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Strong convergence results for convex minimization and monotone variational inclusion problems in Hilbert space

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Abstract
In this paper, we propose a new modification of the Gradient Projection Algorithm and the Forward–Backward Algorithm. Using our proposed algorithms, we establish two strong convergence theorems for solving convex minimization problem, monotone variational inclusion problem and fixed point problem for demicontractive mappings in a real Hilbert space. Furthermore, we apply our results to solve split feasibility and optimal control problems. We also give two numerical examples of our algorithm in real Euclidean space of dimension 4 and in an infinite dimensional Hilbert space, to show the efficiency and advantage of our results.

Keywords Minimization problem · Monotone inclusion problem · Fixed point problem · Inverse strongly monotone · Maximal monotone operators

Mathematics Subject Classification 47H06 · 47H09 · 47J05 · 47J25

1 Introduction

In this paper, we assume that H is a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$.

Let C be a nonempty, closed and convex subset of H. A point $x \in C$ is said to be a fixed point of the mapping $T : C \to C$, if $Tx = x$. We denote by $F(T)$ the fixed points set of T.

Definition 1.1 A mapping $T : C \to C$ is said to be

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(i) *contractive*, if there exists $k \in (0, 1)$ such that

$$||Tx - Ty|| \leq k||x - y|| \quad \forall x, y \in C,$$

if $k = 1$, then T is said to be nonexpansive,

(ii) *quasi-nonexpansive,* if

$$\|Tx - Tp\| \leq \|x - p\| \;\; \forall x \in C, \quad p \in F(T),$$

(iii) *demicontractive,* (or ψ-demicontractive) if there exists $0 < \psi < 1$ such that

$$\|Tx - Tp\|^2 \leq \|x - p\|^2 + \psi\|x - Tx\|^2 \;\; \forall x \in C, \quad p \in F(T),$$

(iv) *monotone,* if

$$\langle x - y, Tx - Ty\rangle \geq 0 \quad \forall\, x, y \in C,$$

(v) *β-strongly monotone,* if there exists $\beta > 0$ such that

$$\langle x - y, Tx - Ty\rangle \geq \beta\|x - y\|^2 \quad \forall\, x, y \in C,$$

(vi) *ν-inverse strongly monotone (for short ν-ism)*, if there exists $\nu > 0$ such that

$$\langle x - y, Tx - Ty\rangle \geq \nu\|Tx - Ty\|^2 \quad \forall\, x, y \in C.$$

Definition 1.2 A mapping $T : H \to H$ is said to be firmly nonexpansive if and only if $2T - I$ is nonexpansive or equivalently

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2 \quad \forall x, y \in H.$$

Alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive. In this connection, see [1, Proposition 11.2].

It can easily be seen that if T is nonexpansive, then $I - T$ is monotone where I is an identity operator. Inverse strong monotone (also referred to as co-coercive) operators have been widely used to solve practical problem in various fields, for instance, in traffic assignment problems (see, for example [2,3] and the references therein).

Remark 1.3 The class of demicontractive mappings is of central importance in optimization since it contains many common types of operators arising in optimization (see [4–7] and the references therein). More precisely, the class of demicontractive mappings contains the class of quasi-nonexpansive mappings (which contains the class of nonexpansive mappings with nonempty fixed point sets) and is more desirable for fixed point methods in image recovery (see [5,7–9]). More so, the class of demicontractive mappings contains the class of firmly nonexpansive mappings which in turn contains the class of metric resolvent and projections, known as important tools for solving optimization problems (see [6,10,11] and the references therein).

Let $x \in H$, there exists a unique point $P_C x \in C$ such that

$$\|x - P_C x\| \leq \|x - y\| \quad \forall y \in C,$$

where P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping from H onto C. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \quad \forall x, y \in H. \tag{1.1}$$

Furthermore, $P_C x$ is characterized by the properties $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y \rangle \geq 0 \quad \forall y \in C. \tag{1.2}$$

In a real Hilbert space, metric projections are examples of firmly nonexpansive mappings. For more information on metric projections, see [1] and the references therein.

Definition 1.4 A mapping $T : H \to H$ is said to be an averaged mapping, if it can be written as the average of the identity mapping I and a nonexpansive mapping; that is

$$T = (1 - \alpha)I + \alpha S, \tag{1.3}$$

where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive. More precisely, when (1.3) holds, we say that T is α-averaged. Thus, firmly nonexpansive mappings (in particular, projections) are $\frac{1}{2}$-averaged mappings. The term "averaged mapping" was coined by Baillon–Bruck–Reich [12].

Consider the following constrained convex minimization problem:

$$\min_{x \in C} g(x), \tag{1.4}$$

where C is a closed convex subset of H and $g : C \to \mathbb{R}$ is a real-valued convex function. We say that the minimization problem (1.4) is consistent, if it has a solution. In the sequel, we shall denote the set of solutions of problem (1.4) by Υ. If g is Fréchet differentiable functional, then the Gradient-Projection Algorithm (GPA) generates a sequence $\{x_n\}$ according to the recursive formula

$$x_{n+1} = Proj_C(I - \lambda \nabla g)(x_n), \quad n \geq 0; \tag{1.5}$$

or more generally

$$x_{n+1} = Proj_C(I - \lambda_n \nabla g)(x_n), \quad n \geq 0; \tag{1.6}$$

where in both (1.5) and (1.6), the initial guess x_1 is taken from C arbitrarily, and the parameters, λ and λ_n, are positive real numbers. The convergence of algorithms (1.5) and (1.6) depends on the behaviour of the gradient ∇g. As a matter of fact, it is known that if ∇g is α-strongly monotone and L-Lipschizian, that is,

$$\langle \nabla g(x) - \nabla g(y), x - y \rangle \geq \alpha \|x - y\|^2 \quad \forall x, y \in C \tag{1.7}$$

and

$$\|\nabla g(x) - \nabla g(y)\| \leq L\|x - y\| \quad \forall x, y \in C, \tag{1.8}$$

then for $0 < \gamma < \frac{2\alpha}{L^2}$, the operator

$$T := P_C(I - \lambda \nabla f) \tag{1.9}$$

is a contraction. Hence, the sequence $\{x_n\}$ defined by the algorithm (1.5) converges in norm to the unique solution of the minimization problem (1.4). More generally, if the sequence $\{\lambda_n\}$ is chosen to satisfy the property

$$0 < \liminf \lambda_n \leq \limsup \lambda_n < \frac{2\alpha}{L^2}, \tag{1.10}$$

then the sequence $\{x_n\}$ defined by the algorithm (1.6) converges in norm to the unique minimizer of (1.4). However, if the gradient ∇g fails to be strongly monotone, then the operator T defined by (1.9) would fail to be contractive. Consequently, the sequence $\{x_n\}$

generated by algorithm (1.6) may fail to converge strongly (see [13]). If ∇g is Lipschizian, then algorithm (1.5) and (1.6) can still converge in the weak topology under certain conditions.

Xu [13] gave an alternative operator-oriented approach to algorithm (1.6); namely an average mapping approach. He gave his averaged mapping approach to the gradient-projection algorithm (1.6) and the relaxed gradient-projection algorithm. Moreover, he constructed a counter example which shows that algorithm (1.5) does not converge in norm in an infinite-dimensional space, and he also presented two modification of gradient-projection algorithm which are shown to have strong convergence. Furthermore, he regularized the minimization problem (1.4) to devise an iterative scheme that generates a sequence converging in norm to the minimum-norm solution of (1.4) in the consistent case.

Recently, Cai and Shehu [14] introduced the following iterative algorithm for finding a fixed point of a strictly pseudocontractive mapping which is also a solution of a constrained convex minimization problem for a convex and continuously Fréchet differentiable functional g in a real Hilbert space and prove the strong convergence of the sequence generated by their scheme in a real Hilbert space.

Theorem 1.5 [14] *Let C be a nonempty, closed and convex subset of real Hilbert space H. Suppose that the minimization problem (1.4) is consistent and let Υ denote its solution set. Assume that the gradient ∇g is L-Lipschitzian with constant $L > 0$. Let T be a k-strictly pseudo-contractive mapping on C into itself such that $F(T) \cap \Gamma \neq \emptyset$. Let $\{t_n\}$ be a sequence in $(0, 1)$, $\{\alpha_n\}$ in $(0, (1-k)(1-t_n)) \subset (0, 1)$, and $\{\lambda_n\}$ a sequence in $(0, \frac{2}{L})$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} t_n = 0$;
(ii) $\sum_{n=1}^{\infty} t_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1-k$;
(iv) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{L}$.

Then the sequences $\{u_n\}$ and $\{x_n\}$ generated for fixed $u \in C$ by $u_1, x_1 \in C$

$$\begin{cases} x_n = P_C(u_n - \lambda_n \nabla f(u_n)), \\ u_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n - t_n(x_n - u), \ n \geq 1, \end{cases} \tag{1.11}$$

converges strongly to $x^* \in F(T) \cap \Gamma$, where $x^* = P_{F(T)\cap\Upsilon} u$.

Remark 1.6 Having searched the literature, we observe that, to prove strong convergence results for the GPA problem and other related optimization problems, the CQ (modified Haugazeau) algorithms are often used. In some other cases (where algorithms other than the CQ algorithms are used), some compactness conditions are assumed on the operators under consideration, or the proof maybe divided into two cases which may result to a very long proof.

Motivated by the above works and Remark 1.6, we propose a new modification of the GPA and the FBA by adopting the idea of algorithm (1.11). Using our proposed algorithms, we establish two strong convergence theorems for solving convex minimization problem, monotone variational inclusion problem and fixed point problem for demicontractive mappings in a real Hilbert space. Furthermore, we apply our results to solve split feasibility and optimal control problems. We also give two numerical examples of our algorithm in real Euclidean space of dimension 4 and in an infinite dimensional Hilbert space, to show the efficiency and advantage of our results.

2 Preliminaries

Lemma 2.1 [15,16] *Let H be a real Hilbert space. Then the following hold:*

(i) $||x+y||^2 \leq ||y||^2 + 2\langle x, x+y\rangle$ *for all* $x, y \in H$.
(ii) $||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x-y||^2$ *for all* $x, y \in H$ *and* $\alpha \in (0, 1)$.
(iii) $||x+y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2$.

Lemma 2.2 ([17]) *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at* 0*, (i.e., if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$).*

Lemma 2.3 [13,18] *Let C be a nonempty subset of H. Then, the following statements hold:*

i *If $T : C \to H$ is α-averaged, then for any $z \in Fix(T)$ and for all $x \in C$,*

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \frac{1-\alpha}{\alpha}\|Tx - x\|^2.$$

ii *If $T_1 : H \to H$ and $T_2 : H \to H$ are α_1 and α_2-averaged respectively. Then T_1T_2 is $(\alpha_1 + \alpha_2 - \alpha_1\alpha_2)$-averaged.*

Lemma 2.4 [19] *Let $\{a_n\}$ be a sequence of non-negative number such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n r_n,$$

where $\{r_n\}$ is a sequence of real numbers bounded from above and $\{\alpha_n\} \subset [0, 1]$ satisfies $\sum \alpha_n = \infty$. Then

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} r_n.$$

3 Main Results

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a real Hilbert space H and f be a contraction mapping on C with coefficient $k \in (0, 1)$. Let $T : C \to C$ be a ψ-demicontractive mapping with $\psi \in [0, 1)$. Suppose that the minimization problem (1.4) is consistent and Υ denotes its solution set such that $\Theta := F(T) \cap \Upsilon \neq \emptyset$. Assume that the gradient ∇g is L-Lipschitzian with constant $L > 0$. Let the sequence $\{x_n\}$ be generated for fixed $x_1 \in C$ by*

$$\begin{cases} x_{n+1} = (1-\beta_n)y_n + \beta_n z_n; \\ z_n = (1-t_n)y_n + t_n T_\kappa y_n; \\ y_n = P_C(I - \lambda_n \nabla g)w_n; \\ w_n = (1-\alpha_n)x_n + \alpha_n f(x_n), \quad n \geq 1; \end{cases} \tag{3.1}$$

where $T_\kappa = \kappa I + (1-\kappa)T$, $\kappa \in [\psi, 1)$ such that T is demiclosed at 0*, $\{\alpha_n\}$, $\{t_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, \frac{2}{L})$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{L}$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iv) $0 < \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n < 1$.

Then $\{x_n\}$ *converges strongly to* $z \in \Theta$*, where* $z = P_\Theta f(z)$.

Proof It is well known that $z \in C$ solves the minimization problem (1.4) if and only if z solves the fixed point equation:

$$z = P_C(I - \lambda \nabla g)z,$$

where $\lambda > 0$ is any fixed positive number. We may assume that (due to condition (ii))

$$0 < a \leq \lambda_n \leq b < \frac{2}{L}, \quad n \geq 1,$$

where a and b are constant. Furthermore it is well known that the gradient ∇g is $\frac{1}{L}$ -ism, $(I - \lambda_n \nabla g)$ is nonexpansive and that $P_C(I - \lambda \nabla g)$ is $\frac{2+\lambda L}{4}$-averaged for $0 < \lambda < \frac{2}{L}$. Hence, we find that for each n, $P_C(I - \lambda_n \nabla g)$ is $\frac{2+\lambda_n L}{4}$-averaged. Therefore we can write

$$P_C(I - \lambda_n \nabla g) = \frac{2 - \lambda L}{4} + \frac{2 + \lambda_n L}{4} S_n = (1 - \mu_n)I + \mu_n S_n, \tag{3.2}$$

where S_n is nonexpansive for each $n \geq 1$, $\mu_n = \frac{2+\lambda_n L}{4} \in [a_1, b_1] \subset (0, 1)$, $a_1 = \frac{2+aL}{4}$ and $b_1 = \frac{2+bL}{4} < 1$. Let $y_n = P_C(I - \lambda_n \nabla g)w_n$. Then by (3.2), we obtain

$$y_n = P_C(I - \lambda_n \nabla g)w_n = (1 - \mu_n)w_n + \mu_n S_n w_n. \tag{3.3}$$

Firstly, we show that if T is ψ- demicontractive, T_κ is quasi-nonexpansive. Let $x \in C,\ y \in F(T)$ and $0 < \psi < \kappa \leq 1$, then we have

$$\begin{aligned}
\|T_\kappa x - y\|^2 &\leq \kappa \|x - y\|^2 + (1 - \kappa)\|Tx - y\|^2 - \kappa(1 - \kappa)\|x - Tx\|^2 \\
&\leq \kappa \|x - y\|^2 + (1 - \kappa)\left[\|x - y\|^2 + \psi \|x - Tx\|^2\right] - \kappa(1 - \kappa)\|x - Tx\|^2 \\
&= \|x - y\|^2 + (1 - \kappa)(\psi - \kappa)\|x - Tx\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

Hence, T_κ is quasi-nonexpansive. Furthermore, we know that $F(T_\kappa) = F(T)$.

Next, we show that $\{x_n\}$ is bounded. Let $z \in \Theta$, from (3.3), we obtain

$$\begin{aligned}
\|z_n - z\|^2 &= \|(1 - \mu_n)w_n + \mu_n S_n w_n - z\|^2 \\
&= (1 - \mu_n)\|w_n - z\|^2 + \mu_n \|S_n w_n - z\|^2 - \mu_n(1 - \mu_n)\|w_n - S_n w_n\|^2 \\
&\leq \|w_n - z\|^2 - \mu_n(1 - \mu_n)\|w_n - S_n w_n\|^2.
\end{aligned} \tag{3.4}$$

Since T_κ is quasi-nonexpansive, we obtain from (3.1) that

$$\begin{aligned}
\|y_n - z\|^2 &= \|(1 - t_n)y_n + t_n T_\kappa y_n - z\|^2 \\
&\leq (1 - t_n)\|y_n - z\|^2 + t_n \|T_\kappa y_n - z\|^2 - t_n(1 - t_n)\|y_n - T_\kappa y_n\|^2 \\
&\leq (1 - t_n)\|y_n - z\|^2 + t_n \|y_n - z\|^2 - t_n(1 - t_n)\|y_n - T_\kappa y_n\|^2 \\
&\leq \|y_n - z\|^2.
\end{aligned} \tag{3.5}$$

Now, we obtain from (3.1) and Lemma 2.1 (ii) that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|(1 - \beta_n)y_n + \beta_n z_n - z\|^2 \\
&= (1 - \beta_n)\|y_n - z\|^2 + \|z_n - z\|^2 - \beta_n(1 - \beta_n)\|y_n - z_n\|^2.
\end{aligned} \tag{3.6}$$

But from (3.1), we have

$$z_n - y_n = \frac{1}{\beta_n}(x_{n+1} - y_n). \tag{3.7}$$

Similarly, we have from (3.3) that

$$S_n w_n - w_n = \frac{1}{\mu_n}(y_n - w_n). \tag{3.8}$$

Therefore, from (3.6), (3.7) and (3.8), we obtain

$$\begin{aligned} \|x_{n+1} - z\|^2 &\le \|y_n - z\|^2 - \frac{1}{\beta_n}(1-\beta_n)\|x_{n+1} - y_n\|^2 \\ &\le \|w_n - z\|^2 - \frac{1}{\mu_n}(1-\mu_n)\|y_n - w_n\|^2 - \frac{1}{\beta_n}(1-\beta_n)\|x_{n+1} - y_n\|^2. \end{aligned} \tag{3.9}$$

Using (3.1) and (3.9), we get

$$\begin{aligned} \|x_{n+1} - z\| &\le \|w_n - z\| \\ &= \|\alpha_n(f(x_n) - f(z)) + \alpha_n(f(z_n) - z) + (1-\alpha_n)(x_n - z)\| \\ &\le \alpha_n k\|x_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)(x_n - z)\| \\ &= (1 - \alpha_n(1-k))\|x_n - z\| + \alpha_n\|f(z) - z\| \\ &\le \max\left\{\|x_n - z\|, \frac{\|f(z) - z\|}{1-k}\right\} \\ &\ \ \vdots \\ &\le \max\left\{\|x_1 - z\|, \frac{\|f(z) - z\|}{1-k}\right\}, \end{aligned}$$

which shows that $\{x_n\}$ is bounded and consequently, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$.

Furthermore, we have from (3.7) that

$$\begin{aligned} \|z_n - y_n\|^2 &= \left\|\frac{1}{\beta_n}(x_{n+1} - y_n)\right\|^2 \\ &= \frac{1}{\beta_n^2}\|x_{n+1} - y_n\|^2 \\ &= \frac{\alpha_n}{\beta_n}\left(\frac{\|x_{n+1} - y_n\|^2}{\alpha_n\beta_n}\right). \end{aligned} \tag{3.10}$$

Also, from (3.8), we obtain

$$\|w_n - S_n w_n\|^2 = \frac{\alpha_n}{\mu_n}\left(\frac{\|y_n - w_n\|^2}{\alpha_n\mu_n}\right). \tag{3.11}$$

Using Lemma 2.1 and (3.1) (noting that $\alpha_n \in (0, 1)$), we have

$$\begin{aligned} \|w_n - z\|^2 &= \|\alpha_n(f(x_n) - z) + (1-\alpha_n)(x_n - z)\|^2 \\ &\le \alpha_n^2\|f(x_n) - z\|^2 + 2\alpha_n(1-\alpha_n)\langle f(x_n) - z, x_n - z\rangle + (1-\alpha_n)^2\|x_n - z\|^2 \\ &\le (1-\alpha_n)^2\|x_n - z\|^2 + \alpha_n^2\|f(x_n) - z\|^2 + 2k\alpha_n(1-\alpha_n)\|x_n - z\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle f(z) - z, x_n - z\rangle \\ &\le (1 - 2\alpha_n(1-k))||x_n - z||^2 + \alpha_n^2||x_n - z||^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle f(z) - z, x_n - z)\rangle + \alpha_n^2||f(x_n) - z||^2. \end{aligned} \tag{3.12}$$

Putting (3.12) in (3.9), we obtain

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le (1-2\alpha_n(1-k))||x_n-z||^2+\alpha_n^2||x_n-z||^2\\&\quad+2\alpha_n(1-\alpha_n)\langle f(z)-z,x_n-z)\rangle+\alpha_n^2||f(x_n)-z||^2\\&\quad-\frac{1}{\mu_n}(1-\mu_n)\|y_n-w_n\|^2-\frac{1}{\beta_n}(1-\beta_n)\|x_{n+1}-y_n\|^2\\&=(1-2\alpha_n(1-k))||x_n-z||^2\\&\quad-2\alpha_n(1-k)\left(-\frac{\alpha_n}{2(1-k)}\left[||x_n-z||^2+||f(x_n)-z||^2\right]\right.\\&\quad+\frac{(1-\alpha_n)}{(1-k)}\langle f(z)-z,z-x_n\rangle+\frac{1}{2\alpha_n\mu_n(1-k)}(1-\mu_n)\|y_n-w_n\|^2\\&\quad\left.+\frac{1}{2\alpha_n\beta_n(1-k)}(1-\beta_n)\|x_{n+1}-y_n\|^2\right).\end{aligned}\tag{3.13}$$

Let

$$\begin{aligned}\Gamma_n:=&-\frac{\alpha_n}{2(1-k)}\left[||x_n-z||^2+||f(x_n)-z||^2\right]+\frac{(1-\alpha_n)}{(1-k)}\langle f(z)-z,z-x_n\rangle\\&+\frac{1}{2\alpha_n\mu_n(1-k)}(1-\mu_n)\|y_n-w_n\|^2+\frac{1}{2\alpha_n\beta_n(1-k)}(1-\beta_n)\|x_{n+1}-y_n\|^2.\end{aligned}\tag{3.14}$$

Then, (3.13) becomes

$$\|x_{n+1}-z\|^2\le(1-2\alpha_n(1-k))\|x_n-z\|^2-2\alpha_n(1-k)\Gamma_n.\tag{3.15}$$

Since $\{x_n\}$ is bounded an so it is bounded below. Hence, Γ_n is bounded below. Furthermore, using Lemma 2.4 and condition (i) of Theorem 3.1 in (3.15), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\|x_n-z\|^2&\le\limsup_{n\to\infty}(-\Gamma_n)\\&=-\liminf_{n\to\infty}\Gamma_n.\end{aligned}\tag{3.16}$$

Therefore, $\liminf_{n\to\infty}\Gamma_n$ is a finite. We have from (3.14) that

$$\begin{aligned}\liminf_{n\to\infty}\Gamma_n=\liminf_{n\to\infty}&\left((1-k)^{-1}\langle f(z)-z,z-x_n\rangle+\frac{(1-k)^{-1}}{2\alpha_n\beta_n}(1-\beta_n)\|x_{n+1}-y_n\|^2\right.\\&\left.+\frac{(1-k)^{-1}}{2\alpha_n\mu_n}(1-\mu_n)\|y_n-w_n\|^2\right).\end{aligned}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup q\in H$ and

$$\begin{aligned}\liminf_{n\to\infty}\Gamma_n=\lim_{k\to\infty}&\left((1-k)^{-1}\langle f(z)-z,z-x_{n_k}\rangle+\frac{(1-k)^{-1}}{2\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k})\|x_{n_k+1}-y_{n_k}\|^2\right.\\&\left.+\frac{(1-k)^{-1}}{2\alpha_{n_k}\mu_{n_k}}(1-\mu_{n_k})\|y_{n_k}-w_{n_k}\|^2\right).\end{aligned}\tag{3.17}$$

Since $\{x_n\}$ is bounded and $\liminf\limits_{n\to\infty}\Gamma_n$ is finite, we have that $\left\{\frac{1}{\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k})\|x_{n_k+1}-y_{n_k}\|^2\right\}$ and $\left\{\frac{1}{\alpha_{n_k}\mu_{n_k}}(1-\mu_{n_k})\|y_{n_k}-w_{n_k}\|^2\right\}$ are bounded. Also, by assumption (iii), we have that there exists $b\in(0,1)$ such that $\beta_n\le b<1$ and this implies that $\frac{1}{\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k})\ge$

$\frac{1}{\alpha_{n_k}\beta_{n_k}}(1-b) > 0$ and so we have that $\left\{\frac{1}{\alpha_{n_k}\beta_{n_k}}\|x_{n_k+1}-y_{n_k}\|^2\right\}$ is bounded. Similarly, we obtain that $\frac{1}{\alpha_{n_k}\mu_{n_k}}(1-\mu_{n_k}) \geq \frac{1}{\alpha_{n_k}\mu_{n_k}}(1-b_1) > 0$ and $\left\{\frac{1}{\alpha_{n_k}\mu_{n_k}}\|y_{n_k}-w_{n_k}\|^2\right\}$ is bounded. Observe from assumptions (i) and (iii) that there exists $a \in (0,1)$ such that

$$0 < \frac{\alpha_{n_k}}{\beta_{n_k}} \leq \frac{\alpha_{n_k}}{a} \to 0,\ k \to \infty.$$

This implies that $\frac{\alpha_{n_k}}{\beta_{n_k}} \to 0,\ k \to \infty$. Therefore, we obtain from (3.10) and $\frac{\alpha_{n_k}}{\beta_{n_k}} \to 0,\ k \to \infty$ that

$$\lim_{k\to\infty}\|z_{n_k}-y_{n_k}\| = 0. \tag{3.18}$$

From (3.1) and (3.18), we obtain

$$\|T_\kappa y_{n_k}-y_{n_k}\| = \frac{1}{t_n}\|y_{n_k}-z_{n_k}\| \to 0,\ k \to \infty. \tag{3.19}$$

Following the same argument as in above, we obtain that

$$\frac{\alpha_{n_k}}{\mu_{n_k}} \leq \frac{\alpha_{n_k}}{a_1} \to 0,\ k \to \infty$$

and that

$$\lim_{k\to\infty}\|S_{n_k}w_{n_k}-w_{n_k}\| = 0. \tag{3.20}$$

From (3.7) and (3.18), we have that

$$\|x_{n_k+1}-y_{n_k}\| = \beta_{n_k}\|z_{n_k}-y_{n_k}\| \to 0,\ k \to \infty.$$

Furthermore, from (3.1) and assumption (i), we obtain

$$\|w_{n_k}-x_{n_k}\| = \alpha_{n_k}\|f(x_{n_k})-x_{n_k}\| \to 0,\ k \to \infty. \tag{3.21}$$

Also, from (3.4) and (3.20), we obtain

$$\|y_{n_k}-w_{n_k}\| = \|P_C(I-\lambda\nabla g)w_{n_k}-w_{n_k}\| = \mu_{n_k}\|S_{n_k}w_{n_k}-w_{n_k}\| \to 0,\quad k \to \infty. \tag{3.22}$$

So, we get that

$$\|y_{n_k}-x_{n_k}\| \leq \|y_{n_k}-w_{n_k}\| + \|w_{n_k}-x_{n_k}\| \to 0,\ k \to \infty.$$

Hence,

$$\|x_{n_k+1}-x_{n_k}\| \leq \|x_{n_k+1}-y_{n_k}\| + \|y_{n_k}-x_{n_k}\| \to 0,\ k \to \infty.$$

Observe that $w_{n_k} \rightharpoonup x^* \in C,\ k \to \infty$ since $w_{n_k} - x_{n_k} \to 0,\ k \to \infty$ and $x_{n_k} \rightharpoonup x^* \in C,\ k \to \infty$. We may assume that $\lambda_{n_k} \rightharpoonup \lambda$; then we have $0 < \lambda < \frac{2}{L}$. Set $S := P_C(I-\lambda\nabla g)$, then S is nonexpansive and we get from (3.18) that

$$\begin{aligned}\|P_C(I-\lambda\nabla g)w_{n_k}-w_{n_k}\| &\leq \|P_C(I-\lambda\nabla g)w_{n_k}-P_C(I-\lambda_{n_k}\nabla g)w_{n_k}\|\\ &\quad+\|P_C(I-\lambda_{n_k}\nabla g)w_{n_k}-w_{n_k}\|\\ &\leq \|(I-\lambda\nabla g)w_{n_k}-(I-\lambda_{n_k}\nabla g)w_{n_k}\|\\ &\quad+\|P_C(I-\lambda_{n_k}\nabla g)w_{n_k}-w_{n_k}\|\\ &= |\lambda_{n_k}-\lambda|\|\nabla g(w_{n_k})\| + \|P_C(I-\lambda_{n_k}\nabla g)w_{n_k}-w_{n_k}\| \to 0.\end{aligned}$$

It then follows from Lemma 2.2 that $x^* \in F(S)$. But $F(S) = \Upsilon$. Therefore we have that $x^* \in \Upsilon$.

Moreover, since $\{x_{n_k}\}$ converges weakly to $x^* \in C$, and that $y_{n_k} - x_{n_k} \to 0,\ k \to \infty$, then there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ that converges weakly to $x^* \in C$. Hence by (3.19) and the demicloseness of T_κ at the origin, we obtain that $x^* \in F(T_\kappa) = F(T)$. Hence $x^* \in \Theta$. Now, we obtain from (3.17) and the property of P_Θ that

$$\begin{aligned}
\liminf_{n\to\infty} \Gamma_n &= \lim_{k\to\infty} \Bigg((1-k)^{-1}\langle f(z) - z, z - x_{n_k}\rangle + \frac{(1-k)^{-1}}{2\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k})\|x_{n_k+1} - y_{n_k}\|^2 \\
&\quad + \frac{(1-k)^{-1}}{2\alpha_{n_k}\mu_{n_k}}(1-\mu_{n_k})\|y_{n_k} - w_{n_k}\|^2 \Bigg) \\
&\geq (1-k)^{-1} \lim_{k\to\infty} \langle f(z) - z, z - x_{n_k}\rangle \\
&= (1-k)^{-1}\langle f(z) - z, z - x^*\rangle \geq 0.
\end{aligned} \tag{3.23}$$

Thus from (3.16), we have that

$$\limsup_{n\to\infty} \|x_n - z\|^2 \leq -\liminf_{n\to\infty} \Gamma_n \leq 0.$$

Therefore, $\lim_{n\to\infty} \|x_n - z\| = 0$ and this implies that $\{x_n\}$ converges strongly to z. □

If T is a strictly pseudocontractive mapping, then we obtain the following result.

Corollary 3.2 *Let C be a nonempty, closed and convex subset of a real Hilbert space H and f be a contraction mapping on C with coefficient $k \in (0, 1)$. Let $T : C \to C$ be a ψ-strictly pseudocontractive mapping with $\psi \in [0, 1)$. Suppose that the minimization problem (1.4) is consistent and Υ denote its solution set such that $\Theta := F(T) \cap \Upsilon \neq \emptyset$. Assume that the gradient ∇g is L-Lipschitzian with constant $L > 0$. Let the sequence $\{x_n\}$ be generated for fixed $x_1 \in C$ by*

$$\begin{cases} x_{n+1} = (1-\beta_n)y_n + \beta_n z_n; \\ z_n = (1-t_n)y_n + t_n T_\kappa y_n; \\ y_n = P_C(I - \lambda_n \nabla g)w_n; \\ w_n = (1-\alpha_n)x_n + \alpha_n f(x_n), \quad n \geq 1; \end{cases} \tag{3.24}$$

where $T_\kappa = \kappa I + (1-\kappa)T,\ \ \kappa \in [\psi, 1)$, $\{\alpha_n\}$, $\{t_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, \frac{2}{L})$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{L}$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iv) $0 < \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n < 1$.

Then $\{x_n\}$ converges strongly to $z \in \Theta$, where $z = P_\Theta f(z)$.

By setting $f(x) = u\ \forall x \in C$ in Theorem 3.1, we obtain the following result.

Corollary 3.3 *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to C$ be a ψ- demicontractive mapping with $\psi \in [0, 1)$. Suppose that the minimization problem (1.4) is consistent and Υ denote its solution set such that $\Theta := F(T) \cap \Upsilon \neq \emptyset$.*

Assume that the gradient ∇g is L-Lipschitzian with constant $L > 0$. Let the sequence $\{x_n\}$ be generated for fixed $x_1,\ u \in C$ by

$$\begin{cases} x_{n+1} = (1-\beta_n)y_n + \beta_n z_n; \\ z_n = (1-t_n)y_n + t_n T_\kappa y_n; \\ y_n = P_C(I - \lambda_n \nabla g)w_n; \\ w_n = (1-\alpha_n)x_n + \alpha_n f(x_n), \quad n \geq 1; \end{cases} \tag{3.25}$$

where $T_\kappa = \kappa I + (1-\kappa)T,\ \ \kappa \in [\psi, 1)$ such that T is demiclosed at 0, $\{\alpha_n\}$, $\{t_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, \frac{2}{L})$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{L}$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iv) $0 < \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n < 1$.

Then $\{x_n\}$ converges strongly to $z \in \Theta$, where $z = P_\Theta u$.

We next investigate the problem of finding a zero of the sum of two monotone operators, which is formulated as the following monotone variational inclusion problem: Find $x \in H$ such that

$$0 \in (A+B)x, \tag{3.26}$$

where $A : H \to H$ and $B : H \to 2^H$ are two monotone operators in Hilbert space H.

Lemma 3.4 *Let H be a real Hilbert space, then the following well-known identity holds:*

$$||x+y||^2 = ||x||^2 + ||y||^2 + 2\langle x, y\rangle \quad \forall x, y \in H.$$

Lemma 3.5 *Let C be a nonempty subset of H, $\nu \in \mathbb{R}^+$, $T : C \to H$ be ν-ism and $\gamma \in (0, 2\nu)$. Then $I - \gamma T$ is $\gamma/2\nu$-averaged.*

Proof Set $N = I - 2\nu T$. Since T is ν-ism, we obtain from Lemma (3.4) that

$$\begin{aligned} \|Nx - Ny\|^2 &= \|(I-2\nu T)x - (I-2\nu T)y\|^2 \\ &= \|(x-y) - 2\nu(Tx - Ty)\|^2 \\ &= \|x-y\|^2 + 4\nu^2\|Tx-Ty\|^2 - 4\nu\langle x-y, Tx-Ty\rangle \\ &\leq \|x-y\|^2 + 4\nu^2\|Tx-Ty\|^2 - 4\nu^2\|Tx-Ty\| \\ &= \|x-y\|^2. \end{aligned}$$

□

Hence, N is nonexpansive. Thus, we obtain that

$$I - \gamma T = (1-\gamma/2\nu)I + (\gamma/2\nu)I - \gamma T = (1-\gamma/2\nu)I + (\gamma/2\nu)N.$$

Since $\gamma \in (0, 2\nu)$, then $\gamma/2\nu \in (0, 1)$, thus we have that $I - \gamma T$ is $\gamma/2\nu$-averaged.

We shall assume that problem (3.26) is consistent, namely its solution set, denoted by Θ is nonempty. We now introduce an iterative algorithm that converges strongly to a solution of (3.26). More accurately, our algorithm starts with an arbitrary initial guess $x_0 \in H$, and generates x_{n+1} according to the recursion process

$$\begin{cases} x_{n+1} = (1-\beta_n)y_n + \beta_n z_n; \\ z_n = (1-t_n)y_n + t_n T_\kappa y_n; \\ y_n = J_{\gamma_n B}(I - \gamma_n A)w_n; \\ w_n = (1-\alpha_n)x_n + \alpha_n f(x_n), \quad n \geq 1, \end{cases} \tag{3.27}$$

where $T_\kappa = \kappa I + (1-\kappa)T$, $\kappa \in [\psi, 1)$ such that T is demiclosed at 0. Noting that $\Theta = F(J_{\gamma_n B}(I - \gamma_n A))$ i.e $x \in \Theta$, if and only if

$$\begin{aligned} 0 \in (A+B)x = Ax + Bx &\Leftrightarrow x - \gamma Ax \in x + \gamma Bx \\ &\Leftrightarrow x \in (I + \gamma B)^{-1}(I - \gamma A)x \\ &\Leftrightarrow x = J_{\gamma B}(I - \gamma A)x. \end{aligned}$$

Let $z \in \Theta$, from Lemma 3.5, we obtain that $I - \gamma_n A$ is $\gamma_n A/2\nu$-averaged for every $n \in \mathbb{N}$. Since $J_{\gamma_n B}$ is nonexpansive, then it is $\frac{1}{2}$-averaged. It follows from Lemma 2.3 (ii) that $J_{\gamma_n B}(I - \gamma_n A)$ is $(2\nu + \gamma_n)/4\nu$ - averaged. Let $\mu_n = \frac{2\nu+\gamma_n}{4\nu}$, in view of $\gamma_n \in (0, 2\nu)$, we have that $\mu_n \in (0, 1)$. So $J_{\gamma_n B}(I - \gamma_n A)$ is μ_n-averaged. Hence it follows from Definition 1.2 that

$$J_{\gamma_n B}(I - \gamma_n A) = (1-\mu_n)I + \mu_n T_n, \tag{3.28}$$

where T_n is nonexpansive for every $n \geq 1$. By (3.28), we obtain that

$$F(J_{\gamma_n B}(I - \gamma_n A)) = F(T_n).$$

Hence we obtain the following strong convergence theorem for finding a zero of the sum of two monotone operators.

Theorem 3.6 *Let $A : H \to H$ be a ν-inverse strongly monotone mapping and $B : H \to 2^H$ be a maximal monotone mapping. Let f be a contraction with constant $k \in (0, 1)$ and T be a ψ-demicontractive mapping with $\psi \in [0, 1)$. Let $F(T) \cap \Theta \neq \emptyset$, $\{\gamma_n\}$, $\{\alpha_n\}$, $\{t_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 2\nu$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iv) $0 < \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n < 1$.

Then, the sequence $\{x_n\}$ generated by (3.27) converges strongly to $F(T) \cap \Theta$.

Proof By replacing $J_{\gamma_n B}(I - \gamma_n A)$ with $P_C(I - \lambda_n \nabla g)$ in Algorithm 3.1, and following similar proof as in the proof of Theorem 3.1, we get that $\{x_n\}$ generated by (3.27) converges strongly to $z \in F(T) \cap \Theta$, where $z = P_{F(T)\cap\Theta} f(z)$. □

Remark 3.7 We now point out some differences between the presentation of our method of proof of Theorem 3.1 and that of Theorem 2.4 of [14] viz:

(i) The major key in proving Theorem 3.1 is to show that $\limsup_{n\to\infty}(-\Gamma_n) \leq 0$ as given in (3.23) and using Lemma 2.4 in (3.15).
(ii) In our convergence analysis, we did not make use of Lemma 2.3 of [14], which was used in the convergence analysis of proof of Theorem 2.4 in [14]; rather we used Lemma 2.4 of this paper.
(iii) If we replace "$f(x_n)$" by "u" (for arbitrary $u \in C$) in Algorithm 3.1 (which is of viscosity-type), then Algorithm 3.1 becomes of Halpern-type (see Corollary 3.3), and the conclusion of Theorem 3.1 will still hold. However, we use a viscosity-type algorithm instead of an Halpern-type due to the fact that viscosity-type algorithms have higher rate of convergence than Halpern-type. Moreover, it was established in [20,21] that Halpern-type convergence theorems imply viscosity convergence theorems for weak contractions.

(iv) Observe from the characterization of metric projection that,

$$z = P_\Theta f(z) \Longleftrightarrow \langle f(z) - z, z - y\rangle \geq 0 \quad \forall y \in C. \tag{3.29}$$

Therefore, one advantage of adopting Algorithm (3.1) for our convergence analysis, is that it also converges strongly to a solution of the variational inequality (3.29) (see for example [22]).

(v) As stated in Remark 1.6, in establishing strong convergence results for the GPA problems and other related optimization problems, the CQ algorithms (modified Haugazeau or an Halpern-CQ modifications) are often used. However, these algorithms require at each step of the iteration process, the computation of two subsets C_n and Q_n, the computation of their intersection $C_n \cap Q_n$ and the computation of the projection of the initial starting point onto this intersection; thus, leading to an increase in the computational cost of the iteration. Therefore, algorithms that does not involve the constructions of C_n and Q_n (as in our case) are more interesting and of practical computational importance since they are easy to compute than those that involve these computations.

Based on the above remark, Corollaries 3.2 and 3.3, our results improve and extend the results of Cai and Shehu [14], and many other important results in this direction.

4 Applications

In this section, we give applications of Theorem 3.1 to solve split feasibility and optimal control problems.

4.1 Split feasibility problem

The Split Feasibility Problem (SFP) was introduced by Censor and Elfving [23] and has gained much attention of several authors due to its applications to image reconstruction, signal processing, and intensity-modulated radiation therapy.

The SFP is finding a point x such that

$$x \in C \ \ and \ \ Bx \in Q, \tag{4.1}$$

where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively and $B : H_1 \to H_2$ is a bounded linear operator.

Clearly x^* is a solution to the SFP (4.1) if and only if $x^* \in C$ and $Bx^* - P_Q Bx^* = 0$. Several iterative methods have been developed for solving the SFP and its related optimization problems (see for example, [24–34]).

The proximity function g is defined by

$$g(x) = \frac{1}{2}\|Bx - P_Q Bx\|^2 \tag{4.2}$$

and we consider the constrained convex minimization problem

$$\min_{x\in C} g(x) = \min_{x\in C} \frac{1}{2}\|Bx - P_Q Bx\|^2. \tag{4.3}$$

Then x^* solves the SFP (4.1) if and only if x^* solves the minimization problem (4.3). In [35], the following CQ algorithm was introduced to solve the SFP,

$$x_{n+1} = P_C(I - \lambda B^*(I - P_Q)B)x_n, \qquad n \geq 0 \tag{4.4}$$

where $0 < \lambda < \frac{2}{\|B\|^2}$ and B^* is the adjoint of B. It was proved that the sequence generated by (4.4) converges weakly to a solution of the SFP.

We now state the following Theorem as an application of Theorem 3.1 to solve the SFP (4.1) and fixed point problem for ψ-demicontractive mapping.

Theorem 4.1 *Let C and Q be nonempty, closed and convex subset of real Hilbert spaces H_1 and H_2 respectively, and $B : H_1 \to H_2$ be bounded linear operator. Let f be a contraction with constant $k \in (0, 1)$ and T be a ψ-demicontractive mapping with $\psi \in [0, 1)$. Let $g(x) = \frac{1}{2}\|Bx - P_Q Bx\|^2$ and let Υ denotes the solution set of problem (4.1) such that $\Upsilon \cap F(T) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated for fixed $x_1 \in C$ by*

$$\begin{cases} x_{n+1} = (1-\beta_n)y_n + \beta_n z_n; \\ z_n = (1-t_n)y_n + t_n T_\kappa y_n; \\ y_n = P_C(I - \lambda_n B^*(I - P_Q)B)w_n; \\ w_n = (1-\alpha_n)x_n + \alpha_n f(x_n), \quad n \geq 1; \end{cases} \tag{4.5}$$

where $T_\kappa = \kappa I + (1-\kappa)T$, $\kappa \in [\psi, 1)$ such that T is demiclosed at 0, $\{\alpha_n\}$, $\{t_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, \frac{2}{\|B\|^2})$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{B}$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iv) $0 < \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n < 1$.

Then $\{x_n\}$ converges strongly to $z \in F(T) \cap \Upsilon$, where $z = P_{F(T)\cap\Upsilon} f(z)$.

Proof By the definition of proximity function, we have that $\nabla g = B^*(I - P_Q)B$ and ∇g is $||B||^2$-Lipschitz continuous. Hence, by setting $\nabla g = B^*(I - P_Q)B$ in Algorithm (3.1), we obtain the desired result. □

4.2 Optimal control problem

Let $L_2([0, \alpha], \mathbb{R}^m)$ be the Hilbert space of square integrable and measurable vector functions u defined from $[0, \alpha]$ into $\mathbb{R}^m$, which is endowed with inner product

$$\langle u, v\rangle = \int_0^\alpha \langle u(t), v(t)\rangle dt$$

and norm

$$||u|| = \sqrt{\langle u, u\rangle}.$$

Now, consider the following optimal control problem:

$$u^*(t) = \operatorname{argmin}\{g(u) : u \in V\}, \tag{4.6}$$

where V is the set of admissible controls in the form of an m-dimensional box and consists of piecewise continuous functions:

$$V = \{u(t) \in L_2([0, \alpha], \mathbb{R}^m) : u_i(t) \in [u_i^-, u_i^+],\ i = 1, 2 \ldots, m\} \text{ (see [37])}.$$

Assuming that such a control exists. The terminal objective has the form:

$$g(u) = \Phi(x(\alpha)),$$

where Φ is a convex and differentiable function (see [36]). If the trajectory $x(t) \in L_2([0, \alpha])$ satisfies constrains in the of a system of linear differential equation:

$$\dot{x}(t) = D(t)x(t) + B(t)u(t), \quad x(0) = x_0, \ t \in [0, \alpha],$$

where $D(t) \in \mathbb{R}^{m\times n}, \ B(t) \in \mathbb{R}^{n\times m}$ are given continuous matrices for every $t \in [0, \alpha]$. Then, by the Pontryagin maximum principle (see [36]), there exists a function $p^* \in L_2([0, \alpha])$ such that (x^*, p^*, u^*) solves for a.e. $t \in [0, \alpha]$, the following system:

$$\begin{cases} \dot{x}^*(t) = D(t)x^*(t) + B(t)u^*(t) \\ x^*(0) = x_0, \end{cases} \tag{4.7}$$

$$\begin{cases} \dot{p}^*(t) = -D(t)^T p^*(t) \\ p^*(\alpha) = \nabla h(x(\alpha)), \end{cases} \tag{4.8}$$

$$0 \in B(t)^T p^*(t) + N_V(u^*(t)), \tag{4.9}$$

where $N_V(u)$ is the normal cone to V at u defined by

$$N_V(u) := \begin{cases} \{w \in H : \langle w, v - u\rangle \le 0, \quad \forall v \in V\}, & \text{if} \quad u \in V \\ \emptyset, & \text{if} \quad u \notin V. \end{cases}$$

Letting $Gu(t) := B(t)^T p(t)$, we have that Gu is the gradient of the objective function g (see [37,38]). More so, (4.9) can be rewritten as

$$\langle Gu^*, v - u^*\rangle \ge 0 \quad \forall v \in V. \tag{4.10}$$

But we know that u^* solves (4.10) if and only if $u^* = P_V(I - \lambda G)u^*$, for any $\lambda > 0$. Therefore, by setting $\nabla g = G$ in Theorem 3.1, we can apply Theorem 3.1 to solve (4.9).

5 Numerical examples

In this section, we present two numerical examples of our algorithm in real Euclidean space of dimension 4 and in an infinite dimensional Hilbert space, to show its efficiency and advantage.

Throughout this section, we shall take $\alpha_n = \frac{1}{3n+1}$, $\beta_n = \frac{n+1}{3n}$, $t_n = \frac{2n+3}{5n+1}$ and $\lambda_n = \frac{n}{25n+3}$.

Example 5.1 Here, we present a numerical example in $\mathbb{R}^4$ to illustrate the performance of our algorithm. Let $H_1 = H_2 = \mathbb{R}^4$ and $\nabla g(x) = B^*(I - P_Q)Bx$, where $Bx = (3x_1 + x_2 + x_3 - x_4, -2x_1 - x_2 - 3x_3 + 2x_4, -4x_1 - x_2 + 5x_3 - 2x_4, x_1 - x_2 + x_3 - x_4)$, $Q = \{x \in \mathbb{R}^4 : \langle w, x\rangle = b\}$, $w = (-1, 2, 4, 7)^T$, $b = 2$, $P_Q(x) = \max\left\{0, \frac{b-\langle w,x\rangle}{||w||^2}\right\} w + x$. Since B is a bounded linear operator and P_Q is a metric projection onto Q, then ∇g is L-Lipschitz continuous with $L = ||B||^2 = 50$. Let $C = \{x \in \mathbb{R}^4 : \langle y, x\rangle \ge a\}$, $y = (2, -5, -7, 1)^T$, $a = 3$, $P_C(x) = \frac{a-\langle y,x\rangle}{||y||^2} y + x$. Define $T(x) = \frac{-3}{2}x$ and $f(x) = \frac{1}{3}x$, then T is a ψ-demicontractive mapping with $\psi = \frac{1}{5}$ and f is a contraction. Thus, we can take $\kappa = \frac{1}{2}$, so that $T_\kappa = \frac{-1}{4}x$. Hence, Algorithm (4.5) becomes

$$\begin{cases} x_{n+1} = (1 - \frac{n+1}{3n})y_n + \frac{n+1}{3n}z_n; \\ z_n = (1 - \frac{2n+3}{5n+1})y_n - \frac{(2n+3)}{4(5n+1)}y_n; \\ y_n = P_C(I - \lambda_n B^*(I - P_Q)B)w_n; \\ w_n = (1 - \frac{1}{3n+1})x_n + \frac{1}{3(3n+1)}x_n, \quad n \ge 1. \end{cases} \tag{5.1}$$

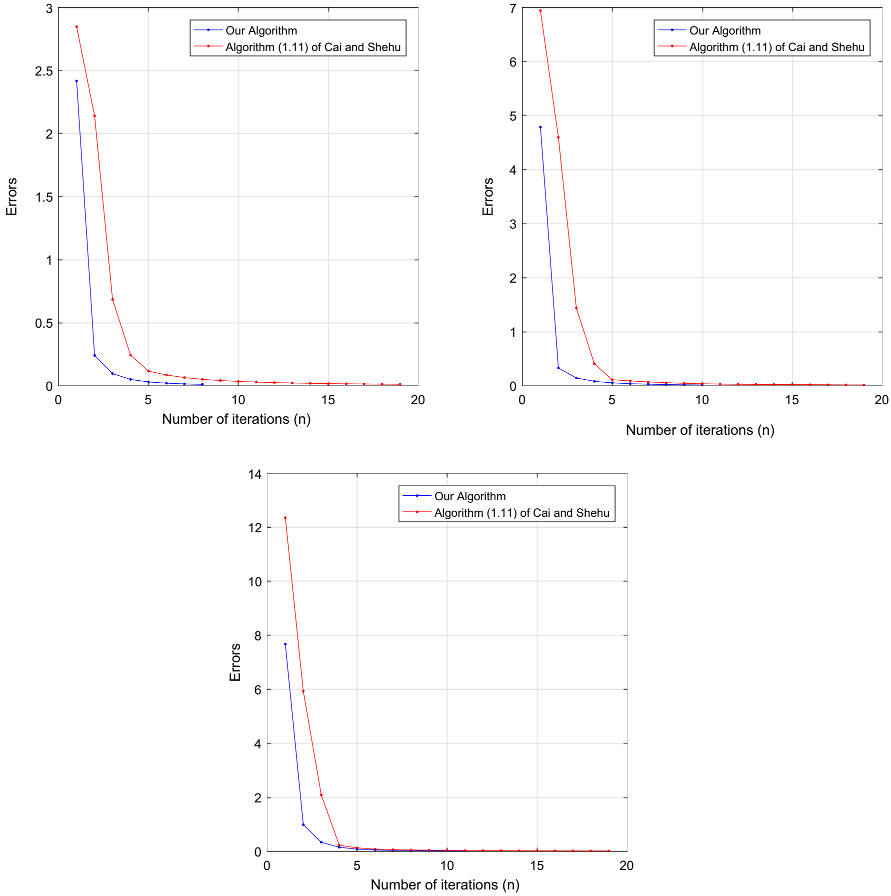

Fig. 1 Errors versus iteration numbers for Example 1: Case 1 (top left); Case 2 (top right); Case 3 (bottom)

We consider the following cases for Example 5.1.
Case I Take $x_1 = (-1, 2, 1, 0.5)^T$
Case II Take $x_1 = (-8, 2, 7, 1)^T$
Case III Take $x_1 = (1, 7, -5, 3)^T$

Example 5.2 We now give an example in an infinite dimensional Hilbert space to further show the efficiency and advantage of our results. Let $H_1 = H_2 = L_2([0, 2\pi])$ be endowed with inner product

$$\langle x, y\rangle = \int_0^{2\pi} x(t)y(t)dt \quad \forall\, x, y \in L_2([0, 2\pi])$$

and norm

$$||x|| := \left(\int_0^{2\pi} |x(t)|^2 dt\right)^{\frac{1}{2}} \quad \forall\, x, y \in L_2([0, 2\pi]).$$

Let $C = \{x \in L_2([0, 2\pi]) : \langle y, x\rangle \leq a\}$, where $y = e^{2t}$ and $a = 3$. Then,

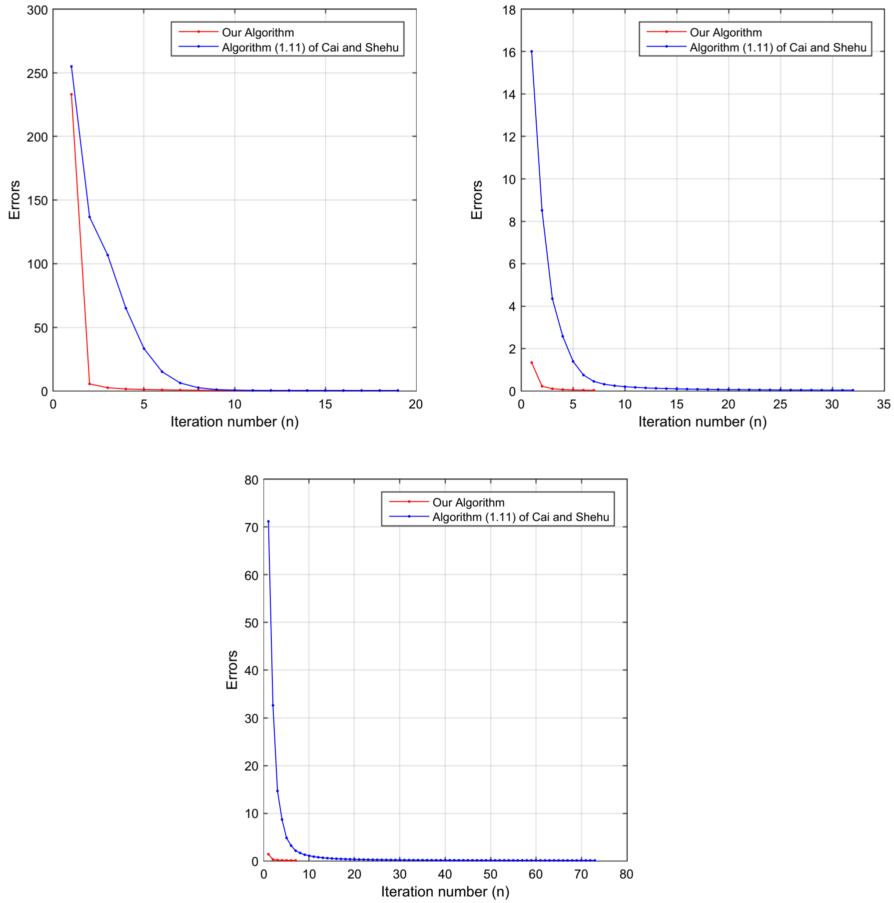

Fig. 2 Errors versus iteration numbers for Example 2: Case 1 (top left); Case 2 (top right); Case 3 (bottom)

$$P_C(x) = \begin{cases} \frac{a-\langle y,x\rangle}{||y||_{L_2}^2} y + x, & \text{if } \langle y, x\rangle > a, \\ x, & \text{if } \langle y, x\rangle \leq a. \end{cases}$$

Again, let $Q = \{x \in L_2([0, 2\pi]) : ||x - d||_{L_2} \leq r\}$, where $d = \sin(t)$ and $r = 16$. Then,

$$P_Q(x) = \begin{cases} d + r\frac{x-d}{||x-d||_{L_2}}, & \text{if } ||x - d||_{L_2} > r, \\ x, & \text{if } ||x - d||_{L_2} \leq r. \end{cases}$$

Now, let $B, f, T : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ be defined by $Bx(t) = x(t)$, $fx(t) = \frac{x(t)}{3}$ and $Tx(t) = \frac{-5}{2}x(t)$. Then, B is a bounded linear operator with adjoint $B^*x(t) = x(t)$ and $||B|| = 1$, f is a contraction and T is ψ-demicontractive with $\psi = \frac{21}{49}$.

We now consider the following cases for Example 5.2 (Figs. 1, 2).

Case 1 Take $x_1(t) = t^3$.

Case 2 Take $x_1(t) = \sin t$.

Case 3 Take $x_1(t) = \cos t$.

Remark 5.3 Using Examples 5.1 and 5.2, we compare our algorithm with Algorithm (1.11) of Cai and Shehu [14], by considering in each example, 4 different initial points. As seen from the graphs below, our viscosity-type algorithm converges faster than the Halpern-type algorithm studied by Cai and Shehu [14]. This shows that our algorithm works well and have competitive advantages over the algorithm of Cai and Shehu [14].

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