4$ and $\Sigma$ a complete intersection, so the bound of Corollary 6 is sharp.

From the point of view of the jacobian ideal to get a curve $C$ with $\tau = 1$ we may argue as follows. Let $\mathbb{P}$ denote the blowing-up of $\mathbb{P}^2$ at a point. We have $\mathbb{P} = \mathbb{F}_1 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ (see for ex. [2]). Denote by $h$, $f$ the classes of $\mathcal{O}_{\mathbb{F}_1}(1)$ and of a fiber in $Pic(\mathbb{F}_1)$. We have $h^2 = 1 = hf$, $f^2 = 0$. The exceptional divisor is $E = h - f$. For any $a \geq 1$, the linear system $|ah + 2f|$ contains a smooth irreducible curve, $C'$, such that $C'.E = 2$. The image of $C'$ in $\mathbb{P}^2$ is a curve, $C$, of degree $a + 2$ with $\tau(C) = 1$ (for $a = 1$ $C$ is a nodal cubic).

Other examples with $m = 4$ and $\Sigma$ complete intersection can be obtained by taking $C = A \cup B$ where $A, B$ are smooth curves, of degrees $a, b$, intersecting transversally. We have $d = a + b$, $\tau = ab$ and $\Sigma$ is a complete intersection $(a, b)$. Assume $a \geq 2$ then, arguing as above, we get $d_1 = d - 2$, $d_2 = d_3 = d_4 = d - 1$, $b_1 = d + a - 2$, $b_2 = d + b - 2$ and the corresponding resolution of $H^0_*(E)$ is minimal.

Another consequence of Lemma 7:

**Corollary 11** *With notations as in Sect. 2 (in particular $m \geq 3$, see Remark 2) we have:*

(i) $d_1 + d_i \geq d + m - 3, \forall i \geq 2$

(ii) *$Z$ is a complete intersection if and only if $m = 3$. In that case $Z$ is a complete intersection of type $(d_1 + d_2 - d + 1, d_1 + d_3 - d + 1)$.*

***Proof*** (i) This follows from (6) and Lemma 7.

(ii) Follows from (iii) of Lemma 3.

□

***Remark 12*** Part (i) is proved also in [7] and (ii) is Prop. 3.1. of [6]. The proofs are different.

If $m = 3$ and $d_1 + d_2 = d$, following [6] one says that $C$ is a *plus one generated curve*. We see that $C$ is a plus one generated curve if and only if $Z$ (of degree $d_3 - d_2 + 1$) is contained in a line. We recover the fact that $C$ is *nearly free* (i.e. $Z$ is a point) if, moreover, $d_3 = d_2$.

## 5 Around the extremal cases in du Plessis–Wall's theorem

We recall the bound given by du Plessis–Wall ([9], see [11] for a different proof, valid also for q.c.i.): $(d-1)(d-1-d_1) \leq \tau \leq (d-1)(d-1-d_1) + d_1^2$.

**Theorem 13** *With notations as in Sect. 2 (in particular $m \geq 3$).*

(i) *We have* $d_1 \leq d_2 \leq d_3 \leq d-1$.
(ii) *We have* $d+1 \geq m$.
(iii) *We have* $(d-1)(d-1-d_1) = \tau$ *if and only if* $\Sigma$ *is a complete intersection of type* $(d-1, d-1-d_1)$. *In this case* $m=3$ *and* $d_2 = d_3 = d-1$.
(iv) *Assume* $\tau = (d-1)(d-1-d_1)+1$. *If* $\tau > 1$, *then* $m = 4$ *and* $\{d_i\} = \{d_1, d-1, d-1, d-3+d_1\}$ *or* $d_1 = 1$, $m = 2$ *and* $E$ *splits like* $\mathcal{O}(-1) \oplus \mathcal{O}(d-2)$.

***Proof*** (i) Let us denote by $g_1, g_2, g_3$ the generators of degrees $d_1, d_2, d_3$ of $H^0_*(E)$. We will coonsider the $g_i$'s as relations among the partials.

Consider the Koszul relations: $K_z = (f_y, -f_x, 0)$, $K_y = (f_z, 0, -f_x)$, $K_x = (0, f_z, -f_y)$. We have:

$$f_z K_z - f_y K_y + f_x K_x = 0 \tag{9}$$

The relations $K_x, K_y, K_z$ correspond to sections $s_x, s_y, s_z$ of $E(d-1)$. It follows that $d_1 \leq d-1$. We also clearly have $d_2 \leq d-1$. Indeed otherwise $K_x, K_y, K_z$ are multiple of $g_1 = (u_1, v_1, w_1)$, which is impossible ($P(u_1, v_1, w_1) = (f_y, -f_x, 0)$ implies $w_1 = 0$ and going on this way we get $g_1 = 0$). If $d_3 \geq d$, these sections are combinations of $g_1, g_2$ only. Now (9) yields a relation involving only $g_1$ and $g_2$. We claim that this relation is non trivial.

Indeed let $s_x = ag_1 + bg_2$, $s_y = a'g_1 + b'g_2$, $s_z = a''g_1 + b''g_2$. Then (9) becomes: $g_1(af_x - a'f_y + a''f_z) + g_2(bf_x - b'f_y + b''f_z) = 0$. Assume $af_x - a'f_y + a''f_z = 0$ and $bf_x - b'f_y + b''f_z = 0$. Then $\alpha = (a, -a', a'')$ determines a section of $E(d-1-d_1)$ and $\beta = (b, -b', b'')$ a section of $E(d-1-d_2)$. Since $d-1-d_2 \leq d_1 - 1$ (Corollary 11), we get $\beta = 0$, hence $b = b' = b'' = 0$. Since $d-1-d_1 \leq d_2 - 1$ (Corollary 11), we see that $\alpha$ is a multiple of $g_1$: $(a, -a', a'') = P(u_1, v_1, w_1)$. It follows that $a = Pu_1$. Moreover $s_x = (0, f_z, -f_y) = ag_1 = (Pu_1^2, Pu_1v_1, Pu_1w_1)$ and it follows that $Pu_1 = 0 = a$, hence $s_x = 0$, which is impossible.

So we have a non trivial relation $Ag_1 = Bg_2$. We may assume $(A, B) = 1$ (otherwise just divide by the common factors). It follows that $B$ divides every components $u_1, v_1, w_1$ of $g_1$ and we get a relation $(u_1', v_1', w_1')$ of degree $< d_1$, against the minimality of $d_1$. We conclude that $d_3 \leq d-1$.

(ii) From (i) we have $2d-2 \geq d_1 + d_3$. We conclude with Corollary 11.

(iii) Assume $\tau = (d-1)(d-1-d_1)$. Since $\mathcal{I}_\Sigma(d-1)$ is generated by global sections we can link $\Sigma$ to a subscheme $\Gamma$ by a complete intersection $F \cap G$ of type $(d-1, d-1)$. Clearly $\deg(\Gamma) = (d-1)^2 - \tau = d_1(d-1)$. By mapping cone we have (after simplifications): $0 \to \mathcal{O} \to E(d-1) \to \mathcal{I}_\Gamma(d-1) \to 0$. Twisting by $1-d+d_1$ we get: $0 \to \mathcal{O}(1-d+d_1) \to E(d_1) \to \mathcal{I}_\Gamma(d_1) \to 0$. Since $\tau > 0$, $d_1 < d-1$, hence $h^0(\mathcal{I}_\Gamma(d_1)) \neq 0$. It follows that $\Gamma$ is contained in a complete intersection $(d_1, d-1)$. Indeed the base locus of the linear system of curves of degree $d-1$ containing $\Gamma$ has dimension zero (consider $F \cap G$) and $d_1 < d-1$. For degree reasons $\Gamma$ is a complete intersection $(d_1, d-1)$ and we have $0 \to \mathcal{O}(1-d-d_1) \to \mathcal{O}(-d_1) \oplus \mathcal{O}(1-d) \to \mathcal{I}_\Gamma \to 0$. By mapping cone again: $0 \to \mathcal{O}(1-d) \oplus \mathcal{O}(d_1 - 2d + 2) \to \mathcal{O}(d_1 + 1 - d) \oplus 2.\mathcal{O}(1-d) \to \mathcal{I}_\Sigma \to 0$. We claim that we can cancel the repeated term $\mathcal{O}(1-d)$. Indeed, since $\dim(F \cap G) = 0$, we may assume that $F$ or $G$ is not a multiple of $S$, the curve of degree $d_1$ containing $\Gamma$, hence $F$ or $G$ is a minimal generator of $H^0_*(\mathcal{I}_\Gamma)$. It follows that $\Sigma$ is a complete intersection. We conclude with Proposition 4.

Conversely if $\Sigma$ is a complete intersection $(d-1, d-1-d_1)$, from Proposition 4 we get $m = 3$ and $d_2 = d_3 = d-1$.

(iv) We argue as above. The assumption $\tau > 1$ makes sure that $h^0(\mathcal{I}_\Gamma(d_1)) \neq 0$. This time we find that $\Gamma$ is linked to one point by a complete intersection $(d-1, d_1)$. By mapping cone we get: $0 \to 2.\mathcal{O}(-d-d_1+2) \to \mathcal{O}(-d-d_1+3) \oplus \mathcal{O}(-d_1) \oplus \mathcal{O}(-d+1) \to \mathcal{I}_\Gamma \to 0$.

This resolution is minimal except if $d_1 = 1$ in which case we have: $0 \to \mathcal{O}(1-d) \to \mathcal{O}(2-d) \oplus \mathcal{O}(-1) \to \mathcal{I}_\Gamma \to 0$. As we have seen above $\Gamma = (s)_0$ where $s \in H^0(E(d-1))$. If $s$ is a minimal generator of $H^0_*(E)$, then $H^0_*(\mathcal{I}_Z)$ has $m-1$ minimal generators, otherwise it has $m$ minimal generators. So if $d_1 > 1, 3 \leq m \leq 4$. By mapping cone we go back to $\Sigma$. If $d_1 > 1$ we get: $0 \to \mathcal{O}(-d+d_1-1) \oplus \mathcal{O}(-2d+2+d_1) \to 2.\mathcal{O}(-d+d_1) \oplus \mathcal{O}(1-d) \to \mathcal{I}_\Sigma \to 0$. From Proposition 4 we conclude that $m = 4$ and $\{d_i\} = \{d_1, d-1, d-1, d-3+d_1\}$. If $d_1 = 1$, by mapping cone we get $0 \to \mathcal{O}(-2d+3) \oplus \mathcal{O}(-d) \to 3.\mathcal{O}(1-d) \to \mathcal{I}_\Sigma \to 0$. This resolution is minimal. Hence $m = 2$ and $E$ splits like $\mathcal{O}(-d+2) \oplus \mathcal{O}(-1)$. □

***Remark 14*** See [6] for a different proof of part (i). Point (ii) is proved in [7].

Since the minimal free resolution of sets of points of low degree are known (see for example [12] for a list), the analysis above can be extended to the cases $\tau = (d-1)(d-1-d_1) + x$, for small $x$.

It is easy to show that if $\tau$ reaches the upper-bound in the first part of du Plessis–Wall's Theorem, then $E$ splits (because $c_2(E(d_1)) = 0$ and $h^0(E(d_1)) \neq 0$) i.e. $\Sigma$ is an almost complete intersection (or $C$ is a *free* curve). However there is a second part in du Plessis–Wall's theorem: under the assumption $2d_1+1 > d$ (which amounts to say that $E$ is stable), we have a better upper-bound: $\tau \leq \tau_+ := (d-1)(d-1-d_1)+d_1^2 - \frac{1}{2}(2d_1+1-d)(2d_1+2-d)$. Notice that this holds true also for q.c.i. [11].

In [7] Thm. 3.1, the authors prove that this bound is reached if and only if we have:

$$0 \to (m-2).\mathcal{O}(-d_1-1) \to m.\mathcal{O}(-d_1) \to E \to 0 \tag{10}$$

with $m = 2d_1 - d + 3$.

This can be proved as follows. From the exact sequence (3) we have $h^0(\mathcal{I}_Z(2d_1-d)) = 0$ (observe that $Z \neq \emptyset$ because $2r+1 > d$). It follows that $\deg(Z) \geq h^0(\mathcal{O}(2d_1-d))$. The assumption $\tau = \tau_+$ implies [use (4)] that we have equality: $\deg(Z) = h^0(\mathcal{O}(2d_1-d))$. This implies $h^1(\mathcal{I}_Z(2d_1-d)) = 0$. It follows (Castelnuovo–Mumford's lemma or numerical character) that the minimal free resolution of $\mathcal{I}_Z$ is: $0 \to s.\mathcal{O}(-s-1) \to (s+1).\mathcal{O}(-s) \to \mathcal{I}_Z \to 0$, with $s = 2d_1 - d + 1$. We conclude with Lemma 3.

Conversely if we have (10), by Lemma 3 we get that $\mathcal{I}_Z$ has a linear resolution and $\deg(Z) = h^0(\mathcal{O}(2d_1-d))$. This implies $\tau = \tau_+$.

Then the authors ask ([7] Conjecture 1.2) if for any integer $d \geq 3$ and for any integer $r$, $d/2 \leq r \leq d-1$, there exists $\Sigma$ with $d_1 = r$ and $\tau = \tau_+$. I don't know the answer in general but, in the framework of q.c.i., the answer is yes:

**Proposition 15** *With notations as above, for every $d \geq 3$ and for every integer $r$, $d/2 \leq r \leq d-1$, there exists a q.c.i. subscheme $\Sigma \subset \mathbb{P}^2$, of degree $\tau_+$, with $d_1 = r$*

***Proof*** We recall that a general set of $s(s+1)/2$ points has a linear resolution:

$$0 \to s.\mathcal{O}(-s-1) \to (s+1).\mathcal{O}(-s) \to \mathcal{I}_Z \to 0 \tag{11}$$

Actually to have such a resolution is equivalent to have $h^0(\mathcal{I}_Z(s-1)) = 0$. Since the Cayley–Bachararch condition CB$(s-3)$ (see for instance [3]) is obviously satisfied we may associate a rank two vector bundle to $\mathcal{I}_Z(s)$: $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_Z(s) \to 0$. We have $c_1(\mathcal{E}) = s$ and $c_2(\mathcal{E}) = s(s+1)/2 = \deg(Z)$. Since $h^1(\mathcal{O}) = 0$ and $\mathcal{I}_Z(s)$ and $\mathcal{O}$ are globally generated, $\mathcal{E}$ also is globally generated. For $a \geq 0$ let us consider a section of $\mathcal{E}(a)$: $0 \to \mathcal{O} \to \mathcal{E}(a) \to \mathcal{I}_\Gamma(2a+s) \to 0$. For $k \geq a+s$, $\mathcal{I}_\Gamma(k)$ is globally generated and we can link $\Gamma$ to $\Sigma$ by a complete intersection of type $(k,k)$. By mapping cone we get, if $k = 2a+s$:

$$0 \to \mathcal{E}(-3a-2s) \to 3.\mathcal{O}(-2a-s) \to \mathcal{I}_\Sigma \to 0 \tag{12}$$

We have $c_2(\mathcal{E}(a)) = as + s(s+1)/2 + a^2 = \deg(\Gamma)$. It follows that $\tau := \deg(\Sigma) = 3a^2 + 3as + s(s-1)/2$. Since $d_1 = a + s$ ($E := \mathcal{E}(-a-s)$), it is easy to check that $\tau = \tau_+$.

Let $d$ be an integer. Assume $d$ odd, $d = 2\delta + 1$. For $1 \le \rho \le \delta$, set $a = \delta - \rho$, $s = 2\rho$, $d_1 = a + s$ and $d = 2a + s + 1$. Then the construction above yields $\Sigma$ of degree $\tau_+$, q.c.i. of three curves of degree $d - 1$, with $d_1 = a + s$. We have $\delta + 1 \le d_1 \le 2\delta$.

If $d = 2\delta$, for $0 \le \rho \le \delta - 1$, set $a = \delta - \rho - 1$ and $s = 2\rho + 1$ ($d_1 = a + s$). □

***Remark 16*** It is not clear at all that there are examples with $\Sigma$ a jacobian set. For some partial results see [7], Section 4.

More generally to characterize the zero-dimensional subschemes that are jacobian sets seems quite a challenge.

It is possible to give a little improvement, namely:

**Proposition 17** *Assume $2d_1 + 1 > d$ and $\tau = \tau_+ - 1$. Set $s := 2d_1 - d$. Then we have two possibilities:*

(a) *The minimal free resolution of $\mathcal{I}_Z$ is:*

$$0 \to \mathcal{O}(-s-2) \oplus (s-2).\mathcal{O}(-s-1) \to s.\mathcal{O}(-s) \to \mathcal{I}_Z \to 0 \qquad (13)$$

*In this case $m = 2d_1 - d + 1$ and $d_i = d_1, \forall i$.*

(b) *The minimal free resolution of $\mathcal{I}_Z$ is:*

$$0 \to \mathcal{O}(-s-2) \oplus (s-1).\mathcal{O}(-s-1) \to \mathcal{O}(-s-1) \oplus s.\mathcal{O}(-s) \to \mathcal{I}_Z \to 0 \qquad (14)$$

*In this case $m = 2d_1 - d + 2$ and $d_i = d_1, 2 \le i < m, d_m = d_1 + 1$.*

***Proof*** Arguing exactly as above this time we have $\deg Z = h^0(\mathcal{O}(s-1)) + 1, h^0(\mathcal{I}_Z(s-1)) = 0$, hence $h^1(\mathcal{I}_Z(s-1)) = 1$. Let $0 \to \bigoplus^t \mathcal{O}(-\beta_j) \to \bigoplus^{t+1} \mathcal{O}(-\alpha_i) \to \mathcal{I}_Z \to 0$ denote the minimal free resolution of $\mathcal{I}_Z$. Since $\beta^+ > \alpha^+$ ($\beta^+ = max\{\beta_j\}$ and the same for $\alpha^+$) and since $\beta^+ - 3 = max\{k \mid h^1(\mathcal{I}_Z(k)) \neq 0\}$, we see that $\beta^+ = s + 2$ (with coefficient equal to 1 because $h^1(\mathcal{I}_Z(s-1)) = 1$). It follows that $H^0_*(\mathcal{I}_Z)$ is generated in degrees $\le s + 1$. Of course we have $s$ minimal generators of degree $s$ and in general nothing else (it is easy to produce examples for any $s$). We conclude that in this case the resolution is like in (a).

What about generators of degree $s + 1$? If there at least two such generators, then the matrix of the resolution has two rows of the form $(L, 0, \dots, 0)$. By erasing another row, we get a maximal minor which is zero, but this is impossible (the maximal minors are the generators). So there is at most one generator of degree $s + 1$. In this case the resolution is like in (b). Examples exist for any $s$: take $s + 1$ points on a line and the remaining ones in general position. □

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