Nef vector bundles on a projective space with first Chern class three

Masahiro Ohno[1]

Abstract

We classify nef vector bundles on a projective space with first Chern class three over an algebraically closed field of characteristic zero; we see, in particular, that these nef vector bundles are globally generated if the second Chern class is less than eight, and that there exist nef but non-globally generated vector bundles with second Chern class eight and nine on a projective plane.

Keywords Nef vector bundles · Fano bundles · Spectral sequences

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1 Introduction

Let $\mathcal{E}$ be a nef vector bundle of rank r on a projective space $\mathbb{P}^n$ over an algebraically closed field K of characteristic zero. Let c_1 be the first Chern class of $\mathcal{E}$. Then c_1 is non-negative since $\mathcal{E}$ is nef. In [15, Theorem 1], Peternell–Szurek–Wiśniewski classified such $\mathcal{E}$'s in case $c_1 \leq 2$, based on the study [18] of Szurek–Wiśniewski. If $c_1 \leq 2$ and $n \geq 2$, then $\mathbb{P}(\mathcal{E})$ is a Fano manifold, and their proof is based on analysis of contraction morphisms of extremal rays. In [11, §6], a different proof of the classification in case $c_1 \leq 2$ is given, based on analysis of some twist $\mathcal{E}(d)$ of $\mathcal{E}$ with the full strong exceptional sequence $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)$ of line bundles.

In this paper we continue our approach to classify nef vector bundles in the next case $c_1 = 3$. Note here that if $c_1 = 3$ then the anti-canonical bundle of $\mathbb{P}(\mathcal{E})$ is nef if $n \geq 2$ and ample if $n \geq 3$. Moreover if $c_1 = 3$ then $0 \leq c_2 \leq c_1^2 = 9$, where c_2 denotes the second Chern class of $\mathcal{E}$. The main result of this paper is as follows.

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Theorem 1 *Let $\mathcal{E}$ be a nef vector bundle of rank r on a projective space $\mathbb{P}^n$ with $c_1 = 3$. Then c_2 and $\mathcal{E}$ satisfy one of the following:*

(1) $c_2 = 0$ *and* $\mathcal{E} \cong \mathcal{O}(3) \oplus \mathcal{O}^{\oplus r-1}$;
(2) $c_2 = 2$ *and* $\mathcal{E} \cong \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-2}$;
(3) $c_2 = 3$ *and* $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}^{\oplus r-3}$;
(4) $c_2 = 3$, $n = 2$, *and* $\mathcal{E} \cong T_{\mathbb{P}^2} \oplus \mathcal{O}^{\oplus r-2}$;

(In the following, $\mathcal{E}$ fits in one of the following exact sequences.)

(5) $c_2 = 3$ *and* $0 \to \mathcal{O}(-1) \to \mathcal{O}(2) \oplus \mathcal{O}^{\oplus r} \to \mathcal{E} \to 0$;
(6) $c_2 = 4$ *and* $0 \to \mathcal{O}(-1) \to \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-1} \to \mathcal{E} \to 0$;
(7) $c_2 = 4$, $n = 3$, *and* $0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus 4} \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r+2} \to \mathcal{E} \to 0$;
(8) $c_2 = 4$, $n = 4$, *and*

$$0 \to \mathcal{O}(-3) \to \mathcal{O}(-2)^{\oplus 5} \to \mathcal{O}(-1)^{\oplus 10} \to \mathcal{O}^{\oplus r+6} \to \mathcal{E} \to 0;$$

(9) $c_2 = 5$ *and* $0 \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r+1} \to \mathcal{E} \to 0$;
(10) $c_2 = 5$, $n = 3$ *or* 4, *and* $0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus 5} \to \mathcal{O}^{\oplus r+4} \to \mathcal{E} \to 0$;
(11) $c_2 = 6$ *and* $0 \to \mathcal{O}(-1)^{\oplus 3} \to \mathcal{O}^{\oplus r+3} \to \mathcal{E} \to 0$;
(12) $c_2 = 6$ *and* $0 \to \mathcal{O}(-2) \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r} \to \mathcal{E} \to 0$;
(13) $c_2 = 7$ *and* $0 \to \mathcal{O}(-2) \oplus \mathcal{O}(-1) \to \mathcal{O}^{\oplus r+2} \to \mathcal{E} \to 0$;
(14) $c_2 = 8$, $n = 2$, *and* $0 \to \mathcal{O}(-2)^{\oplus 2} \to \mathcal{O}^{\oplus r+1} \oplus \mathcal{O}(-1) \to \mathcal{E} \to 0$;
(15) $c_2 = 9$ *and* $0 \to \mathcal{O}(-3) \to \mathcal{O}^{\oplus r+1} \to \mathcal{E} \to 0$;
(16) $c_2 = 9$, $n = 2$, *and* $0 \to \mathcal{O}(-2)^{\oplus 3} \to \mathcal{O}^{\oplus r} \oplus \mathcal{O}(-1)^{\oplus 3} \to \mathcal{E} \to 0$;
(17) $c_2 = 9$, $n = 2$, *and*

$$0 \to \mathcal{O}(-3)^{\oplus r} \to \mathcal{O}(-2)^{\oplus 3r+3} \to \mathcal{O}(-1)^{\oplus 3r+3} \to \mathcal{E} \to 0;$$

(18) $c_2 = 9$, $n = 2$, *and* $0 \to \mathcal{O}(-2)^{\oplus 4} \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-3} \oplus \mathcal{O}(-1)^{\oplus 6} \to \mathcal{E} \to 0$;
(19) $c_2 = 9$, $n \geq 3$, $c_3 = 27$, *and* $h^0(\mathcal{E}(-1)) = 1$;
(20) $c_2 = 9$, $n \geq 4$, $c_3 = 27$, $h^0(\mathcal{E}(-1)) = 0$, *and* $h^{n-3}(\mathcal{E}(2-n)) = 1$.

Note that every case except the cases (18), (19), and (20) in Theorem 1 is effective: examples of $\mathcal{E}$ in case (7) are $\mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-4} \oplus \Omega_{\mathbb{P}^3}(2)$ and $\mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-3} \oplus \mathcal{N}(1)$ where $\mathcal{N}$ is a null correlation bundle on $\mathbb{P}^3$ (see Remark 8.1); in case (8), $\mathcal{E}$ is given by a locally free resolution in terms of $\mathcal{O}(-3)$-twist of the full strong exceptional sequence $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(4)$ in accordance with [11], but $\mathcal{E}$ is in fact isomorphic to $\Omega_{\mathbb{P}^4}(2) \oplus \mathcal{O}^{\oplus r-4}$ (see Remark 8.2); in case (10), if $n = 4$, $\mathcal{E}$ is nothing but an extension of the Tango bundle [19] (see also [14, Chap. I §4.3]) on $\mathbb{P}^4$ by a trivial bundle $\mathcal{O}^{\oplus r-3}$, so that $\Omega_{\mathbb{P}^4}^2(3)$ is a typical example (see Remark 8.3), and if $n = 3$, the restriction of such bundle to a hyperplane $\mathbb{P}^3$ is an example; the case (14) comes from Proposition 2 below (see [12] for details); an example in case (16) is given in Example 1; the exact sequence in case (17) in fact derives from Example 3. On the other hand, it is uncertain whether nef vector bundles exist or not in case (18), (19), or (20). Note that neither in case (19) nor in case(20) exist nef vector bundles if they does not exist in case (18), since the restriction to a plane of a nef vector bundle in case (19) or (20) lies in the case (18).

Anghel–Manolache [1] and Sierra–Ugaglia [17] classified globally generated vector bundles on a projective space with first Chern class three. Since global generation implies nefness, Theorem 1 is a generalization of their results. We also note that Langer [9] classified smooth Fano 4-folds with adjunction theoretic scroll structure and $b_2 = 2$; his classification includes that of nef and big rank 2 bundles with $c_1 = 3$ on $\mathbb{P}^2$ and $\mathbb{P}^3$.

As in the proof in case $c_1 \leq 2$ in [11], the main feature of our proof of Theorem 1 is an application of the spectral sequence deduced in [13, Theorem 1] from Bondal's theorem [4, Theorem 6.2]. Besides this spectral sequence, some of the key ingredients of our proof are the Riemann–Roch formula, the Kawamata–Viehweg vanishing theorem [7,22], and the non-negativity of Chern classes of nef vector bundles (see, e.g., [10, Theorem 8.2.1]).

Whereas global generation implies nefness, the converse is not true in general; indeed, the (scheme-theoretic) support of the cokernel of the evaluation map $H^0(\mathcal{E}) \otimes \mathcal{O} \to \mathcal{E}$ in cases (14), (16), and (17) in Theorem 1 is, respectively, a point w, a cubic curve E, and the whole $\mathbb{P}^2$. (If there exists an example in case (18), its corresponding support is also the whole $\mathbb{P}^2$. Hence examples in cases (18), (19), or (20) shall be also nef but non-globally generated if they exist.) In fact, the evaluation map in case (14) fits in the exact sequence in the following proposition.

Proposition 2 *Given an integer $r \geq 2$ and a point w in a projective plane $\mathbb{P}^2$, there exists a vector bundle $\mathcal{E}$ fitting in an exact sequence*

$$0 \to \mathcal{O}(-3) \to \mathcal{O}^{\oplus r+1} \to \mathcal{E} \to k(w) \to 0$$

where $k(w)$ denotes the residue field of the point w. Moreover a vector bundle $\mathcal{E}$ fitting in the sequence above is nef but non-globally generated with $c_1 = 3$ and $c_2 = 8$.

In [12, Theorem 1.1], we classified nef vector bundles on a projective space with $c_1 = 3$ and $c_2 = 8$ (based on the previous version of this manuscript): we showed that such bundles exist only on a projective plane and every such bundle derives from the exact sequence given in Proposition 2 and fits in the exact sequence (14) in Theorem 1. Note that the parts of the previous version of the manuscript on which the argument in [12] are relied are kept the same in this major revised manuscript.

This paper is organized as follows. In §2, we recall Bondal's theorem [4, Theorem 6.2] and its related results including the spectral sequence deduced in [13, Theorem 1]. The results in §2 are fundamental throughout this paper. In §3, we begin our proof(s) of Theorem 1 (and of Theorem 3 below); based on results in [11], we reduce the problem to the case where $H^0(\mathcal{E}(-2)) = 0$. We also show in §3 that this reduction enables us to assume that $3 \leq c_2$. Several other formulas—such as the Riemann–Roch formulas—used repeatedly in this paper are also presented in §3. In §4, we collect some lemmas needed later. In §5, we give a key lemma, Lemma 5.1, which together with exact sequence (3.16) in §3.1 is crucial for the whole proof. In §6, we show that nef vector bundles on a projective space with $c_1 = 3$ and $c_2 \leq 7$ are globally generated (Theorem 3). In §7, we give a proof of Theorem 1; we omit the proof in case $c_2 \leq 7$, since we have Theorem 3 and globally generated vector bundles with $c_1 = 3$ are classified in [1] and [17]. Similarly the proof in case $c_2 = 8$ is omitted since it is already given in [12]; thus we only give a proof in case $c_2 = 9$. Note that our presentation of the classification in case $c_2 \leq 7$ is different from both of [1,17]; the situation is the same as in [1,17]: the presentations of the classification in [1,17] are different because their methods of proofs are different; similarly our presentation differs from those of [1,17] because of our new proof. A reader who wonders where the presentation of Theorem 1 in case $c_2 \leq 7$ comes from may find its proof in the previous version (arXiv 1604.05847 version 4) of the manuscript. In §8 we give several remarks related to the cases (7), (8), and (10) and an example of the case (16) in Theorem 1. We end §8 with a question about the cokernel of the evaluation map $H^0(\mathcal{E}) \otimes \mathcal{O} \to \mathcal{E}$ of a nef vector bundle in case (16) of Theorem 1. In §9, we give a proof of Proposition 2. Finally note that, since globally generated vector bundles are nef, some properties of globally generated vector bundles also hold more generally for nef

vector bundles, but some do not; in §10, we give two examples which illustrate that a typical exact sequence for globally generated vector bundles—related to the degeneracy locus of general global sections—does not necessarily exist for nef vector bundles.

1.1 Notation and conventions

Throughout this paper we work over an algebraically closed field K of characteristic zero. Basically we follow the standard notation and terminology in algebraic geometry. For a vector bundle $\mathcal{E}$, $\mathbb{P}(\mathcal{E})$ denotes $\mathrm{Proj} S(\mathcal{E})$, where $S(\mathcal{E})$ denotes the symmetric algebra of $\mathcal{E}$. The tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is also denoted by $H(\mathcal{E})$. We denote by $\mathcal{E}^{\vee}$ the dual of $\mathcal{E}$. For a coherent sheaf $\mathcal{F}$ on a smooth projective variety X, we denote by $c_i(\mathcal{F})$ the ith Chern class of $\mathcal{F}$. In particular, c_i stands for $c_i(\mathcal{E})$ of the nef vector bundle $\mathcal{E}$ we are dealing with. We say that a vector bundle is (non-)globally generated if it is (not) generated by global sections. For coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, $h^q(\mathcal{F})$ denotes $\dim H^q(\mathcal{F})$, and $\hom(\mathcal{F}, \mathcal{G})$ denotes $\dim \mathrm{Hom}(\mathcal{F}, \mathcal{G})$. For any closed subscheme Z in $\mathbb{P}^n$, denote by $\mathcal{I}_Z$ the ideal sheaf of Z in $\mathbb{P}^n$. Finally we refer to [10] for the definition and basic properties of nef vector bundles.

2 Preliminaries

In our proof of Theorem 1, we shall apply repeatedly a spectral sequence deduced in [13, Theorem 1]. So we shall recall this sequence in this section. Note that this sequence is a corollary of Bondal's theorem [4, Theorem 6.2] as can be seen below.

Let X be a smooth projective variety over K, $D^b(X)$ the bounded derived category of the abelian category of coherent sheaves on X. Assume that there exists a full strong exceptional sequence $G_0, \ldots, G_m$ in $D^b(X)$, and let G be the direct sum $\oplus_{j=0}^m G_j$ in $D^b(X)$.

Recall that if $X = \mathbb{P}^n$ then $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)$ is a full strong exceptional sequence in $D^b(\mathbb{P}^n)$ by Beilinson's theorem [3, Theorem].

Let A be the endomorphism ring $\mathrm{End}_{D^b(X)}(G)$ of G, and let e_j be the composite of the projection $G \to G_j$ and the inclusion $G_j \to G$. Then $e_j \in A$. Define a right A-module P_j by $P_j = e_j A$. The natural isomorphism $A \cong \oplus_{j=0}^m P_j$ of right A-modules implies that P_j is a projective right A-module. Let S_j be the simple right A-module such that $S_j e_j \cong K$ and $S_j e_k \cong 0$ for all $k \neq j$ and $0 \leq k \leq m$.

Let $D^b(\mathrm{mod} A)$ be the bounded derived category of the abelian category $\mathrm{mod} A$ of finitely generated right A-modules. Bondal's theorem [4, Theorem 6.2] states that $\mathrm{RHom}(G, \bullet) : D^b(X) \to D^b(\mathrm{mod} A)$ is an exact equivalence. Since $\bullet \otimes^L G : D^b(\mathrm{mod} A) \to D^b(X)$ is a quasi-inverse of $\mathrm{RHom}(G, \bullet)$, we have a natural isomorphism

$$\mathrm{RHom}(G, \bullet) \otimes^L G \cong \bullet$$

of functors on $D^b(X)$. If we write down this isomorphism for a coherent sheaf F on X in terms of a spectral sequence, we obtain the following spectral sequence ([13, Theorem 1])

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\mathrm{Ext}^q(G, F), G) \Rightarrow E^{p+q} = \begin{cases} F & \text{if } p+q=0 \\ 0 & \text{if } p+q \neq 0. \end{cases} \tag{2.1}$$

We call this sequence the Bondal spectral sequence. In practice, in order to apply the Bondal spectral sequence (2.1), we need to compute $E_2^{p,q}$. One way to compute $E_2^{p,q}$ is by definition: $E_2^{p,q} = \mathcal{T}or_{-p}^A(\mathrm{Ext}^q(G, F), G)$, i.e., through a projective resolution of the right A-module $\mathrm{Ext}^q(G, F)$. Recall here (see [11, Lemma 2.1] for a proof) that a finitely generated right

A-module $\mathrm{Ext}^q (G, F)$ has a projective resolution of the form

$$0 \to P_0^{\oplus e_{m,0}} \to \cdots \to \bigoplus_{j=0}^{m-l} P_j^{\oplus e_{l,j}} \to \cdots \to \bigoplus_{j=0}^{m} P_j^{\oplus e_{0,j}} \to \mathrm{Ext}^q (G, F) \to 0 \qquad (2.2)$$

where $e_{0,j} = \dim \mathrm{Ext}^q (G_j, F)$ for $0 \le j \le m$ and $e_{l,j}$ is determined inductively for $l \ge 1$ and $j \le m - l$ by the following formula:

$$e_{l,j} = \sum_{j < k} e_{l-1,k} \hom(G_j, G_k).$$

We shall freely use the following isomorphism:

$$P_j \otimes^L_A G = P_j \otimes_A G \cong G_j.$$

This isomorphism together with (2.2) implies that $E_2^{p,q}$ is the $(-p)$th homology of the following complex

$$0 \to G_0^{\oplus e_{m,0}} \to \cdots \to \bigoplus_{j=0}^{m-l} G_j^{\oplus e_{l,j}} \to \cdots \to \bigoplus_{j=0}^{m} G_j^{\oplus e_{0,j}} \to 0.$$

Throughout this paper, we set $X = \mathbb{P}^n$, $m = n$, and $G_j = \mathcal{O}(j)$ for $0 \le j \le n$, and we fix the notation G_j, G, A, P_j, and S_j as above.

A typical projective resolution of the form (2.2) in this paper is in the case where $q = 0$ and $F = \mathcal{E}(d)$ for some non-negative integer d. For example, if $\hom(\mathcal{O}(2), \mathcal{E}(d)) = 0$, we frequently and sometimes implicitly consider a projective resolution of the form

$$0 \to P_0^{\oplus (n+1)e_{0,1}} \to P_1^{\oplus e_{0,1}} \oplus P_0^{\oplus e_{0,0}} \to \mathrm{Hom}(G, \mathcal{E}(d)) \to 0 \qquad (2.3)$$

where $e_{0,0} = \hom(\mathcal{O}, \mathcal{E}(d))$ and $e_{0,1} = \hom(\mathcal{O}(1), \mathcal{E}(d))$.

Finally note that the Bott formula [14, p. 8] implies $\mathrm{RHom}(G, \Omega^j_{\mathbb{P}^n}(j)) \cong S_j[-j]$ for $0 \le j \le n$. Hence we have isomorphisms

$$S_j \otimes^L_A G \cong \Omega^j_{\mathbb{P}^n}(j)[j]. \qquad (2.4)$$

for $0 \le j \le n$. In particular,

$$S_n \otimes^L_A G \cong \mathcal{O}(-1)[n]. \qquad (2.5)$$

Note that this gives another way to compute $E_2^{p,q} = \mathcal{H}^p(\mathrm{Ext}^q(G, \mathcal{E}(d)) \otimes^L_A G)$: through a filtration of $\mathrm{Ext}^q(G, \mathcal{E}(d))$ with subquotients the direct sums of the simple modules S_j. For example, if $\mathrm{Ext}^q(G, \mathcal{E}(d)) \cong S_j$ then $E_2^{p,q} = \mathcal{H}^p(\Omega^j_{\mathbb{P}^n}(j)[j])$, and thus $E_2^{p,q} = 0$ if $p \ne -j$ and $E_2^{-j,q} = \Omega^j_{\mathbb{P}^n}(j)$. In particular, if $\mathrm{Ext}^q(G, \mathcal{E}(d)) \cong S_n$ then $E_2^{-n,q} = \mathcal{O}(-1)$ and $E_2^{p,q} = 0$ if $p \ne -n$. We shall also use these formulas frequently.

3 Set-up and formulas for the proofs of Theorems 1 and 3

In this section, we give some preparatory parts of our proofs of Theorems 1 and 3, and we also collect several formulas needed later.

Let $\mathcal{E}$ be a nef vector bundle of rank r on a projective space $\mathbb{P}^n$ with $c_1 = 3$. If $\mathrm{Hom}(\mathcal{O}(3), \mathcal{E}) \ne 0$, then $\mathcal{E} \cong \mathcal{O}(3) \oplus \mathcal{O}^{\oplus r-1}$ by [11, Proposition 5.2 and Remark 5.3]. This is the case (1) of Theorem 1. Assume that $\mathrm{Hom}(\mathcal{O}(3), \mathcal{E}) = 0$. Then $r \ge 2$. If

$\mathrm{Hom}(\mathcal{O}(2), \mathcal{E}) \neq 0$, then it follows from [11, Theorem 6.4] that $\mathcal{E}$ is in the case (2) or (5) of Theorem 1.

In the rest of our proof, we always assume that

$$H^0(\mathcal{E}(-2)) = \mathrm{Hom}(\mathcal{O}(2), \mathcal{E}) = 0. \tag{3.1}$$

Recall that the Kodaira vanishing theorem implies that

$$H^q(\mathcal{E}|_{L^l}(3-k)) = 0 \tag{3.2}$$

for all $q > 0$, all l-dimensional linear subspaces $L^l \subseteq \mathbb{P}^n$, and all $k \leq l$, since $\mathcal{E}$ is a nef vector bundle with $c_1 = 3$ (see [11, Lemma 4.1 (1)] for a proof). If $H(\mathcal{E}|_{L^l})$ is big in addition, then the Kawamata-Viehweg vanishing theorem implies that

$$H^q(\mathcal{E}|_{L^l}(2-k)) = 0 \tag{3.3}$$

for all $q > 0$ and all $k \leq l$ (see [11, Lemma 4.1 (2)] for a proof).

Since $\mathcal{E}$ is nef, $H^q(\mathcal{E}|_L(1)) = 0$ for all $q > 0$ and all lines L in $\mathbb{P}^n$. Together with (3.2), this implies that $H^2(\mathcal{E}|_{L^2}) = 0$ for any plane $L^2 \subseteq \mathbb{P}^n$. This vanishing $H^2(\mathcal{E}|_{L^2}) = 0$ then implies that $H^2(\mathcal{E}|_{L^2}(-1)) = 0$ since $H^q(\mathcal{E}|_L) = 0$ for all $q > 0$. Moreover we have $H^2(\mathcal{E}|_{L^2}(-2)) = 0$ since $H^q(\mathcal{E}|_L(-1)) = 0$ for all $q > 0$. Summing up, we have

$$H^2(\mathcal{E}|_{L^2}(-k)) = 0 \tag{3.4}$$

for all $k \leq 2$ and any plane L^2 in $\mathbb{P}^n$.

If $n = 2$, the Riemann–Roch formula for a twisted vector bundle $\mathcal{E}(t)$ is

$$\chi(\mathcal{E}(t)) = \frac{1}{2}(rt+6)(t+3) + r - c_2. \tag{3.5}$$

Recall that this formula is for $c_1 = c_1(\mathcal{E}) = 3$. It follows from this formula that $\chi(\mathcal{E}(-2)) = 3 - c_2$. For arbitrary $n \geq 2$, the vanishing (3.4) then implies that

$$h^0(\mathcal{E}|_{L^2}(-2)) - h^1(\mathcal{E}|_{L^2}(-2)) = 3 - c_2(\mathcal{E}|_{L^2}) \tag{3.6}$$

for any plane L^2 in $\mathbb{P}^n$.

We claim here that

$$c_2 \geq 3$$

on $\mathbb{P}^n$. Suppose, to the contrary, that $c_2 \leq 2$. Then $\chi(\mathcal{E}|_{L^2}(-2)) = 3 - c_2(\mathcal{E}|_{L^2}) \geq 1$ since $c_2 = c_2(\mathcal{E}|_{L^2})$. Hence we obtain $h^0(\mathcal{E}|_{L^2}(-2)) \neq 0$. As we have seen, this implies that $\mathcal{E}|_{L^2}$ lies either in the case (1), (2), or (5) of Theorem 1; if $\mathcal{E}|_{L^2}$ lies in the case (5) then $c_2(\mathcal{E}|_{L^2}) = 3$, which contradicts that $c_2 \leq 2$. Thus $\mathcal{E}|_{L^2}$ actually lies either in the case (1) or (2) of Theorem 1. In particular it splits into a direct sum of line bundles. Hence $\mathcal{E}$ also splits by the splitting criterion [14, Theorem 2.3.2] of Horrocks. Since $h^0(\mathcal{E}|_{L^2}(-2)) \neq 0$, this implies that $h^0(\mathcal{E}(-2)) \neq 0$, which contradicts the assumption (3.1). Hence the claim follows.

Since $\mathcal{E}|_{L^2}$ is nef, we have an inequality $0 \leq H(\mathcal{E}|_{L^2})^{r+1}$. Since $H(\mathcal{E}|_{L^2})^{r+1}$ is equal to $c_1(\mathcal{E}|_{L^2})^2 - c_2(\mathcal{E}|_{L^2})$ (see [6, §3.1 and §3.2]), we obtain $c_2(\mathcal{E}|_{L^2}) \leq 9$. Hence we have

$$c_2 \leq 9.$$

Recall here the well-known non-negativity (see, e.g., [10, Theorem 8.2.1]) of the top Chern class of a nef vector bundle $\mathcal{E}$ that

$$c_n \geq 0. \tag{3.7}$$

We shall divide the proof of Theorem 1 according to the value of $c_2 \geq 3$.

Note that $H(\mathcal{E}|_{L^2})$ is nef and big if $c_2 < 9$. The vanishing (3.3) then implies that

$$H^q(\mathcal{E}|_{L^2}) = 0 \tag{3.8}$$

for any $q > 0$ and any plane L^2 in $\mathbb{P}^n$. Together with the vanishing (3.2), this implies that

$$H^q(\mathcal{E}|_{L^3}(-1)) = 0 \tag{3.9}$$

for any $q \geq 2$ and any three-dimensional linear subspace $L^3 \subset \mathbb{P}^n$.

3.1 Set-up for the two-dimensional case with $h^1(\mathcal{E}) = 0$

Note that the vanishing $h^1(\mathcal{E}) = 0$ holds if $c_2 < 9$ by (3.8). It follows from (3.1) and (3.6) that

$$h^1(\mathcal{E}(-2)) = c_2 - 3. \tag{3.10}$$

The Riemann–Roch formula (3.5) shows that $\chi(\mathcal{E}(-1)) = 6 - c_2$ and $\chi(\mathcal{E}) = 9 + r - c_2$. Since we have the vanishings (3.4) and $h^1(\mathcal{E}) = 0$, these formulas imply

$$h^0(\mathcal{E}(-1)) - h^1(\mathcal{E}(-1)) = 6 - c_2, \tag{3.11}$$
$$h^0(\mathcal{E}) = 9 + r - c_2. \tag{3.12}$$

Since we have an exact sequence $0 \to \mathcal{E}(-2) \to \mathcal{E}(-1) \to \mathcal{E}|_L(-1) \to 0$, by taking cohomology, we see that

$$h^1(\mathcal{E}(-2)) \geq h^1(\mathcal{E}(-1)). \tag{3.13}$$

Now we apply to $\mathcal{E}$ the Bondal spectral sequence (2.1). It is clear that $E_2^{p,q} = 0$ if $q < 0$ or $p > 0$. The vanishing (3.4) shows that $E_2^{p,q} = 0$ if $q \geq 2$. Since $H^1(\mathcal{E}) = 0$ by assumption, the right A-module $\mathrm{Ext}^1(G, \mathcal{E})$ fits in an exact sequence

$$0 \to S_1^{\oplus h^1(\mathcal{E}(-1))} \to \mathrm{Ext}^1(G, \mathcal{E}) \to S_2^{\oplus h^1(\mathcal{E}(-2))} \to 0.$$

Since $S_1 \otimes^L_A G \cong \Omega_{\mathbb{P}^2}(1)[1]$ and $S_2 \otimes^L_A G \cong \mathcal{O}(-1)[2]$ by (2.4), the sequence above induces the following distinguished triangle in $D^b(X)$:

$$\mathcal{O}(-1)^{\oplus h^1(\mathcal{E}(-2))}[1] \to \Omega_{\mathbb{P}^2}(1)^{\oplus h^1(\mathcal{E}(-1))}[1] \to \mathrm{Ext}^1(G, \mathcal{E}) \otimes^L_A G \to .$$

Since $E_2^{p,1} = \mathcal{H}^p(\mathrm{Ext}^1(G, \mathcal{E}) \otimes^L_A G)$, the triangle above shows that $E_2^{p,1} = 0$ unless $p = -2$ or -1 and that $E_2^{-2,1}$ and $E_2^{-1,1}$ fit in the following exact sequence of coherent sheaves:

$$0 \to E_2^{-2,1} \to \mathcal{O}(-1)^{\oplus h^1(\mathcal{E}(-2))} \xrightarrow{\mu} \Omega_{\mathbb{P}^2}(1)^{\oplus h^1(\mathcal{E}(-1))} \to E_2^{-1,1} \to 0. \tag{3.14}$$

It follows from (3.1) that the right A-module $\mathrm{Hom}(G, \mathcal{E})$ has, as in (2.3), a projective resolution of the form

$$0 \to P_0^{\oplus 3e_{0,1}} \to P_1^{\oplus e_{0,1}} \oplus P_0^{\oplus e_{0,0}} \to \mathrm{Hom}(G, \mathcal{E}) \to 0$$

where $e_{0,0} = h^0(\mathcal{E})$ and $e_{0,1} = h^0(\mathcal{E}(-1))$. In particular, we see that $E_2^{p,0} = 0$ if $p < -1$. We have the following exact sequence

$$0 \to E_2^{-2,1} \to E_2^{0,0} \to E_3^{0,0} \to 0. \tag{3.15}$$

Now the Bondal spectral sequence implies that $E_2^{-1,0} = 0$ and that $\mathcal{E}$ fits in an exact sequence

$$0 \to E_3^{0,0} \to \mathcal{E} \to E_2^{-1,1} \to 0. \tag{3.16}$$

Since $E_2^{-1,0} = 0$, $E_2^{0,0}$ fits in an exact sequence

$$0 \to \mathcal{O}^{\oplus 3e_{0,1}} \to \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus e_{0,0}} \to E_2^{0,0} \to 0. \tag{3.17}$$

The following lemma shall be applied repeatedly throughout this paper.

Lemma 3.1 *For any finite morphism $C \to \mathbb{P}^2$ from a smooth projective curve C, the pullback $E_2^{-1,1}|_C$ of the sheaf $E_2^{-1,1}$ cannot admit a line bundle of negative degree as a quotient. In particular, $E_2^{-1,1}$ cannot admit the following sheaves as a quotient:*

(1) $\Omega_{\mathbb{P}^2}(1)$;
(2) $\mathcal{I}_p$, where p is a point;
(3) $\mathcal{I}_Z(1)$, where Z is a 0-dimensional closed subscheme of $\mathrm{length}\,Z \geq 2$;
(4) $\mathcal{I}_Z(2)$, where Z is a 0-dimensional closed subscheme of $\mathrm{length}\,Z \geq 5$;
(5) $\mathcal{O}_L(-p)$, where L is a line passing through a point p;
(6) $\mathcal{O}_C(-p)$, where C is a conic passing through a point p.

Proof The first statement follows from the sequence (3.16) since $\mathcal{E}$ is nef. The second statement is almost obvious from the first, so that we only give a proof in case (4). If there exists a line L such that $\mathrm{length}\,Z \cap L \geq 3$, then the double twist $\mathcal{I}_Z \cdot \mathcal{O}_L(2)$ of the inverse image ideal sheaf $\mathcal{I}_Z \cdot \mathcal{O}_L$ has negative degree, which contradicts the first statement. If $\mathrm{length}\,Z \cap L \leq 2$ for any line L, then Lemma 4.4 below shows that there exists a smooth conic C such that $\mathrm{length}\,Z \cap C \geq 5$. Hence $\mathcal{I}_Z \cdot \mathcal{O}_C(2)$ has negative degree, which contradicts the first statement again. Therefore $E_2^{-1,1}$ cannot admit $\mathcal{I}_Z(2)$ as a quotient if $\mathrm{length}\,Z \geq 5$. $\square$

Lemma 3.1 and the exact sequence (3.14) indicate that there are some relations between $h^1(\mathcal{E}(-2))$ and $h^1(\mathcal{E}(-1))$; we shall explore these relations in Lemma 5.1.

3.2 Set-up for the three-dimensional case

The Riemann–Roch formula for a twisted vector bundle $\mathcal{E}(t)$ on $\mathbb{P}^3$ is

$$\chi(\mathcal{E}(t)) = \frac{1}{2}\{9 - 3c_2 + c_3 + (9 - 2c_2)(t + 2) + (3t^2 + 12t + 11)\}$$
$$+ \frac{r}{6}(t + 3)(t + 2)(t + 1).$$

Recall that this formula is for $c_1 = c_1(\mathcal{E}) = 3$. By this formula, we have

$$\chi(\mathcal{E}(-3)) = 1 - \frac{1}{2}(c_2 - c_3), \tag{3.18}$$

$$\chi(\mathcal{E}(-2)) = 4 - \frac{1}{2}(3c_2 - c_3), \tag{3.19}$$

$$\chi(\mathcal{E}(-1)) = 10 - \frac{1}{2}(5c_2 - c_3). \tag{3.20}$$

By taking into account the vanishing (3.2), we also have

$$h^0(\mathcal{E}) = 19 - \frac{1}{2}(7c_2 - c_3) + r, \tag{3.21}$$

$$h^0(\mathcal{E}(1)) = 31 - \frac{1}{2}(9c_2 - c_3) + 4r. \tag{3.22}$$

Recall here that $H(\mathcal{E})^{r+2} = c_3 - 2c_1c_2 + c_1^3$ (see [6, §3.1 and §3.2]). Since $c_1 = 3$ and $H(\mathcal{E})^{r+2} \geq 0$, this implies that

$$c_3 \geq 6c_2 - 27. \tag{3.23}$$

4 Lemmas

In this section, we collect some lemmas applied in the proofs of Theorems 1 or 3.

Lemma 4.1 *Let V be an $(n+1)$-dimensional vector space, $(x_0, x_1, \ldots, x_n)$ a basis of V, and $(x_0^*, \ldots, x_n^*)$ the basis of the dual vector space V^* corresponding to $(x_0, x_1, \ldots, x_n)$. Suppose that a non-zero element s of $H^0(T_{\mathbb{P}(V)}(-1))$ corresponds to $\sum_{i=0}^{n} a_i x_i^*$ for some $a_i \in K$ via the isomorphism from $H^0(T_{\mathbb{P}(V)}(-1))$ to V^*. Then the zero locus $(s)_0$ of s is a (reduced) point p whose homogeneous coordinates $(x_0(p): \cdots : x_n(p))$ are $(a_0: a_1: \cdots : a_n)$.*

Proof See [16, Remark 2.3]. □

Lemma 4.2 *Let i be any integer such that $1 \leq i \leq n$. For each non-zero morphism $\varphi : \Omega_{\mathbb{P}(V)}^i(i) \to \Omega_{\mathbb{P}(V)}^{i-1}(i-1)$, there exists a unique element $s \in H^0(T_{\mathbb{P}(V)}(-1))$ such that φ is the morphism appearing in the Koszul complex induced from s. In particular, every non-zero morphism $\varphi : \Omega_{\mathbb{P}^3}^2(2) \to \Omega_{\mathbb{P}^3}(1)$ determines a unique non-zero element $s \in H^0(T_{\mathbb{P}^3}(-1))$ whose corresponding Koszul complex is an exact sequence*

$$0 \to \mathcal{O}(-1) \to \Omega_{\mathbb{P}^3}^2(2) \xrightarrow{\varphi} \Omega_{\mathbb{P}^3}(1) \to \mathcal{I}_p \to 0,$$

where $\{p\} = (s)_0$.

Proof We have an isomorphism $H^0(T_{\mathbb{P}(V)}(-1)) \cong \operatorname{Hom}(\Omega_{\mathbb{P}(V)}^i(i), \Omega_{\mathbb{P}(V)}^{i-1}(i-1))$ by [3, Lemma 2], and this isomorphism indeed sends a global section s of $T_{\mathbb{P}(V)}(-1)$ to the morphism $\varphi : \Omega_{\mathbb{P}(V)}^i(i) \to \Omega_{\mathbb{P}(V)}^{i-1}(i-1)$ appearing in the Koszul complex associated to s. Note that a non-zero section s is regular by Lemma 4.1, so that the corresponding Koszul complex is exact. □

Lemma 4.3 *Let W be a 0-dimensional closed subscheme of $\mathbb{P}^2$ and w a point in $\mathbb{P}^2$. Suppose that we have the following non-split exact sequence of coherent sheaves*

$$0 \to \mathcal{I}_W(d) \to \mathcal{D} \to k(w) \to 0,$$

where d is an integer and $k(w)$ denotes the residue field of the point w. Then w is an associated point of W and $\mathcal{D}$ is isomorphic to $\mathcal{I}_Z(d)$ where Z is a closed subscheme of W with $\operatorname{length} Z = \operatorname{length} W - 1$.

Proof First note that w is not an associated point of $\mathcal{D}$, since the exact sequence above does not split. Hence $\mathcal{D}$ is a torsion-free sheaf of rank one, and its double dual is isomorphic to $\mathcal{O}(d)$. Therefore $\mathcal{D}$ is isomorphic to $\mathcal{I}_Z(d)$, where Z is a closed subscheme of W with $\operatorname{length} Z = \operatorname{length} W - 1$, and w is an associated point of W. □

Lemma 4.4 *Let Z be a 0-dimensional closed subscheme of length 5 in $\mathbb{P}^2$, and suppose that $\operatorname{length}(Z \cap L) \leq 2$ for any line L in $\mathbb{P}^2$. Then there exists a smooth conic C passing through Z.*

Proof See, e.g., the second paragraph of page 75 of [12]. □

5 Key Lemma

The following lemma together with the exact sequence (3.16) plays a crucial role in the proofs of Theorems 1 and 3.

Lemma 5.1 *Let*

$$0 \to E_2^{-2,1} \to \mathcal{O}(-1)^{\oplus a} \xrightarrow{\mu} \Omega_{\mathbb{P}^2}(1)^{\oplus b} \to E_2^{-1,1} \to 0$$

be the exact sequence (3.14) on $\mathbb{P}^2$, where $a = h^1(\mathcal{E}(-2))$ and $b = h^1(\mathcal{E}(-1))$.

(1) *If $b \geq 1$, consider a surjection $\pi : \Omega_{\mathbb{P}^2}(1)^{\oplus b} \to \Omega_{\mathbb{P}^2}(1)$, and let*

$$\varphi : \mathcal{O}(-1)^{\oplus a} \to \Omega_{\mathbb{P}^2}(1)$$

be the composite of μ and π. Then $H^0(\varphi(1)) : H^0(\mathcal{O}^{\oplus a}) \to H^0(\Omega_{\mathbb{P}^2}(2))$ is surjective; consequently $a \geq 3$ and φ is surjective. Moreover we have an exact sequence

$$0 \to E_2^{-2,1} \to \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus a-3} \xrightarrow{\mu_1} \Omega_{\mathbb{P}^2}(1)^{\oplus b-1} \to E_2^{-1,1} \to 0.$$

(2) *If $b \geq 2$, consider a surjection $q : \Omega_{\mathbb{P}^2}(1)^{\oplus b-1} \to \Omega_{\mathbb{P}^2}(1)$, and let*

$$\varphi_1 : \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus a-3} \to \Omega_{\mathbb{P}^2}(1)$$

be the composite of μ_1 and q. Then the image of $H^0(\varphi_1(1))$ has dimension two or three; consequently $a \geq 5$.

  (a) *If $\dim \operatorname{Im} H^0(\varphi_1(1)) = 2$, then we have the following exact sequence*

$$0 \to E_2^{-2,1} \to \mathcal{O}(-3) \oplus \mathcal{O}(-1)^{\oplus a-5} \xrightarrow{\nu_2} \Omega_{\mathbb{P}^2}(1)^{\oplus b-2}$$
$$\to E_2^{-1,1} \to k(w) \to 0$$

    for some point w in $\mathbb{P}^2$, where $k(w)$ denotes the residue field of w.

  (b) *If $\dim \operatorname{Im} H^0(\varphi_1(1)) = 3$, then $a \geq 6$ and we have the following exact sequence*

$$0 \to E_2^{-2,1} \to \mathcal{O}(-2)^{\oplus 2} \oplus \mathcal{O}(-1)^{\oplus a-6} \xrightarrow{\mu_2} \Omega_{\mathbb{P}^2}(1)^{\oplus b-2} \to E_2^{-1,1} \to 0.$$

Proof (1) First note that there exists a surjection $E_2^{-1,1} \to \operatorname{Coker}\varphi$.

If $H^0(\varphi(1)) = 0$, then $\varphi = 0$ and $\operatorname{Coker}\varphi \cong \Omega_{\mathbb{P}^2}(1)$. This however contradicts Lemma 3.1. Therefore $H^0(\varphi(1)) \neq 0$.

Suppose that the image of $H^0(\varphi(1))$ has dimension one, and let s be a non-zero element in the image in $H^0(\Omega_{\mathbb{P}^2}(2)) \cong H^0(T_{\mathbb{P}^2}(-1))$. Then the zero locus of s is a (reduced) point p by Lemma 4.1 and $\operatorname{Coker}\varphi$ is isomorphic to the ideal sheaf $\mathcal{I}_p$ of p, but this cannot occur by Lemma 3.1. Therefore the image of $H^0(\varphi(1))$ has dimension ≥ 2.

Suppose that the image of $H^0(\varphi(1))$ has dimension two, and let (s, t) be a basis of the image. As above, let p be the zero locus of s. Then t induces an injection $\mathcal{O} \to \mathcal{I}_p(1)$, and this gives a line L passing through p. The quotient of the $\mathcal{O}(-1)$-twist of this injection is $\mathcal{O}_L(-p)$ and is isomorphic to $\operatorname{Coker}\varphi$. However this is impossible by Lemma 3.1. Therefore the image of $H^0(\varphi(1))$ has dimension ≥ 3, i.e., $H^0(\varphi(1))$ is surjective. Hence φ is surjective and $\operatorname{Ker}\varphi \cong \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus a-3}$. Now $\operatorname{Ker}\pi \cong \Omega_{\mathbb{P}^2}(1)^{\oplus b-1}$, and the morphism $\mu_1 : \operatorname{Ker}\varphi \to \operatorname{Ker}\pi$ induced by μ extends to the exact sequence in the statement.

(2) Note that there exists a surjection $E_2^{-1,1} \to \operatorname{Coker}\varphi_1$. Denote by ι the inclusion $\mathcal{O}(-1)^{\oplus a-3} \to \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus a-3}$, and consider the composite $\varphi_1 \circ \iota : \mathcal{O}(-1)^{\oplus a-3} \to$

$\Omega_{\mathbb{P}^2}(1)$. We see that φ_1 induces a morphism $\bar{\varphi}_1 : \text{Coker}(\iota) \to \text{Coker}(\varphi_1 \circ \iota)$ and that $\text{Coker}(\iota) \cong \mathcal{O}(-2)$. Moreover we have the following long exact sequence

$$0 \to \text{Ker}(\varphi_1 \circ \iota) \to \text{Ker}\varphi_1 \to \mathcal{O}(-2) \xrightarrow{\bar{\varphi}_1} \text{Coker}(\varphi_1 \circ \iota) \to \text{Coker}\varphi_1 \to 0 \qquad (5.1)$$

by the snake lemma. In particular, $\text{Coker}\varphi_1 \cong \text{Coker}\bar{\varphi}_1$.

If $H^0(\varphi_1(1)) = 0$, then $\varphi_1 \circ \iota = 0$ and thus $\text{Coker}(\varphi_1 \circ \iota) \cong \Omega_{\mathbb{P}^2}(1)$. If $\bar{\varphi}_1 = 0$, then $\text{Coker}\varphi_1 \cong \text{Coker}(\varphi_1 \circ \iota)$. Hence $E_2^{-1,1}$ has $\Omega_{\mathbb{P}^2}(1)$ as a quotient, which is absurd by Lemma 3.1. Therefore $\bar{\varphi}_1 : \mathcal{O}(-2) \to \Omega_{\mathbb{P}^2}(1)$ is a non-zero morphism. Suppose that $\bar{\varphi}_1$ is decomposed as $\bar{\varphi}_1 = sl$, where l is an inclusion $\mathcal{O}(-2) \to \mathcal{O}(-1)$ and s is a non-zero morphism $\mathcal{O}(-1) \to \Omega_{\mathbb{P}^2}(1)$. The cokernel of the inclusion $l : \mathcal{O}(-2) \to \mathcal{O}(-1)$ is isomorphic to $\mathcal{O}_L(-1)$, where L is the line defined by l, and the cokernel of the morphism $s : \mathcal{O}(-1) \to \Omega_{\mathbb{P}^2}(1)$ is isomorphic to $\mathcal{I}_p$, where p is a point; thus $\text{Coker}\bar{\varphi}_1$ fits in an exact sequence

$$0 \to \mathcal{O}_L(-1) \to \text{Coker}\bar{\varphi}_1 \to \mathcal{I}_p \to 0.$$

Therefore $\text{Coker}\varphi_1$ (and hence $E_2^{-1,1}$) has $\mathcal{I}_p$ as a quotient, which cannot happen by Lemma 3.1. Hence $\bar{\varphi}_1$ does not factor as $\bar{\varphi}_1 = sl$. Let $\bar{s}$ be the non-zero section of $H^0(\Omega_{\mathbb{P}^2}(3))$ determined by $\bar{\varphi}_1$. Then $\bar{s}$ is a regular section of $\Omega_{\mathbb{P}^2}(3) \cong T_{\mathbb{P}^2}$, i.e., the zero locus Z of $\bar{s}$ has dimension ≤ 0. Since $c_2(T_{\mathbb{P}^2}) = 3$, Z is thereby a 0-dimensional closed subscheme of length three. Note that $\text{Coker}\bar{\varphi}_1 \cong \mathcal{I}_Z(1)$; this however contradicts Lemma 3.1. Therefore $H^0(\varphi_1(1)) \neq 0$.

Suppose that the image of $H^0(\varphi_1(1))$ has dimension one. Then $\text{Coker}(\varphi_1 \circ \iota)$ is isomorphic to the ideal sheaf $\mathcal{I}_p$ of some point p, as is shown in the proof of (1). If $\bar{\varphi}_1 = 0$, then $\text{Coker}\bar{\varphi}_1 \cong \mathcal{I}_p$, which is absurd by Lemma 3.1. Hence $\bar{\varphi}_1 : \mathcal{O}(-2) \to \mathcal{I}_p$ is non-zero. Then $\text{Coker}\bar{\varphi}_1$ is isomorphic to $\mathcal{O}_C(-p)$ for some conic C on $\mathbb{P}^2$, which again contradicts Lemma 3.1. Therefore the image of $H^0(\varphi_1(1))$ has dimension at least two.

(2) (a) Suppose that $\dim \text{Im} H^0(\varphi_1(1)) = 2$. Then $\text{Coker}(\varphi_1 \circ \iota)$ is isomorphic to $\mathcal{O}_L(-p)$ for some line L in $\mathbb{P}^2$ and some point p in L, as is shown in the proof of (1). If $\bar{\varphi}_1 = 0$, then $\text{Coker}\bar{\varphi}_1 \cong \mathcal{O}_L(-p)$, which cannot happen by Lemma 3.1. Hence $\bar{\varphi}_1 : \mathcal{O}(-2) \to \mathcal{O}_L(-p)$ is non-zero. Then $\text{Coker}\bar{\varphi}_1 \cong k(w)$ for some point w on $\mathbb{P}^2$, and $\text{Ker}\bar{\varphi}_1 \cong \mathcal{O}(-3)$. The long exact sequence (5.1) induces an exact sequence

$$0 \to \text{Ker}(\varphi_1 \circ \iota) \to \text{Ker}\varphi_1 \to \text{Ker}\bar{\varphi}_1 \to 0.$$

Note here that $\text{Ker}\varphi_1 \circ \iota \cong \mathcal{O}(-1)^{\oplus a-5}$; the above sequence then implies that $\text{Ker}\varphi_1 \cong \mathcal{O}(-3) \oplus \mathcal{O}(-1)^{\oplus a-5}$. Now we get the exact sequence in the statement by the snake lemma.

(2) (b) Suppose that $\dim \text{Im} H^0(\varphi_1(1)) \geq 3$. Then $H^0(\varphi_1(1))$ is surjective, and thus $\varphi_1 \circ \iota$ is surjective. Hence $\text{Ker}\varphi_1 \circ \iota \cong \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus a-6}$. Then the long exact sequence (5.1) gives an exact sequence

$$0 \to \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus a-6} \to \text{Ker}\varphi_1 \to \mathcal{O}(-2) \to 0.$$

Therefore $\text{Ker}\varphi_1 \cong \mathcal{O}(-2)^{\oplus 2} \oplus \mathcal{O}(-1)^{\oplus a-6}$. Now $\text{Ker}q \cong \Omega_{\mathbb{P}^2}(1)^{\oplus b-2}$, and μ_1 induces a morphism $\text{Ker}\varphi_1 \to \text{Ker}q$, which extends to the exact sequence in the statement. $\qquad\square$

6 Global generation of $\mathcal{E}$ in case $c_2 \leq 7$

Theorem 3 *Let $\mathcal{E}$ be a nef vector bundle of rank r on a projective space $\mathbb{P}^n$ with first Chern class $c_1 = 3$ and second Chern class $c_2 \leq 7$. Then $\mathcal{E}$ is globally generated.*

Proof Consider first the case $n = 2$. Note that $\mathcal{E}$ is 0-regular in the sense of Castelnuovo–Mumford if $h^1(\mathcal{E}(-1)) = 0$ since we have the vanishing (3.4). Since 0-regularity implies global generation, we may concentrate our attention to the case $h^1(\mathcal{E}(-1)) \neq 0$. We shall work in the setting in §3.1 to apply the Bondal spectral sequence (2.1). Then it follows from Lemma 5.1 (1) and formula (3.10) that $c_2 \geq 6$. Moreover the assumption $c_2 \leq 7$ implies that

$$h^1(\mathcal{E}(-1)) = 1 \qquad (6.1)$$

by Lemma 5.1 (2). Lemma 5.1 (1) then shows that $E_2^{-1,1} = 0$. Therefore $E_2^{0,0} \cong \mathcal{E}$ by the exact sequence (3.16). Note that $E_3^{0,0}$ is a quotient of $E_2^{0,0}$ since $E_2^{p,q} = 0$ if $p > 0$. Since $E_2^{0,0}$ is globally generated by (3.17), this shows that $\mathcal{E}$ is globally generated if $n = 2$ and $c_2 \leq 7$.

Consider next the case $n \geq 3$. Recall here that $\mathcal{E}$ is globally generated if $h^1(\mathcal{E}(-1)) = 0$ and $\mathcal{E}|_H$ is globally generated for any hyperplane H in $\mathbb{P}^n$ by [15, Lemma 3].

Suppose now that $n = 3$. Since $\mathcal{E}|_H$ is globally generated for any plane H as we have seen above, we may concentrate our attention to the case $h^1(\mathcal{E}(-1)) \neq 0$. Then it follows from (3.3) that $H(\mathcal{E})$ is not big, and hence $c_3 = 6c_2 - 27$ by (3.23). Thus c_3 is odd. Hence c_2 is odd by (3.20). Since c_3 is non-negative by (3.7), we see that $c_2 \geq 5$ by equality $c_3 = 6c_2 - 27$; thus $(c_2, c_3) = (5, 3)$ or $(7, 15)$. Suppose that $(c_2, c_3) = (7, 15)$. Then $h^1(\mathcal{E}|_H(-1)) = 1$ by (6.1). Hence $h^0(\mathcal{E}|_H(-1)) = 0$ by (3.11). Since $h^0(\mathcal{E}(-2)) = 0$ by (3.1), this implies that $h^0(\mathcal{E}(-1)) = 0$. Note here that $H^q(\mathcal{E}) = 0$ for all $q > 0$ by (3.2) and that $H^q(\mathcal{E}|_H) = 0$ for all $q > 0$ by (3.8). Hence $H^q(\mathcal{E}(-1)) = 0$ unless $q = 1$. Thus

$$-h^1(\mathcal{E}(-1)) = \chi(\mathcal{E}(-1)) = (c_3 - 15)/2 = 0$$

by (3.20). This contradicts the assumption $h^1(\mathcal{E}(-1)) \neq 0$; thus the case where $(c_2, c_3) = (7, 15)$ is ruled out. In §6.1, we shall show that the case $(c_2, c_3) = (5, 3)$ does not happen. Therefore $h^1(\mathcal{E}(-1)) = 0$ and $\mathcal{E}$ is globally generated.

If $n \geq 4$, then $h^1(\mathcal{E}(-1)) = 0$ by (3.2) and $\mathcal{E}|_H$ is globally generated for any hyperplane H in $\mathbb{P}^n$ as we have seen above. Therefore $\mathcal{E}$ is globally generated. $\qquad \square$

6.1 The case where $(n, c_2, c_3) = (3, 5, 3)$

We first claim that

$$H^q(\mathcal{E}|_{L^2}(-k)) = 0 \text{ for all } q > 0 \text{ and all } k \leq 2 \text{ unless } (q, k) = (1, 2); \qquad (6.2)$$
$$h^1(\mathcal{E}|_{L^2}(-2)) = 2, \qquad (6.3)$$

where L^2 denotes a plane in $\mathbb{P}^3$. Indeed we have $h^1(\mathcal{E}|_{L^2}(-2)) = 2$ by (3.10), and $H^q(\mathcal{E}|_{L^2}(-k))$ vanishes for any $k \leq 0$ and any $q > 0$ by (3.8), for $(k, q) = (1, 2)$ and $(2, 2)$ by (3.4), and for $(k, q) = (1, 1)$ by Lemma 5.1 (1) and (3.10). Since $H^q(\mathcal{E}(-k)) = 0$ for all $q > 0$ if $k \leq 0$ by (3.2), the claims above imply that

$$H^q(\mathcal{E}(-k)) = 0 \text{ for all } q \geq 2 \text{ and all } k \leq 3 \text{ unless } (q, k) = (2, 3); \qquad (6.4)$$
$$h^2(\mathcal{E}(-3)) \leq 2. \qquad (6.5)$$

Set

$$a = h^2(\mathcal{E}(-3)). \qquad (6.6)$$

Then

$$a = h^1(\mathcal{E}(-3)) \qquad (6.7)$$

by (3.1), (3.18), and (6.4). It follows from (3.1), (3.19), and (6.4) that

$$h^1(\mathcal{E}(-2)) = 2. \tag{6.8}$$

We also have

$$\chi(\mathcal{E}(-1)) = -1$$

by (3.20). By (3.21), we have

$$h^0(\mathcal{E}) = r + 3. \tag{6.9}$$

Note that there exists an exact sequence

$$0 \to H^0(\mathcal{E}(-1)) \to H^0(\mathcal{E}|_{L^2}(-1)) \to H^1(\mathcal{E}(-2)) \to H^1(\mathcal{E}(-1)) \to 0$$

and that $h^0(\mathcal{E}|_{L^2}(-1)) = 1$ by (3.11) and (6.2). Therefore we have two cases, corresponding to $h^0(\mathcal{E}(-1)) = 0$ or 1. If $h^0(\mathcal{E}(-1)) = 1$, then $h^1(\mathcal{E}(-1)) = 2$, and we shall consider the following exact sequence

$$0 \to \mathcal{O}(1) \to \mathcal{E} \to \mathcal{F} \to 0 \tag{6.10}$$

of coherent sheaves on $\mathbb{P}^3$. Note here that $\mathrm{Ext}^q(G, \mathcal{E}) \cong \mathrm{Ext}^q(G, \mathcal{F})$ if $q > 0$, that $\mathrm{Hom}(\mathcal{O}(i), \mathcal{F}) = 0$ if $i \geq 2$, that $\mathrm{Hom}(\mathcal{O}(1), \mathcal{F}) = 0$ by the assumption $h^0(\mathcal{E}(-1)) = 1$, and that $\hom(\mathcal{O}, \mathcal{F}) = r - 1$ by (6.9).

6.1.1 Set-up using the Bondal spectral sequence

Suppose that $(h^0(\mathcal{E}(-1)), h^1(\mathcal{E}(-1))) = (0, 1)$ (resp. $(1, 2)$). We apply to $\mathcal{E}$ (resp. $\mathcal{F}$) the Bondal spectral sequence (2.1). It follows from (6.4) that $E_2^{p,q} = 0$ for all p if $q \geq 3$. We have $\mathrm{Ext}^2(G, \mathcal{E}) \cong S_3^{\oplus a}$ by (6.4) and (6.6). Hence $E_2^{p,2} = 0$ unless $p = -3$, and $E_2^{-3,2} = \mathcal{O}(-1)^{\oplus a}$ by (2.5). It follows from (6.7), (6.8), (3.2), and the assumption $h^1(\mathcal{E}(-1)) = 1$ (resp. 2) that the right A-module $\mathrm{Ext}^1(G, \mathcal{E})$ (resp. $\mathrm{Ext}^1(G, \mathcal{F})$) has the filtration which induces the following exact sequences of right A-modules

$$0 \to F \to \mathrm{Ext}^1(G, \mathcal{E}) \to S_3^{\oplus a} \to 0;$$
$$0 \to S_1 \to F \to S_2^{\oplus 2} \to 0.$$
$$(\text{resp. } 0 \to F \to \mathrm{Ext}^1(G, \mathcal{F}) \to S_3^{\oplus a} \to 0;$$
$$0 \to S_1^{\oplus 2} \to F \to S_2^{\oplus 2} \to 0.)$$

Correspondingly we obtain the following distinguished triangles

$$F \otimes^L_A G \to \mathrm{Ext}^1(G, \mathcal{E}) \otimes^L_A G \to \mathcal{O}(-1)^{\oplus a}[3] \to; \tag{6.11}$$
$$\Omega_{\mathbb{P}^3}(1)[1] \to F \otimes^L_A G \to \Omega_{\mathbb{P}^3}^2(2)^{\oplus 2}[2] \to \tag{6.12}$$
$$(\text{resp. } F \otimes^L_A G \to \mathrm{Ext}^1(G, \mathcal{F}) \otimes^L_A G \to \mathcal{O}(-1)^{\oplus a}[3] \to; \tag{6.13}$$
$$\Omega_{\mathbb{P}^3}(1)^{\oplus 2}[1] \to F \otimes^L_A G \to \Omega_{\mathbb{P}^3}^2(2)^{\oplus 2}[2] \to) \tag{6.14}$$

by (2.4). We see that $\mathcal{H}^{-1}(F \otimes^L_A G) \cong E_2^{-1,1}$ by (6.11) [resp. (6.13)]. Moreover we have $\mathcal{H}^p(F \otimes^L_A G) = 0$ unless $p = -2$ or -1 by (6.12) [resp. (6.14)]. Set $\mathcal{M} = \mathcal{H}^{-2}(F \otimes^L_A G)$. Then it follows from (6.11) and (6.12) [resp. (6.13 and (6.14)] that $\mathcal{M}$ fits in the following

exact sequences

$$0 \to E_2^{-3,1} \to \mathcal{O}(-1)^{\oplus a} \to \mathcal{M} \to E_2^{-2,1} \to 0; \tag{6.15}$$

$$0 \to \mathcal{M} \to \Omega_{\mathbb{P}^3}^2(2)^{\oplus 2} \xrightarrow{\varphi} \Omega_{\mathbb{P}^3}(1) \to E_2^{-1,1} \to 0. \tag{6.16}$$

$$(\text{resp. } 0 \to \mathcal{M} \to \Omega_{\mathbb{P}^3}^2(2)^{\oplus 2} \xrightarrow{\varphi} \Omega_{\mathbb{P}^3}(1)^{\oplus 2} \to E_2^{-1,1} \to 0.) \tag{6.17}$$

It follows from (6.9) and the assumption $h^0(\mathcal{E}(-1)) = 0$ (resp. 1) that the right A-module $\mathrm{Hom}(G, \mathcal{E})$ (resp. $\mathrm{Hom}(G, \mathcal{F})$) is isomorphic to $S_0^{\oplus r+3}$ (resp. $S_0^{\oplus r-1}$). Hence $E_2^{p,0} = 0$ unless $p = 0$, and $E_2^{0,0}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^3}^{\oplus r+3}$ (resp. $\mathcal{O}_{\mathbb{P}^3}^{\oplus r-1}$) by (2.4). Therefore $E_2^{p,q} = 0$ unless $(p + q, q) = (0, 0), (0, 1), (-1, 1), (-1, 2)$, or $(-2, 1)$. Then it follows from (2.1) that $E_2^{-3,1} = 0$, and consequently the sequence (6.15) becomes the following exact sequence

$$0 \to \mathcal{O}(-1)^{\oplus a} \to \mathcal{M} \to E_2^{-2,1} \to 0. \tag{6.18}$$

Besides (6.16) [resp. (6.17)] and (6.18), the spectral sequence (2.1) induces the following exact sequences

$$0 \to E_2^{-2,1} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus r+3} \to E_3^{0,0} \to 0; \tag{6.19}$$

$$(\text{resp. } 0 \to E_2^{-2,1} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus r-1} \to E_3^{0,0} \to 0;) \tag{6.20}$$

$$0 \to E_3^{-3,2} \to \mathcal{O}(-1)^{\oplus a} \to E_2^{-1,1} \to E_3^{-1,1} \to 0; \tag{6.21}$$

$$0 \to E_3^{-3,2} \to E_3^{0,0} \to E_4^{0,0} \to 0; \tag{6.22}$$

$$0 \to E_4^{0,0} \to \mathcal{E} \to E_3^{-1,1} \to 0. \tag{6.23}$$

$$(\text{resp. } 0 \to E_4^{0,0} \to \mathcal{F} \to E_3^{-1,1} \to 0.) \tag{6.24}$$

For $i = 1$ (resp. $i = 1, 2$) and $j = 1, 2$, denote by φ_{ij} the composite of the jth inclusion $\Omega_{\mathbb{P}^3}^2(2) \hookrightarrow \Omega_{\mathbb{P}^3}^2(2)^{\oplus 2}$ and the morphism φ appearing in the exact sequence (6.16) [resp. (6.17)] and the ith projection.

We claim here that φ_{11} and φ_{12} (resp. φ_{i1} and φ_{i2} for each i, and φ_{1j} and φ_{2j} for each j) are linearly independent in $\mathrm{Hom}(\Omega_{\mathbb{P}^3}^2(2), \Omega_{\mathbb{P}^3}(1))$. Indeed, if they are not, then we may assume that $\varphi_{12} = 0$ by some row or column elementary transformations; thus φ (resp. the composite of φ and the first projection $\Omega_{\mathbb{P}^3}(1)^{\oplus 2} \to \Omega_{\mathbb{P}^3}(1)$) factors through φ_{11} via the first projection $\Omega_{\mathbb{P}^3}^2(2)^{\oplus 2} \to \Omega_{\mathbb{P}^3}^2(2)$. Note here that the cokernel of φ_{11} is isomorphic to the ideal sheaf $\mathcal{I}_p$ for some point $p \in \mathbb{P}^3$ by Lemma 4.2 if $\varphi_{11} \neq 0$ and to $\Omega_{\mathbb{P}^3}(1)$ if $\varphi_{11} = 0$. Hence there exists a surjection $E_2^{-1,1} \to \mathcal{I}_p$ by (6.16) [resp. (6.17)] (even if $\varphi_{11} = 0$). Consider the composite of the morphism $\mathcal{O}(-1)^{\oplus a} \to E_2^{-1,1}$ in (6.21) and the surjection $E_2^{-1,1} \to \mathcal{I}_p$. The cokernel of the composite $\mathcal{O}(-1)^{\oplus a} \to \mathcal{I}_p$ is a quotient of $E_3^{-1,1}$. The composite determines a linear subspace L' containing p and of codimension at most a in $\mathbb{P}^3$, and its cokernel is the ideal sheaf $\mathcal{J}_p$ of p in L'. Since $a \leq 2$ by (6.5) and (6.6), we have $\dim L' \geq 3 - a \geq 1$. Hence we can take a line L such that $p \in L \subseteq L'$. Since the restriction $\mathcal{J}_p \to \mathcal{O}_L(-p) = \mathcal{O}(-1)$ is surjective, we infer that $E_3^{-1,1}$ has $\mathcal{O}_L(-1)$ as a quotient. On the other hand, $E_3^{-1,1}$ cannot have a negative degree sheaf as a quotient, since it is a quotient of the nef vector bundle $\mathcal{E}$ by (6.23) [resp. (6.24) and (6.10)]. This is a contradiction. Therefore the claim holds.

In case $(h^0(\mathcal{E}(-1)), h^1(\mathcal{E}(-1))) = (1, 2)$, denote by V the vector space generated by φ_{ij} $(1 \leq i, j \leq 2)$ in the following. If $\dim V \leq 3$, then the claim above implies that there exist $\lambda_{ij} \in K$ $(1 \leq i, j \leq 2)$ such that $\lambda_{21}\varphi_{11} + \lambda_{22}\varphi_{21} = \lambda_{11}\varphi_{12} + \lambda_{12}\varphi_{22} \neq 0$. Therefore, by changing the free basis consisting of two elements of $\Omega_{\mathbb{P}^3}(1)^{\oplus 2}$, we may assume that $\varphi_{21} = \varphi_{12}$ in case $\dim V \leq 3$. Moreover if $\dim V = 2$, we can make φ_{12} to be zero

after some row and column elementary transformations, which contradicts the claim above. Therefore dim $V = 3$ or 4, and we may furthermore assume that $\varphi_{11}, \varphi_{12}$, and φ_{22} are linearly independent.

Set $\varphi_0 = \varphi_{11}$ and $\varphi_1 = \varphi_{12}$. (resp. Set $\varphi_0 = \varphi_{11}$, $\varphi_1 = \varphi_{12}$, $\varphi_2 = \varphi_{21}$, and $\varphi_3 = \varphi_{22}$. Note that $\varphi_1 = \varphi_2$ in case dim $V = 3$.)

Let e_i^* be the image of x_i^* via the isomorphism $H^0(\mathcal{O}(1))^* \to H^0(T_{\mathbb{P}^3}(-1))$. Let s be an element of $H^0(T_{\mathbb{P}^3}(-1))$, and suppose that $s = \sum_{i=0}^{3} a_i e_i^*$ for some $a_i \in K$. Note that $e_0^* = -\sum_{i=1}^{3}(x_i/x_0)e_i^*$ and (e_1^*, e_2^*, e_3^*) is a local basis of $T_{\mathbb{P}^3}(-1)$ over $D_+(x_0)$, where $D_+(x_0)$ is the open set defined by $x_0 \neq 0$. Thus $s = \sum_{i=1}^{3}\{a_i - a_0(x_i/x_0)\}e_i^*$ over $D_+(x_0)$. Let (e_1, e_2, e_3) be the dual basis of $\Omega_{\mathbb{P}^3}(1)|_{D_+(x_0)}$. Then the Koszul morphism φ_s from $\Omega_{\mathbb{P}^3}^2(2)$ to $\Omega_{\mathbb{P}^3}(1)$ corresponding to the section s has the representation matrix

$$
\begin{bmatrix}
0 & a_3 - a_0(x_3/x_0) & -(a_2 - a_0(x_2/x_0)) \\
-(a_3 - a_0(x_3/x_0)) & 0 & a_1 - a_0(x_1/x_0) \\
a_2 - a_0(x_2/x_0) & -(a_1 - a_0(x_1/x_0)) & 0
\end{bmatrix}
\tag{6.25}
$$

with respect to the bases $(e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2)$ and (e_1, e_2, e_3) over $D_+(x_0)$. Denote this matrix by $M(a_0, a_1, a_2, a_3)$.

Lemma 4.2 enables us to assume, by taking homogeneous coordinates $(x_0: x_1: x_2: x_3)$ of $\mathbb{P}^3$ suitably, that φ_i corresponds to the global section e_i^* of $T_{\mathbb{P}^3}(-1)$ for $i = 0, 1$.

In case $(h^0(\mathcal{E}(-1)), h^1(\mathcal{E}(-1))) = (1, 2)$, Lemma 4.2 furthermore enables us to assume the following: if dim $V = 4$, φ_2 and φ_3 corresponds respectively to e_2^* and e_3^*; if dim $V = 3$ φ_3 corresponds to e_3^*. Note here that if dim $V = 3$ then φ_2 corresponds to e_1^* since $\varphi_2 = \varphi_1$.

We claim here that coker($[\varphi_0, \varphi_1]$) : $\Omega_{\mathbb{P}^3}(1) \to \text{Coker}([\varphi_0, \varphi_1])$ is the composite of the two restrictions $\Omega_{\mathbb{P}^3}(1) \to \Omega_{\mathbb{P}^3}(1)|_L$ and $\Omega_{\mathbb{P}^3}(1)|_L \to \Omega_L(1)$ where L is the line defined by $x_2 = x_3 = 0$. Note first that the morphism $[\varphi_0, \varphi_1] : \Omega_{\mathbb{P}^3}^2(2)^{\oplus 2} \to \Omega_{\mathbb{P}^3}(1)$ has the representation matrix

$$
\begin{bmatrix} M(1, 0, 0, 0) & M(0, 1, 0, 0) \end{bmatrix} =
\begin{bmatrix}
0 & -x_3/x_0 & x_2/x_0 & 0 & 0 & 0 \\
x_3/x_0 & 0 & -x_1/x_0 & 0 & 0 & 1 \\
-x_2/x_0 & x_1/x_0 & 0 & 0 & -1 & 0
\end{bmatrix}
$$

with respect to the bases $(e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2, e_2' \wedge e_3', e_3' \wedge e_1', e_1' \wedge e_2')$ and (e_1, e_2, e_3) over $D_+(x_0)$, where (e_1', e_2', e_3') is the dual basis of the second component $\Omega_{\mathbb{P}^3}(1)|_{D_+(x_0)}$ of the direct sum $\Omega_{\mathbb{P}^3}^2(2)^{\oplus 2}$. Since $e_i = x_0 d(x_i/x_0)$ for $i = 1, 2, 3$, the image $\text{Im}[\varphi_0, \varphi_1]|_{D_+(x_0)}$ of $[\varphi_0, \varphi_1]|_{D_+(x_0)}$ is generated by $x_0 d(x_2/x_0)$, $x_0 d(x_3/x_0)$, $x_2 d(x_1/x_0)$, and $x_3 d(x_1/x_0)$. Note that $x_2 d(x_i/x_0) = (x_2/x_0)x_0 d(x_i/x_0)$ and $x_3 d(x_i/x_0) = (x_3/x_0)x_0 d(x_i/x_0)$ $(1 \leq i \leq 3)$ and that the quotient of $\Omega_{\mathbb{P}^3}(1)|_{D_+(x_0)}$ by the submodule generated by $x_2 d(x_i/x_0)$ and $x_3 d(x_i/x_0)$ $(1 \leq i \leq 3)$ is $(\Omega_{\mathbb{P}^3}(1) \otimes \mathcal{O}_L)|_{D_+(x_0)}$, where L is the line defined by $x_2 = x_3 = 0$. Hence Coker($[\varphi_0, \varphi_1]$)$|_{D_+(x_0)}$ is nothing but the quotient $\Omega_L(1)|_{D_+(x_0)}$ of $(\Omega_{\mathbb{P}^3}(1) \otimes \mathcal{O}_L)|_{D_+(x_0)}$ by the submodule generated by $x_0 d(x_2/x_0) \otimes 1$ and $x_0 d(x_3/x_0) \otimes 1$. Similarly Coker($[\varphi_0, \varphi_1]$)$|_{D_+(x_1)}$ is naturally isomorphic to $\Omega_L(1)|_{D_+(x_1)}$. Moreover Coker($[\varphi_0, \varphi_1]$)$|_{D_+(x_i)} = 0$ for $i = 2$ and 3. Therefore the claim holds.

6.1.2 The case where $(h^0(\mathcal{E}(-1)), h^1(\mathcal{E}(-1))) = (0, 1)$

Since $\varphi = [\varphi_0, \varphi_1]$, we see by the claim above that coker(φ) : $\Omega_{\mathbb{P}^3}(1) \to E_2^{-1,1}$ is nothing but the composite $\Omega_{\mathbb{P}^3}(1) \to \Omega_{\mathbb{P}^3}(1)|_L \to \Omega_L(1)$ of the restrictions, where L is a line. Therefore $E_2^{-1,1} \cong \mathcal{O}_L(-1)$. Then the sequence (6.21) implies that $E_3^{-1,1}$ is either $\mathcal{O}_L(-1)$ or zero, but (6.23) and the nefness of $\mathcal{E}$ imply that the former cannot happen. Therefore

$E_3^{-1,1} = 0$ and $E_4^{0,0} \cong \mathcal{E}$; thus we obtain a surjection $\mathcal{O}^{\oplus r+3} \to \mathcal{E}$ by (6.19) and (6.22). In fact, this cannot happen, as can be seen below.

Denote by $\mathcal{H}$ the kernel of the surjection $\mathcal{O}^{\oplus r+3} \to \mathcal{E}$; $\mathcal{H}$ is a vector bundle of rank three. Note here that the dual $\mathcal{H}^\vee$ of $\mathcal{H}$ is globally generated. Moreover $c_1(\mathcal{H}^\vee) = 3$ and

$$c_2(\mathcal{H}^\vee) = c_2(\mathcal{H}) = -c_1(\mathcal{H})c_1 - c_2 = 4.$$

Furthermore

$$c_3(\mathcal{H}^\vee) = -c_3(\mathcal{H}) = c_2(\mathcal{H})c_1 + c_1(\mathcal{H})c_2 + c_3 = 0.$$

Since $\mathcal{H}^\vee$ is a globally generated vector bundle of rank three with $c_3(\mathcal{H}^\vee) = 0$, we have an exact sequence of vector bundles

$$0 \to \mathcal{O} \to \mathcal{H}^\vee \to \mathcal{H}' \to 0.$$

Then it follows from [5, Proposition 2.4] that $\mathcal{H}' \cong \mathcal{O}(1) \oplus \mathcal{O}(2)$. This however contradicts that $c_2(\mathcal{H}^\vee) = 4$. The case $(h^0(\mathcal{E}(-1)), h^1(\mathcal{E}(-1))) = (0, 1)$ thus cannot happen.

6.1.3 The case where $(h^0(\mathcal{E}(-1)), h^1(\mathcal{E}(-1))) = (1, 2)$

As we have shown in §6.1.1, the composite $[\varphi_0, \varphi_1]$ of φ and the first projection $\Omega_{\mathbb{P}^3}(1)^{\oplus 2} \to \Omega_{\mathbb{P}^3}(1)$ has $\Omega_L(1)$ as its cokernel, where L is the line defined by $x_2 = x_3 = 0$. Therefore $E_2^{-1,1}$ has $\mathcal{O}_L(-1)$ as a quotient by (6.17).

It follows from (6.25) that φ has, with respect to the bases $(e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2, e_2' \wedge e_3', e_3' \wedge e_1', e_1' \wedge e_2')$ and $(e_1, e_2, e_3, e_1', e_2', e_3')$ over $D_+(x_0)$, the representation matrix

$$\begin{bmatrix} M(1, 0, 0, 0) & M(0, 1, 0, 0) \\ M(0, 0, 1, 0) & M(0, 0, 0, 1) \end{bmatrix}$$

if $\dim V = 4$, and the representation matrix

$$\begin{bmatrix} M(1, 0, 0, 0) & M(0, 1, 0, 0) \\ M(0, 1, 0, 0) & M(0, 0, 0, 1) \end{bmatrix}$$

if $\dim V = 3$. Therefore $\mathrm{Coker}(\varphi)$ is supported on the quadric surface defined by $x_0x_3 - x_1x_2 = 0$ if $\dim V = 4$ and by $x_2^2 = 0$ if $\dim V = 3$. This implies that φ is injective. Hence $\mathcal{M} = 0$ by (6.17); it follows from (6.18) that $a = 0$ and $E_2^{-2,1} = 0$; thus $E_2^{-1,1} \cong E_3^{-1,1}$ by (6.21). Since $E_2^{-1,1}$ has $\mathcal{O}_L(-1)$ as a quotient, so does $E_3^{-1,1}$. However $E_3^{-1,1}$ cannot have $\mathcal{O}_L(-1)$ as a quotient by (6.10) and (6.24) since $\mathcal{E}$ is nef. Therefore we conclude that the case $(h^0(\mathcal{E}(-1)), h^1(\mathcal{E}(-1))) = (1, 2)$ does not happen.

7 Proof of Theorem 1

As we stated at the end of §1, we always assume that $c_2 = 9$ in this section.

7.1 Proof for the case $n = 2$ and $h^1(\mathcal{E}) = 0$

It follows from (3.10), (3.11), and (3.12) that

$$h^1(\mathcal{E}(-2)) = 6, \tag{7.1}$$

$$h^0(\mathcal{E}(-1)) - h^1(\mathcal{E}(-1)) = -3, \tag{7.2}$$

$$h^0(\mathcal{E}) = r. \tag{7.3}$$

As in §3.1, set $e_{0,1} = h^0(\mathcal{E}(-1))$. Then

$$e_{0,1} \leq 1. \tag{7.4}$$

The proof of this fact goes almost identical to that in case $c_2 = 8$, i.e., that in [12, § 3.1]: what we have to do is just to change several numerals related to $c_2 = 8$ to those related to $c_2 = 9$; so we omit the proof of this fact.

We apply to $\mathcal{E}$ the Bondal spectral sequence (2.1). As we have seen in §3.1 and Lemma 5.1 (2), $E_2^{p,q}$ vanishes unless $(p, q) = (-2, 1), (-1, 1)$ or $(0, 0)$, and $E_2^{-2,1}$ and $E_2^{-1,1}$ fit in one of the following exact sequences:

$$0 \to E_2^{-2,1} \to \mathcal{O}(-3) \oplus \mathcal{O}(-1) \xrightarrow{\nu_2} \Omega_{\mathbb{P}^2}(1)^{\oplus e_{0,1}+1} \to E_2^{-1,1} \to k(w) \to 0; \tag{7.5}$$

$$0 \to E_2^{-2,1} \to \mathcal{O}(-2)^{\oplus 2} \xrightarrow{\mu_2} \Omega_{\mathbb{P}^2}(1)^{\oplus e_{0,1}+1} \to E_2^{-1,1} \to 0, \tag{7.6}$$

where $k(w)$ denotes the residue field of some point w in $\mathbb{P}^2$. Recall that each of the exact sequences is a consequence of the assumption $H^1(\mathcal{E}) = 0$. Note that $E_2^{0,0}$ fits in the following exact sequence by (3.17)

$$0 \to \mathcal{O}^{\oplus 3e_{0,1}} \to \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus r} \to E_2^{0,0} \to 0. \tag{7.7}$$

Lemma 7.1 *If $E_2^{-2,1} = 0$, then $E_2^{0,0} \cong E_3^{0,0} \cong \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus r - 3e_{0,1}}$, and $\mathcal{E}$ fits in the following exact sequence*

$$0 \to \mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus r - 3e_{0,1}} \to \mathcal{E} \to E_2^{-1,1} \to 0. \tag{7.8}$$

Moreover $H^i(E_2^{-1,1}) = 0$ for all i; thus $E_2^{-1,1}$ is the cokernel of the evaluation map $H^0(\mathcal{E}) \otimes \mathcal{O} \to \mathcal{E}$.

Proof Since $E_2^{-2,1} = 0$, $E_2^{0,0} \cong E_3^{0,0}$ by (3.15), and hence $E_2^{0,0}$ is torsion-free by (3.16). Therefore (7.7) shows that $\mathcal{O}(1)$ is a subsheaf of $E_2^{0,0}$ if $e_{0,1} = 1$, and consequently we see that $E_2^{0,0}$ is isomorphic to $\mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus r - 3e_{0,1}}$. Hence it follows from (3.16) that $\mathcal{E}$ fits in the desired exact sequence (7.8). Since we have (7.3), $H^0(\mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus r - 3e_{0,1}})$ is isomorphic to $H^0(\mathcal{E})$, and thus $H^0(E_2^{-1,1})$ vanishes. Furthermore $H^i(E_2^{-1,1}) = 0$ for all i since $h^1(\mathcal{E}) = 0$ by assumption and $h^2(\mathcal{E}) = 0$ by (3.4). Now we have the following commutative diagram

$$
\begin{array}{ccc}
H^0(\mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus r - 3e_{0,1}}) \otimes \mathcal{O} & \longrightarrow & H^0(\mathcal{E}) \otimes \mathcal{O} \\
\downarrow & & \downarrow \\
\mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus r - 3e_{0,1}} & \longrightarrow & \mathcal{E}
\end{array}
$$

where the vertical arrows are the evaluation maps. Since the upper horizontal arrow is an isomorphism, the lower one is injective, and the left vertical arrow is surjective, we infer that $\mathcal{O}(1)^{\oplus e_{0,1}} \oplus \mathcal{O}^{\oplus r - 3e_{0,1}}$ is the image of the evaluation map $H^0(\mathcal{E}) \otimes \mathcal{O} \to \mathcal{E}$. Therefore it follows from (7.8) that $E_2^{-1,1}$ is the cokernel of the evaluation map $H^0(\mathcal{E}) \otimes \mathcal{O} \to \mathcal{E}$. $\square$

Let $\alpha_{ij} : \mathcal{O}(-2) \to \Omega_{\mathbb{P}^2}(1)$ be the composite of the jth inclusion $\mathcal{O}(-2) \to \mathcal{O}(-2)^{\oplus 2}$, μ_2, and the ith projection $\Omega_{\mathbb{P}^2}(1)^{\oplus e_{0,1}+1} \to \Omega_{\mathbb{P}^2}(1)$. The α_{ij}'s form a matrix $[\alpha_{ij}]$, which is another expression of μ_2.

Lemma 7.2 *For each i, α_{i1} and α_{i2} are linearly independent. Similarly, for each j, elements in the jth column are linearly independent. Furthermore the cokernel of the composite $[\alpha_{11}, \alpha_{12}]$ of μ_2 and the first projection $\Omega_{\mathbb{P}^2}(1)^{\oplus e_{0,1}+1} \to \Omega_{\mathbb{P}^2}(1)$ is supported on a cubic curve E, and $[\alpha_{11}, \alpha_{12}]$ and μ_2 are injective. In particular $E_2^{-2,1} = 0$ in* (7.6).

Proof First note that $E_2^{-1,1}$ in (7.6) has as a quotient the cokernel of the composite, say ψ, of μ_2 and any surjection $\Omega_{\mathbb{P}^2}(1)^{e_{0,1}+1} \to \Omega_{\mathbb{P}^2}(1)$. Since $E_2^{-1,1}$ cannot have $\Omega_{\mathbb{P}^2}(1)$ as a quotient by Lemma 3.1, we observe that the composite ψ cannot be zero.

Suppose that elements in one of the rows or the columns are linearly dependent. After some row or column elementary transformations, by replacing α_{ij}, we may then assume that $\alpha_{11} \neq 0$ and that $\alpha_{12} = 0$ by the observation above. The morphism $[\alpha_{11}, 0]$ factors through the first projection $\mathcal{O}(-2)^{\oplus 2} \to \mathcal{O}(-2)$, and $\mathrm{Coker}([\alpha_{11}, 0]) = \mathrm{Coker}(\alpha_{11})$. If α_{11} factors through $\mathcal{O}(-1)$, the cokernel of the induced morphism $\mathcal{O}(-1) \to \Omega_{\mathbb{P}^2}(1)$ is isomorphic to $\mathcal{I}_p$ for some point $p \in \mathbb{P}^2$ and $\mathrm{Coker}(\alpha_{11})$ fits in the following exact sequence

$$0 \to \mathcal{O}_L(-1) \to \mathrm{Coker}(\alpha_{11}) \to \mathcal{I}_p \to 0,$$

where $\mathcal{O}_L(-1)$ is the cokernel of the induced injection $\mathcal{O}(-2) \to \mathcal{O}(-1)$. In particular $\mathcal{O}_L(-1)$ is the torsion subsheaf of $\mathrm{Coker}(\alpha_{11})$, and $\mathcal{I}_p$ is the quotient of $\mathrm{Coker}(\alpha_{11})$ by its torsion subsheaf. This contradicts Lemma 3.1. Therefore α_{11} does not factor through $\mathcal{O}(-1)$; this implies that $\mathrm{Coker}(\alpha_{11})$ is isomorphic to $\mathcal{I}_Z(1)$, where Z is a 0-dimensional closed subscheme of length three on $\mathbb{P}^2$. This however cannot happen by Lemma 3.1. Therefore elements in the same row or column are linearly independent.

Since α_{11} and α_{12} are linearly independent, the morphism $[\alpha_{11}, \alpha_{12}]$ induces a non-zero morphism $\bar{\alpha}_{12} : \mathcal{O}(-2) \to \mathrm{Coker}(\alpha_{11})$, and we see that $\mathrm{Coker}([\alpha_{11}, \alpha_{12}]) \cong \mathrm{Coker}(\bar{\alpha}_{12})$. Denote by $\mathrm{Coker}(\alpha_{11})/\mathrm{tors}$ the quotient of $\mathrm{Coker}(\alpha_{11})$ by its torsion subsheaf. As we have seen above, $\mathrm{Coker}(\alpha_{11})/\mathrm{tors}$ is either $\mathcal{I}_p$ or $\mathcal{I}_Z(1)$. If $\mathrm{Coker}(\alpha_{11})/\mathrm{tors} = \mathcal{I}_p$, then $\mathrm{Coker}(\bar{\alpha}_{12})$ admits a negative degree line bundle on a conic as a quotient, which is impossible by Lemma 3.1. Hence $\mathrm{Coker}(\alpha_{11}) = \mathcal{I}_Z(1)$; thus $\mathrm{Coker}(\bar{\alpha}_{12})$ is supported on a cubic curve E passing through Z. Therefore $\bar{\alpha}_{12}$ is injective. Since α_{11} is also injective, so is $[\alpha_{11}, \alpha_{12}]$. Hence μ_2 is injective, and thus $E_2^{-2,1} = 0$. $\square$

Lemma 7.3 *If $E_2^{-1,1}$ fits in the exact sequence* (7.6), *then it fits in the following exact sequence*

$$0 \to \mathcal{O}(-2)^{\oplus e_{0,1}+3} \to \mathcal{O}(-1)^{\oplus 3e_{0,1}+3} \to E_2^{-1,1} \to 0. \tag{7.9}$$

Proof By Lemma 7.2, $E_2^{-2,1} = 0$, and thus $E_2^{-1,1}$ fits in the following exact sequence

$$0 \to \mathcal{O}(-2)^{\oplus 2} \xrightarrow{\mu_2} \Omega_{\mathbb{P}^2}(1)^{\oplus e_{0,1}+1} \to E_2^{-1,1} \to 0. \tag{7.10}$$

Since $\Omega_{\mathbb{P}^2}(1)$ is isomorphic to $T_{\mathbb{P}^2}(-2)$, the Euler sequence and the sequence above induce the desired exact sequence (7.9). $\square$

Denote by $\mathcal{K}$ the kernel of the surjection $E_2^{-1,1} \to k(w)$ in (7.5).

Lemma 7.4 *The sheaf $\mathcal{K}$ cannot admit the following sheaves as a quotient:*

(1) $\Omega_{\mathbb{P}^2}(1)$;

(2) $\mathcal{I}_W(1)$, *where W is a* 0-*dimensional closed subscheme of* length $W \geq 3$;
(3) $\mathcal{I}_W(2)$, *where W is a* 0-*dimensional closed subscheme of* length $W \geq 6$.

Proof If $\mathcal{K}$ admits $\Omega_{\mathbb{P}^2}(1)$ as a quotient, then $E_2^{-1,1}|_L$ admits $\mathcal{O}_L(-1)$ as a quotient for a line L not passing through w. This contradicts Lemma 3.1. Suppose that $\mathcal{K}$ admits $\mathcal{I}_W(d)$ as a quotient, where length $W \geq 3$ if $d = 1$ and length $W \geq 6$ if $d = 2$. Then we obtain the following exact sequence

$$0 \to \mathcal{I}_W(d) \to \mathcal{D} \to k(w) \to 0$$

for some quotient sheaf $\mathcal{D}$ of $E_2^{-1,1}$. Note that this sequence does not split since $E_2^{-1,1}$ does not admit $\mathcal{I}_W(d)$ as a quotient by Lemma 3.1. Lemma 4.3 then implies that $\mathcal{D} \cong \mathcal{I}_Z(d)$, where Z is a 0-dimensional closed subscheme of length $Z = $ length $W - 1$. However $E_2^{-1,1}$ cannot admit $\mathcal{I}_Z(d)$ as a quotient by Lemma 3.1. $\qquad \square$

Let $\beta_i : \mathcal{O}(-1) \to \Omega_{\mathbb{P}^2}(1)$ be the composite of the inclusion $\mathcal{O}(-1) \to \mathcal{O}(-3) \oplus \mathcal{O}(-1)$, v_2, and the ith projection $\Omega_{\mathbb{P}^2}(1)^{\oplus e_{0,1}+1} \to \Omega_{\mathbb{P}^2}(1)$, and let $\gamma_i : \mathcal{O}(-3) \to \Omega_{\mathbb{P}^2}(1)$ be the composite of the inclusion $\mathcal{O}(-3) \to \mathcal{O}(-3) \oplus \mathcal{O}(-1)$, v_2, and the ith projection $\Omega_{\mathbb{P}^2}(1)^{\oplus e_{0,1}+1} \to \Omega_{\mathbb{P}^2}(1)$. We shall denote by $\mathrm{Coker}(\gamma_i)/\mathrm{tors}$ the quotient of $\mathrm{Coker}(\gamma_i)$ by its torsion subsheaf.

Lemma 7.5 *If $\gamma_i \neq 0$, then* $\mathrm{Coker}(\gamma_i)/\mathrm{tors}$ *is isomorphic to* $\mathcal{I}_W(d)$ *for some* 0-*dimensional closed subscheme W of $\mathbb{P}^2$ and some integer d, and the possible values of the pair* (length W, d) *are* (7, 2), (3, 1), *or* (1, 0).

Proof Suppose that γ_i factors through $\mathcal{O}(-1)$. The cokernel of the induced morphism $\mathcal{O}(-1) \to \Omega_{\mathbb{P}^2}(1)$ is isomorphic to $\mathcal{I}_p$ for some point $p \in \mathbb{P}^2$, and $\mathrm{Coker}(\gamma_i)$ fits in the following exact sequence

$$0 \to \mathcal{O}_C(-1) \to \mathrm{Coker}(\gamma_i) \to \mathcal{I}_p \to 0,$$

where C is a conic and $\mathcal{O}_C(-1)$ is the cokernel of the induced morphism $\mathcal{O}(-3) \to \mathcal{O}(-1)$. Hence (length W, d) = (1, 0) in this case.

Suppose that γ_i does not factor through $\mathcal{O}(-1)$ and that it factors through $\mathcal{O}(-2)$. Then the induced morphism $\mathcal{O}(-2) \to \Omega_{\mathbb{P}^2}(1)$ does not factor through $\mathcal{O}(-1)$ and the cokernel of the morphism $\mathcal{O}(-2) \to \Omega_{\mathbb{P}^2}(1)$ is isomorphic to $\mathcal{I}_Z(1)$ for some closed subscheme $Z \subset \mathbb{P}^2$ of length three. Moreover $\mathrm{Coker}(\gamma_i)$ fits in the following exact sequence

$$0 \to \mathcal{O}_L(-2) \to \mathrm{Coker}(\gamma_i) \to \mathcal{I}_Z(1) \to 0,$$

where L is the line determined by the induced morphism $\mathcal{O}(-3) \to \mathcal{O}(-2)$ and $\mathcal{O}_L(-2)$ is the cokernel of that morphism. Hence (length W, d) = (3, 1) in this case.

Finally suppose that γ_i does not factor through $\mathcal{O}(-1)$ nor $\mathcal{O}(-2)$. Then γ_i induces a regular section $s \in H^0(\Omega_{\mathbb{P}^2}(4))$, whose zero locus $(s)_0$ has length seven. Hence $\mathrm{Coker}(\gamma_i)$ is isomorphic to $\mathcal{I}_W(2)$, where $W = (s)_0$. Therefore (length W, d) = (7, 2) in this case. $\qquad \square$

Lemma 7.6 *For any i, β_i is non-zero.*

Proof Suppose, to the contrary, that $\beta_1 = 0$. Then the composite $[\gamma_1, \beta_1]$ of v_2 and the first projection $\Omega_{\mathbb{P}^2}(1)^{e_{0,1}+1} \to \Omega_{\mathbb{P}^2}(1)$ factors through $\mathcal{O}(-3)$ via the projection $\mathcal{O}(-3) \oplus \mathcal{O}(-1) \to \mathcal{O}(-3)$, and $\mathrm{Coker}([\gamma_1, 0])$ is isomorphic to $\mathrm{Coker}(\gamma_1)$. Note that $\mathrm{Coker}(\gamma_1)$ is a quotient of $\mathcal{K}$. It follows from Lemma 7.4 that $\gamma_1 \neq 0$. Then $\mathrm{Coker}(\gamma_1)/\mathrm{tors} \cong \mathcal{I}_W(d)$ for some 0-dimensional closed subscheme W of $\mathbb{P}^2$, and the possible values of the pair

(length W, d) are (7, 2), (3, 1), or (1, 0) by Lemma 7.5. However the values (7, 2) and (3, 1) are ruled out by Lemma 7.4. Hence (length W, d) = (1, 0) and thus the torsion subsheaf of Coker(γ_1) is isomorphic to $\mathcal{O}_C(-1)$, where C is a conic on $\mathbb{P}^2$. Set $\{p\} = W$. There exists the following commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & E_2^{-1,1} & \longrightarrow & k(w) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Coker}(\gamma_1) & \longrightarrow & \mathcal{G} & \longrightarrow & k(w) & \longrightarrow & 0
\end{array}
$$

for some quotient coherent sheaf $\mathcal{G}$ of $E_2^{-1,1}$. Denote by $\mathcal{D}$ the cokernel of the composite of the two inclusions $\mathcal{O}_C(-1) \to \mathrm{Coker}(\gamma_1)$ and $\mathrm{Coker}(\gamma_1) \to \mathcal{G}$. Then we have the following commutative diagram with exact rows.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Coker}(\gamma_1) & \longrightarrow & \mathcal{G} & \longrightarrow & k(w) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{I}_p & \longrightarrow & \mathcal{D} & \longrightarrow & k(w) & \longrightarrow & 0
\end{array}
$$

Note here that the bottom row of the diagram above does not split since $E_2^{-1,1}$ cannot admit $\mathcal{I}_p$ as a quotient by Lemma 3.1. Therefore Lemma 4.3 implies that $w = p$ and that $\mathcal{D}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}$. Hence $\mathcal{G}$ fits in the following exact sequence

$$0 \to \mathcal{O}_C(-1) \to \mathcal{G} \to \mathcal{O}_{\mathbb{P}^2} \to 0.$$

Let $\mathcal{E}_1$ be the kernel of the composite of the two surjections $\mathcal{E} \to \mathcal{G}$ and $\mathcal{G} \to \mathcal{O}_{\mathbb{P}^2}$. Then $\mathcal{O}_C(-1)$ is a quotient of $\mathcal{E}_1$. Hence $\mathcal{E}_1|_C$ admits $\mathcal{O}_C(-1)$ as a quotient, and it also fits in the following exact sequence

$$0 \to \mathcal{E}_1|_C \to \mathcal{E}|_C \to \mathcal{O}_C \to 0.$$

Since C is a conic, this implies that $\mathcal{E}|_C$ admits a negative degree quotient, which contradicts that $\mathcal{E}$ is nef. $\qquad\square$

There exists the natural composition

$$\mathrm{Hom}(\mathcal{O}(-1), \Omega_{\mathbb{P}^2}(1)) \times \mathrm{Hom}(\mathcal{O}(-3), \mathcal{O}(-1)) \to \mathrm{Hom}(\mathcal{O}(-3), \Omega_{\mathbb{P}^2}(1)),$$

and we have two cases:

- γ_1 has β_1 as a factor;
- γ_1 does not have β_1 as a factor.

Lemma 7.7 *If γ_1 does not have β_1 as a factor, then $E_2^{-2,1} = 0$ and $E_2^{-1,1}$ fits in (7.8) and (7.9).*

Proof If γ_1 does not have β_1 as a factor, then $[\gamma_1, \beta_1]$ induces a non-zero morphism $\bar{\gamma}_1 : \mathcal{O}(-3) \to \mathrm{Coker}(\beta_1)$. Since $\beta_1 \neq 0$ by Lemma 7.6, this implies that $[\gamma_1, \beta_1]$ is injective. Hence ν_2 is injective and thus $E_2^{-2,1} = 0$. Lemma 7.1 then shows that $E_2^{-1,1}$ lies in (7.8). Since $E_2^{-2,1} = 0$, $\mathcal{K}$ fits in the following exact sequence

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-1) \to \Omega_{\mathbb{P}^2}(1)^{\oplus e_{0,1}+1} \to \mathcal{K} \to 0.$$

This together with the isomorphism $\Omega_{\mathbb{P}^2}(1) \cong T_{\mathbb{P}^2}(-2)$ and the Euler sequence induces the following exact sequence

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^{\oplus e_{0,1}+1} \to \mathcal{O}(-1)^{\oplus 3e_{0,1}+2} \to \mathcal{K} \to 0.$$

Since we have an exact sequence

$$0 \to \mathcal{O}(-3) \to \mathcal{O}(-2)^{\oplus 2} \to \mathcal{O}(-1) \to k(w) \to 0, \tag{7.11}$$

the resolution of $\mathcal{K}$ above implies that $E_2^{-1,1}$ fits in the following exact sequence

$$0 \to \mathcal{O}(-3) \xrightarrow{d_2} \mathcal{O}(-3) \oplus \mathcal{O}(-2)^{\oplus e_{0,1}+3} \to \mathcal{O}(-1)^{\oplus 3e_{0,1}+3} \to E_2^{-1,1} \to 0. \tag{7.12}$$

Lemma 7.1 and this sequence (7.12) then show that $H^2(d_2)$ is an isomorphism; we infer that the composite of d_2 and the projection $\mathcal{O}(-3) \oplus \mathcal{O}(-2)^{\oplus e_{0,1}+3} \to \mathcal{O}(-3)$ is an isomorphism. Therefore the exact sequence (7.12) is reduced to the exact sequence (7.9). $\qquad\square$

Now if $E_2^{-1,1}$ lies in (7.6) or if $E_2^{-1,1}$ lies in (7.5) and γ_1 does not have β_1 as a factor, then $E_2^{-1,1}$ lies also in (7.8) and (7.9) by Lemmas 7.1, 7.2, 7.3, and 7.7. Hence if $e_{0,1} = 1$ then we obtain the following exact sequences:

$$0 \to \mathcal{O}(-2)^{\oplus 4} \to \mathcal{O}(-1)^{\oplus 6} \to E_2^{-1,1} \to 0; \tag{7.13}$$

$$0 \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-3} \to \mathcal{E} \to E_2^{-1,1} \to 0. \tag{7.14}$$

These two exact sequences imply that $\mathcal{E}$ fits in the exact sequence in case (18) of Theorem 1. If $e_{0,1} = 0$ then we obtain the following exact sequences:

$$0 \to \mathcal{O}(-2)^{\oplus 3} \to \mathcal{O}(-1)^{\oplus 3} \to E_2^{-1,1} \to 0; \tag{7.15}$$

$$0 \to \mathcal{O}^{\oplus r} \to \mathcal{E} \to E_2^{-1,1} \to 0. \tag{7.16}$$

These two exact sequences show that $\mathcal{E}$ fits in the exact sequence in case (16) of Theorem 1.

Question 1 Does there exist a nef vector bundle fitting in the exact sequence in case (18) of Theorem 1 ?

Remark 7.1 If $E_2^{-1,1}$ is in (7.15), then $E_2^{-1,1}$ is supported on a cubic curve.

Lemma 7.8 *If γ_1 has β_1 as a factor, then $e_{0,1} = 0$, $E_2^{-1,1} \cong \mathcal{O}$, and we have the following non-split exact sequences:*

$$0 \to \mathcal{O}(-3) \to \mathcal{O}^{\oplus r} \to E_3^{0,0} \to 0; \tag{7.17}$$

$$0 \to E_3^{0,0} \to \mathcal{E} \to \mathcal{O} \to 0. \tag{7.18}$$

In particular $E_2^{-1,1}$ is the cokernel of the evaluation map $H^0(\mathcal{E}) \otimes \mathcal{O} \to \mathcal{E}$.

Proof Since γ_1 has β_1 as a factor, by changing the free basis of $\mathcal{O}(-3) \oplus \mathcal{O}(-1)$, we may assume that $\gamma_1 = 0$.

Suppose that $e_{0,1} = 1$. Then we have the following commutative diagram with exact rows.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}(-3) & \longrightarrow & \mathcal{O}(-3) \oplus \mathcal{O}(-1) & \longrightarrow & \mathcal{O}(-1) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \nu_2} & & \downarrow{\scriptstyle \beta_1} & & \\
0 & \longrightarrow & \Omega_{\mathbb{P}^2}(1) & \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & \Omega_{\mathbb{P}^2}(1)^{\oplus 2} & \xrightarrow{[1,0]} & \Omega_{\mathbb{P}^2}(1) & \longrightarrow & 0
\end{array}
$$

Since $\beta_1 \neq 0$ by Lemma 7.6, β_1 is injective and $\text{Coker}(\beta_1) = \mathcal{I}_p$ for some point $p \in \mathbb{P}^2$. Hence the diagram above induces an exact sequence

$$0 \to \text{Coker}(\gamma_2) \to \mathcal{K} \to \mathcal{I}_p \to 0.$$

Denote by $\mathcal{D}$ the quotient of $E_2^{-1,1}$ by $\text{Coker}(\gamma_2)$. Then we obtain the following commutative diagram with exact rows.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & E_2^{-1,1} & \longrightarrow & k(w) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{I}_p & \longrightarrow & \mathcal{D} & \longrightarrow & k(w) & \longrightarrow & 0
\end{array}
$$

Since $E_2^{-1,1}$ cannot admit $\mathcal{I}_p$ as a quotient by Lemma 3.1, the bottom row of the diagram above does not split. Hence $p = w$ and $\mathcal{D}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}$ by Lemma 4.3. The definition of $\mathcal{D}$ induces the following exact sequence

$$0 \to \text{Coker}(\gamma_2) \to E_2^{-1,1} \to \mathcal{O} \to 0.$$

If $\gamma_2 = 0$, then $\text{Coker}(\gamma_2) \cong \Omega_{\mathbb{P}^2}(1)$, and thus $E_2^{-1,1} \cong \Omega_{\mathbb{P}^2}(1) \oplus \mathcal{O}$, but this is absurd by Lemma 3.1. Therefore $\gamma_2 \neq 0$, and it follows from Lemma 7.5 that $\text{Coker}(\gamma_2)/\text{tors}$ is isomorphic to $\mathcal{I}_W(d)$ for some 0-dimensional closed subscheme W of $\mathbb{P}^2$ and that the possible values of the pair (length W, d) are $(7, 2)$, $(3, 1)$, or $(1, 0)$. The sequence above induces an exact sequence

$$0 \to \mathcal{I}_W(d) \to E_2^{-1,1}/\text{tors} \to \mathcal{O} \to 0.$$

Now Lemma 4.4 implies that there exists a smooth rational curve C such that the dth twist $\mathcal{I}_W \cdot \mathcal{O}_C(d)$ of the inverse image ideal sheaf $\mathcal{I}_W \cdot \mathcal{O}_C$ is a negative degree line bundle. This implies that $E_2^{-1,1}|_C$ admits a negative degree line bundle as a direct summand, which contradicts Lemma 3.1. Therefore the case $e_{0,1} = 1$ cannot happen.

Suppose that $e_{0,1} = 0$. Since $\beta_1 \neq 0$ by Lemma 7.6 and $\gamma_1 = 0$ by assumption, the exact sequence (7.5) shows that $E_2^{-2,1} \cong \mathcal{O}(-3)$ and that $\mathcal{K} \cong \text{Coker}(\beta_1) \cong \mathcal{I}_p$. Since the sequence

$$0 \to \mathcal{I}_p \to E_2^{-1,1} \to k(w) \to 0$$

does not split by Lemma 3.1, we see that $p = w$ and that

$$E_2^{-1,1} \cong \mathcal{O}_{\mathbb{P}^2}$$

by Lemma 4.3. Note here that $E_2^{0,0} \cong \mathcal{O}^{\oplus r}$ by (7.7). Hence we obtain the desired exact sequences (7.17) and (7.18) by (3.15) and (3.16). Moreover the sequence (7.18) does not split since $H^0(E_3^{0,0}) \cong H^0(\mathcal{E})$ by (7.3) and (7.17). Furthermore the isomorphism $H^0(E_3^{0,0}) \cong H^0(\mathcal{E})$ together with the global generation of $E_3^{0,0}$ by (7.17) implies that $E_3^{0,0}$ is the image of the evaluation map $H^0(\mathcal{E}) \otimes \mathcal{O} \to \mathcal{E}$ by the same argument at the end of the proof of Lemma 7.1. Therefore $E_2^{-1,1}$, which is isomorphic to $\mathcal{O}$, is the cokernel of the evaluation map $H^0(\mathcal{E}) \otimes \mathcal{O} \to \mathcal{E}$ by (7.18). $\qquad\square$

Suppose that γ_1 has β_1 as a factor. We shall give a resolution of $\mathcal{E}$ in terms of a full strong exceptional collection $(\mathcal{O}(-3), \mathcal{O}(-2), \mathcal{O}(-1))$ of line bundles on $\mathbb{P}^2$. By composing the Euler exact sequence and the dual of that, we obtain the following exact sequence

$$0 \to \mathcal{O}(-3) \to \mathcal{O}(-2)^{\oplus 3} \to \mathcal{O}(-1)^{\oplus 3} \to \mathcal{O} \to 0. \tag{7.19}$$

Hence we obtain the following exact sequence

$$0 \to \mathcal{O}(-3)^{\oplus r} \to \mathcal{O}(-2)^{\oplus 3r} \to \mathcal{O}(-1)^{\oplus 3r} \to \mathcal{O}^{\oplus r} \to 0.$$

The exact sequence (7.17) then induces the following exact sequence

$$0 \to \mathcal{O}(-3)^{\oplus r} \to \mathcal{O}(-2)^{\oplus 3r} \oplus \mathcal{O}(-3) \to \mathcal{O}(-1)^{\oplus 3r} \to E_3^{0,0} \to 0. \qquad (7.20)$$

The sequence (7.18) together with (7.20) and (7.19) induces the following resolution of $\mathcal{E}$:

$$0 \to \mathcal{O}(-3)^{\oplus r+1} \xrightarrow{d_2} \mathcal{O}(-2)^{\oplus 3r+3} \oplus \mathcal{O}(-3) \to \mathcal{O}(-1)^{\oplus 3r+3} \to \mathcal{E} \to 0.$$

Since $h^1(\mathcal{E}) = 0$ by assumption, the linear map $H^2(d_2)$ is surjective. Hence the composite of d_2 and the projection $\mathcal{O}(-2)^{\oplus 3r+3} \oplus \mathcal{O}(-3) \to \mathcal{O}(-3)$ is non-zero and thus a surjection. Therefore the resolution above is reduced to that in case (17) of Theorem 1. This completes the proof of Theorem 1 for the case $n = 2$, $c_2 = 9$, and $h^1(\mathcal{E}) = 0$.

Here we give the following two remarks about the property of $E_2^{-1,1}$ in (7.13), but they do not effect the rest of the proof at all, so the reader may skip them.

Remark 7.2 Since $\mathcal{O}(1)^{\oplus 6}$ is globally generated of rank six, a general morphism from $\mathcal{O}(-2)^{\oplus 4}$ to $\mathcal{O}(-1)^{\oplus 6}$ is a subbundle morphism, and hence its cokernel is a vector bundle. On the other hand, $E_2^{-1,1}$ is not a vector bundle; indeed, since $E_2^{-1,1}$ lies also in (7.14), we see $c_1(E_2^{-1,1}) = 2$ and $c_2(E_2^{-1,1}) = 7$, and if $E_2^{-1,1}$ were a vector bundle, it would be a nef vector bundle, so that $c_2(E_2^{-1,1}) \leq c_1(E_2^{-1,1})^2 = 4$, which is absurd. Therefore the morphism $\mathcal{O}(-2)^{\oplus 4} \to \mathcal{O}(-1)^{\oplus 6}$ in (7.13) is not a general morphism.

Recall that (7.13) comes from the exact sequence (7.10) with $e_{0,1} = 1$:

$$0 \to \mathcal{O}(-2)^{\oplus 2} \xrightarrow{\mu_2} \Omega_{\mathbb{P}^2}(1)^{\oplus 2} \to E_2^{-1,1} \to 0.$$

Remark 7.3 We have the following commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}(-2)^{\oplus 2} & \xrightarrow{\mu_2} & \Omega_{\mathbb{P}^2}(1)^{\oplus 2} & \longrightarrow & E_2^{-1,1} & \longrightarrow & 0 \\
& & \| & & {\scriptstyle [1,0]}\downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-2)^{\oplus 2} & \xrightarrow{[\alpha_{11},\alpha_{12}]} & \Omega_{\mathbb{P}^2}(1) & \longrightarrow & \mathrm{Coker}([\alpha_{11},\alpha_{12}]) & \longrightarrow & 0
\end{array}
$$

and an exact sequence

$$0 \to \Omega_{\mathbb{P}^2}(1) \to E_2^{-1,1} \to \mathrm{Coker}([\alpha_{11},\alpha_{12}]) \to 0.$$

Since α_{11} and α_{12} are linearly independent and so are α_{11} and α_{21}, there exist no elements $\lambda \in K$ and $\lambda_{ij} \in K$ ($1 \leq i, j \leq 2$) such that

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} \end{bmatrix}.$$

Hence no splitting injection ${}^t[1,\lambda]$ ($\lambda \in K$) of the projection $[1,0]$ induces a splitting injection of the surjection $E_2^{-1,1} \to \mathrm{Coker}([\alpha_{11},\alpha_{12}])$.

7.2 The sheaf $E_2^{-1,1}$ in (7.14) is torsion-free

We shall apply the following lemma in §7.3.

Lemma 7.9 *The sheaf $E_2^{-1,1}$ in (7.14) is a torsion-free sheaf of rank two with $c_1(E_2^{-1,1}) = 2$ and $c_2(E_2^{-1,1}) = 7$.*

Proof Denote by $\mathcal{F}$ the cokernel of the composite of the two inclusions $\mathcal{O}(1) \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-3}$ and $\mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-3} \to \mathcal{E}$. Then $c_1(\mathcal{F}) = 2$ and $c_2(\mathcal{F}) = 7$. Moreover $\mathcal{F}$ fits in the following exact sequence

$$0 \to \mathcal{O}^{\oplus r-3} \to \mathcal{F} \to E_2^{-1,1} \to 0.$$

Let $(e_1, e_2, \ldots, e_{r-3})$ be the free basis of $\mathcal{O}^{\oplus r-3}$. For $0 \le s \le r - 3$, denote by $\mathcal{O}^{\oplus s}$ the submodule generated by the set $\{e_i | i \le s\}$ in $\mathcal{O}^{\oplus r-3}$ and by $\mathcal{F}_s$ the quotient of $\mathcal{F}$ by $\mathcal{O}^{\oplus s}$. We have the following exact sequence

$$0 \to \mathcal{O} \to \mathcal{F}_s \to \mathcal{F}_{s+1} \to 0.$$

Note that $\mathcal{F}_0 = \mathcal{F}$ and that $\mathcal{F}_{r-3} = E_2^{-1,1}$.

We shall show that $\mathcal{F}_s$ is torsion-free for all s ($0 \le s \le r - 3$) by induction on s. Now consider the case $s = 0$. Since we have (3.1) and $\mathcal{E}$ is locally free, the singular locus of $\mathcal{F}$ has codimension ≥ 2. Hence it follows from [11, Lemma 5.4] that $\mathcal{F}$ is torsion-free; thus the claim holds for the case $s = 0$.

Suppose that $\mathcal{F}_s$ is torsion-free. We shall show that $\mathcal{F}_{s+1}$ is torsion-free by contradiction. For simplicity, by changing the symbols, we denote $\mathcal{F}_s$ by $\mathcal{F}$ and $\mathcal{F}_{s+1}$ by $\mathcal{F}_+$, and assume that $\mathcal{F}$ is a torsion-free sheaf with $c_1(\mathcal{F}) = 2$ and $c_2(\mathcal{F}) = 7$, that $\mathcal{F}$ cannot admit a negative degree line bundle on a line or a conic as a quotient, and that $\mathcal{F}_+$ is not torsion-free. Let $\mathcal{F}^{\vee\vee}$ be the double dual of $\mathcal{F}$. Then $\mathcal{F}^{\vee\vee}$ is a nef vector bundle with first Chern class two by Lemma 9.1. Denote by $\mathcal{Q}$ the cokernel of the inclusion $\mathcal{F} \to \mathcal{F}^{\vee\vee}$. Then we have

$$\text{length}\,\mathcal{Q} = c_2(\mathcal{F}) - c_2(\mathcal{F}^{\vee\vee}) = 7 - c_2(\mathcal{F}^{\vee\vee}). \tag{7.21}$$

Denote by α the composite of the two inclusions $\mathcal{O} \to \mathcal{F}$ and $\mathcal{F} \to \mathcal{F}^{\vee\vee}$, and let $\mathcal{F}^{\vee\vee}/\mathcal{O}$ be the cokernel of α. If $h^0(\mathcal{F}^{\vee\vee}(-1)) = 0$, then $\mathcal{F}^{\vee\vee}/\mathcal{O}$ is torsion-free by [11, Lemma 5.4]. Since $\mathcal{F}_+$ is a subsheaf of $\mathcal{F}^{\vee\vee}/\mathcal{O}$ and it has a non-zero torsion subsheaf, we infer that $h^0(\mathcal{F}^{\vee\vee}(-1)) \ne 0$. It then follows from [11, Theorem 6.5] that $\mathcal{F}^{\vee\vee}$ satisfies one of the following:

(1) $\mathcal{F}^{\vee\vee} \cong \mathcal{O}(2) \oplus \mathcal{O}^{\oplus f-1}$;
(2) $\mathcal{F}^{\vee\vee} \cong \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus f-2}$;
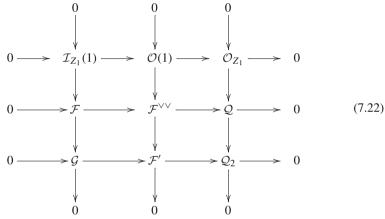(3) $\mathcal{F}^{\vee\vee}$ fits in an exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus f} \to \mathcal{F}^{\vee\vee} \to 0$.

In particular, we see that $c_2(\mathcal{F}^{\vee\vee}) \le 2$. Therefore $\text{length}\,\mathcal{Q} \ge 5$ by (7.21). Since $\mathcal{F}$ does not admit a negative degree line bundle on a line as a quotient, we infer that $\text{length}\,\mathcal{Q}|_L \le 2$ for any line L. Similarly we infer that $\text{length}\,\mathcal{Q}|_C \le 4$ for any smooth conic C by the assumption on $\mathcal{F}$. If $\mathcal{Q}$ is generated by a single element, then $\mathcal{Q}$ is isomorphic to the structure sheaf $\mathcal{O}_Z$ of the scheme-theoretic support Z of $\mathcal{Q}$. Lemma 4.4 then shows the existence of a conic C such that $\text{length}\,Z \cap C \ge 5$. Hence $\text{length}\,\mathcal{Q}|_C = \text{length}\,\mathcal{O}_Z|_C \ge 5$, which is a contradiction. In the following, we assume that $\mathcal{Q}$ is not generated by a single element.

Suppose that the case (1) holds. Let $\mathcal{Q}_1$ be the image of the composite of the inclusion $\mathcal{O}(2) \to \mathcal{F}^{\vee\vee}$ and the surjection $\mathcal{F}^{\vee\vee} \to \mathcal{Q}$. Denote by $\mathcal{Q}_2$ the quotient of $\mathcal{Q}$ by $\mathcal{Q}_1$. Then we have a surjection $\mathcal{O}^{\oplus f-1} \to \mathcal{Q}_2$. Let $\mathcal{G}$ be the kernel of this surjection $\mathcal{O}^{\oplus f-1} \to \mathcal{Q}_2$.

Then we have a surjection $\mathcal{F} \to \mathcal{G}$. Suppose that $\mathcal{Q}_2 \neq 0$. Then there exists a line L such that $\operatorname{length} \mathcal{Q}_2|_L \geq 1$. This implies that the kernel of the surjection $\mathcal{O}_L^{\oplus f-1} \to \mathcal{Q}_2|_L$, i.e., a quotient of $\mathcal{G}|_L$ has negative degree. Since we have a surjection $\mathcal{F}|_L \to \mathcal{G}|_L$, this contradicts the assumption on $\mathcal{F}$. Therefore $\mathcal{Q}_2 = 0$; thus $\mathcal{Q}_1 \cong \mathcal{Q}$. Hence we obtain a surjection $\mathcal{O}(2) \to \mathcal{Q} \cong \mathcal{Q}(2)$, and thus $\mathcal{Q}$ is generated by a single element. This contradicts the assumption. Hence the case (1) does not happen.

Suppose that we are in case (2) or (3). We have an inclusion $\mathcal{O}(1) \to \mathcal{F}^{\vee\vee}$. Denote by $\mathcal{F}'$ the cokernel of the inclusion. If $\mathcal{F}^{\vee\vee}$ lies in the case (2), then $\mathcal{F}' \cong \mathcal{O}(1) \oplus \mathcal{O}^{\oplus f-2}$. If $\mathcal{F}^{\vee\vee}$ lies in the case (3), then $\mathcal{F}'$ is a torsion-free sheaf with $c_1(\mathcal{F}') = 1$ and $c_2(\mathcal{F}') = 1$. Hence its double dual $\mathcal{F}'^{\vee\vee}$ is either $\mathcal{O}(1) \oplus \mathcal{O}^{\oplus f-2}$ or $T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}^{\oplus f-3}$. If $\mathcal{F}'^{\vee\vee} \cong T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}^{\oplus f-3}$, then $c_2(\mathcal{F}') = 1 = c_2(\mathcal{F}'^{\vee\vee})$; thus $\mathcal{F}'$ and $\mathcal{F}'^{\vee\vee}$ are isomorphic. In particular $\mathcal{F}'$ is locally free. If $\mathcal{F}'^{\vee\vee} \cong \mathcal{O}(1) \oplus \mathcal{O}^{\oplus f-2}$, then $\mathcal{F}'$ is $\mathcal{I}_q(1) \oplus \mathcal{O}^{\oplus f-2}$ for some point q. Let $\mathcal{H}_1$ and $\mathcal{Q}_1$ be respectively the kernel and the image of the composite of the inclusion $\mathcal{O}(1) \to \mathcal{F}^{\vee\vee}$ and the projection $\mathcal{F}^{\vee\vee} \to \mathcal{Q}$. Denote by $\mathcal{G}$ the quotient of $\mathcal{F}$ by $\mathcal{H}_1$ and by $\mathcal{Q}_2$ the quotient of $\mathcal{Q}$ by $\mathcal{Q}_1$. Since $\mathcal{H}_1(-1)$ is a coherent subsheaf of $\mathcal{O}$, it is the ideal sheaf $\mathcal{I}_{Z_1}$ of some closed subscheme Z_1 of $\mathbb{P}^2$; thus $\mathcal{Q}_1(-1) \cong \mathcal{O}_{Z_1}$. Since $\mathcal{Q}_1$ is a subsheaf of $\mathcal{Q}$, Z_1 is 0-dimensional. Hence $\mathcal{Q}_1 \cong \mathcal{O}_{Z_1}$. Now we have the following commutative diagram with exact rows and columns.
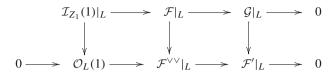
$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{Z_1}(1) & \longrightarrow & \mathcal{O}(1) & \longrightarrow & \mathcal{O}_{Z_1} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}^{\vee\vee} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{Q}_2 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
\tag{7.22}
$$

We claim that $\operatorname{length} \mathcal{Q}_2|_L \leq 1$ for any line L; this follows from the fact that $\mathcal{G}|_L$ cannot admit a line bundle of negative degree as a quotient, but a careful argument is needed if $\mathcal{F}' \cong \mathcal{I}_q(1) \oplus \mathcal{O}^{\oplus f-2}$ and $q \in L$; suppose that $\operatorname{length} \mathcal{Q}_2|_L \geq 1$ for a line L passing through q. Note here that $\mathcal{I}_q(1)|_L$ is isomorphic to $\mathcal{O}_L \oplus k(q)$. Since $\mathcal{G}|_L$ does not admit a line bundle of negative degree as a quotient, we infer that the composite of the inclusion $k(q) \to \mathcal{I}_q(1)|_L \to \mathcal{F}'|_L$ and the surjection $\mathcal{F}'|_L \to \mathcal{Q}_2|_L$ must be surjective. Hence $\operatorname{length} \mathcal{Q}_2|_L = 1$. Therefore the claim holds.

The claim above implies, in particular, that the number of minimal (non-zero) generators of $\mathcal{Q}_{2,p}$ at each point p is at most one. Hence it follows from the claim above that $\mathcal{Q}_2$ is isomorphic to the residue field $k(p)$ of some point p unless $\mathcal{Q}_2$ is zero.

If $\mathcal{Q}_2 = 0$, then $\mathcal{Q} \cong \mathcal{Q}_1 \cong \mathcal{O}_{Z_1}$, which contradicts our assumption that $\mathcal{Q}$ is not generated by a single element. Hence $\mathcal{Q}_2 \cong k(p)$. Then $\operatorname{length} Z_1 \geq 4$. If $p \notin Z_1$, then the support Z of $\mathcal{Q}$ is the disjoint union of Z_1 and p, and we see that $\mathcal{Q} \cong \mathcal{O}_Z$; this case is also ruled out. In the following, we assume that $p \in Z_1$.

We claim that $\operatorname{length} Z_1 \cap L \leq 2$ for any line L. If $\mathcal{O}_{Z_1}|_L \to \mathcal{Q}|_L$ is injective, then the claim holds since $\operatorname{length} \mathcal{Q}|_L \leq 2$. Suppose that $\mathcal{O}_{Z_1}|_L \to \mathcal{Q}|_L$ is not injective. Then $p \in L$.

Now pull back to L the diagram (7.22); the left half of the diagram becomes the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
\mathcal{I}_{Z_1}(1)|_L & \longrightarrow & \mathcal{F}|_L & \longrightarrow & \mathcal{G}|_L & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & \mathcal{O}_L(1) & \longrightarrow & \mathcal{F}^{\vee\vee}|_L & \longrightarrow & \mathcal{F}'|_L & \longrightarrow & 0
\end{array}$$

By the snake lemma, the diagram above induces the following exact sequence

$$\kappa \to \mathcal{O}_{Z_1}|_L \to \mathcal{Q}|_L \to k(p) \to 0, \tag{7.23}$$

where κ denotes the kernel of the morphism $\mathcal{G}|_L \to \mathcal{F}'|_L$. Note here that $\kappa \cong k(p)$; indeed, the defining morphism $\mathcal{G}|_L \to \mathcal{F}'|_L$ of κ also fits in the following commutative diagram with exact rows,

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{G}(-1) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{G}|_L & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & \mathcal{F}'(-1) & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F}'|_L & \longrightarrow & 0
\end{array}$$

and this diagram induces the following exact sequence by the snake lemma

$$0 \to \kappa \to k(p) \to k(p) \to k(p) \to 0;$$

thus the isomorphism $\kappa \cong k(p)$ follows. Since $\text{Ker}(\mathcal{O}_{Z_1}|_L \to \mathcal{Q}|_L) \neq 0$ by assumption, the exact sequence (7.23) together with the isomorphism $\kappa \cong k(p)$ induces the following exact sequence

$$0 \to k(p) \to \mathcal{O}_{Z_1}|_L \to \mathcal{Q}|_L \to k(p) \to 0; \tag{7.24}$$

in particular $k(p) \to \mathcal{O}_{Z_1}|_L$ is injective. Indeed, the exact sequence (7.23) induces a surjection $\kappa \to \text{Ker}(\mathcal{O}_{Z_1}|_L \to \mathcal{Q}|_L)$, and this is a non-zero K-linear from $\kappa \cong k(p) \cong K$; thus the injectivity follows. Now the exact sequence (7.24) shows that $\text{length} Z_1 \cap L = \text{length} \mathcal{O}_{Z_1}|_L = \text{length} \mathcal{Q}|_L \leq 2$.

We claim here that α factors through some subsheaf $\mathcal{O}(1)$ of $\mathcal{F}^{\vee\vee}$. Assume, to the contrary, that α does not factor through any $\mathcal{O}(1)$. Recall and observe here that $\mathcal{F}^{\vee\vee}/\mathcal{O}$ cannot be torsion-free, since its subsheaf $\mathcal{F}_+$ contains a non-zero torsion subsheaf. Suppose that $\mathcal{F}^{\vee\vee} \cong \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus f-2}$. The composite of α and the projection $\mathcal{F}^{\vee\vee} \to \mathcal{O}^{\oplus f-2}$ is zero, since otherwise $\mathcal{F}^{\vee\vee}/\mathcal{O}$ would be locally free. Hence α factors through $\mathcal{O}(1)^{\oplus 2}$. Since α does not factor through any $\mathcal{O}(1)$, the cokernel of the induced morphism $\mathcal{O} \to \mathcal{O}(1)^{\oplus 2}$ is $\mathcal{I}(2)$, where $\mathcal{I}$ is the ideal sheaf of some point. Since it fits in the following exact sequence
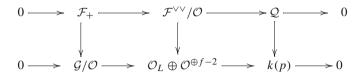
$$0 \to \mathcal{I}(2) \to \mathcal{F}^{\vee\vee}/\mathcal{O} \to \mathcal{F}' \to 0,$$

we infer that $\mathcal{F}^{\vee\vee}/\mathcal{O}$ is torsion-free, but this contradicts the observation above. Suppose that we are in case (3). Then $\mathcal{F}'$ is either $T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}^{\oplus f-3}$ or $\mathcal{I}_p(1) \oplus \mathcal{O}^{\oplus f-2}$. Now α induces an injection $\mathcal{O} \to \mathcal{F}'$, since α does not factor through $\mathcal{O}(1)$. Note here that the cokernel $\mathcal{F}'/\mathcal{O}$ of the injection cannot be torsion-free, since it fits in the following exact sequence

$$0 \to \mathcal{O}(1) \to \mathcal{F}^{\vee\vee}/\mathcal{O} \to \mathcal{F}'/\mathcal{O} \to 0$$

and $\mathcal{F}^{\vee\vee}/\mathcal{O}$ cannot be torsion-free. If $\mathcal{F}' = T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}^{\oplus f-3}$, then it follows from [11, Lemma 5.4] that $\mathcal{F}'/\mathcal{O}$ is torsion-free, which is a contradiction. If $\mathcal{F}' = \mathcal{I}_p(1) \oplus \mathcal{O}^{\oplus f-2}$, then $\mathcal{F}'/\mathcal{O}$ is isomorphic to either $\mathcal{I}_p(1) \oplus \mathcal{O}^{\oplus f-3}$ or $\mathcal{O}_L \oplus \mathcal{O}^{\oplus f-2}$ for some line L containing

p, but the former is ruled out since it is torsion-free. Hence $\mathcal{F}'/\mathcal{O}$ is $\mathcal{O}_L \oplus \mathcal{O}^{\oplus f-2}$. Now we have the following commutative diagram with exact rows.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}_+ & \longrightarrow & \mathcal{F}^{\vee\vee}/\mathcal{O} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G}/\mathcal{O} & \longrightarrow & \mathcal{O}_L \oplus \mathcal{O}^{\oplus f-2} & \longrightarrow & k(p) & \longrightarrow & 0
\end{array}
$$

We see that $(\mathcal{G}/\mathcal{O})|_L$ admits a negative degree line bundle as a quotient. This is a contradiction. Therefore the claim holds.
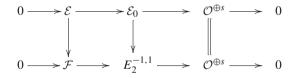
Note here that the composite of α and the surjection $\mathcal{F}^{\vee\vee} \to \mathcal{Q}$ is zero, since α factors through $\mathcal{F}$ by definition. Hence the claim above implies that α factors through not only $\mathcal{O}(1)$ but also $\mathcal{I}_{Z_1}(1)$. Therefore Z_1 lies on a line L. Since length $Z_1 \geq 4$, this contradicts the claim that length $Z_1 \cap L \leq 2$. Therefore we conclude that $\mathcal{F}_+$ is torsion-free. $\qquad\square$

7.3 Proof for the case $n = 2$ and $h^1(\mathcal{E}) > 0$

Set $s = h^1(\mathcal{E})$. Then we have the following exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}_0 \to \mathcal{O}^{\oplus s} \to 0$$

which induces an isomorphism $H^0(\mathcal{O}^{\oplus s}) \cong H^1(\mathcal{E})$. It follows from [10, Theorem 6.2.12 (ii)] that $\mathcal{E}_0$ is a nef vector bundle of rank $r + s$ with first Chern class three, second Chern class nine, and $h^1(\mathcal{E}_0) = 0$. Since $H^0(\mathcal{E}) \cong H^0(\mathcal{E}_0)$, the image $\mathcal{E}_1$ of the evaluation map $H^0(\mathcal{E}_0) \otimes \mathcal{O} \to \mathcal{E}_0$ is contained in $\mathcal{E}$. Denote by $\mathcal{F}$ the quotient of $\mathcal{E}$ by $\mathcal{E}_1$. By abuse of notation, we also denote by $E_2^{-1,1}$ the cokernel of the evaluation map $H^0(\mathcal{E}_0) \otimes \mathcal{O} \to \mathcal{E}_0$. Then we have the following commutative diagram with exact rows.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}_0 & \longrightarrow & \mathcal{O}^{\oplus s} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & E_2^{-1,1} & \longrightarrow & \mathcal{O}^{\oplus s} & \longrightarrow & 0
\end{array}
$$

Since we have a surjection $E_2^{-1,1} \to \mathcal{O}^{\oplus s}$ with $s \geq 1$, Remark 7.1 and Lemmas 7.8 and 7.9 imply that $E_2^{-1,1}$ is either $\mathcal{O}$ or a torsion-free sheaf of rank two with $c_1(E_2^{-1,1}) = 2$ and $c_2(E_2^{-1,1}) = 7$. If $E_2^{-1,1}$ is the latter, then $s = 1$ and $\mathcal{F} \cong \mathcal{I}_Z(2)$ for some 0-dimensional closed subscheme Z of length seven. Lemma 4.4 then shows that $\mathcal{E}$ admits a negative degree quotient, which contradicts that $\mathcal{E}$ is nef. Hence $E_2^{-1,1} \cong \mathcal{O}$, and thus $\mathcal{F} = 0$, $\mathcal{E}_1 = \mathcal{E}$, and $\mathcal{E}_0$ and $\mathcal{E}_1$ fit in the following exact sequences

$$0 \to \mathcal{O}(-3) \to \mathcal{O}^{\oplus r+1} \to \mathcal{E}_1 \to 0;$$
$$0 \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{O} \to 0$$

as in (7.17) and (7.18). Hence we obtain the case (15) of Theorem 1, for $n = 2$.

7.4 Proof for the case $n \geq 3$

What we have to show in case $n \geq 3$ is the following lemma.

Lemma 7.10 *If $n \geq 3$, then $c_3 = 27$ and $h^0(\mathcal{E}(-1)) \leq 1$. Moreover $\mathcal{E}$ satisfies one of the following:*

(1) $h^0(\mathcal{E}(-1)) = 0$, $h^{n-3}(\mathcal{E}(2-n)) = 0$, *and $\mathcal{E}$ lies in the case* (15) *of Theorem 1;*
(2) $h^0(\mathcal{E}(-1)) = 1$*;*
(3) $n \geq 4$, $h^0(\mathcal{E}(-1)) = 0$, *and $h^{n-3}(\mathcal{E}(2-n)) = 1$.*

Proof We shall first show that $h^0(\mathcal{E}(-1)) \leq 1$ by induction on $n \geq 3$. It follows from (3.1) that $h^0(\mathcal{E}(-1)) \leq h^0(\mathcal{E}|_H(-1))$ for any hyperplane H in $\mathbb{P}^n$. Note here that $h^0(\mathcal{E}|_H(-1)) \leq 1$ by induction hypothesis if $n \geq 4$ and by what we have seen in §7.1 and §7.1 if $n = 3$. Hence $h^0(\mathcal{E}(-1)) \leq 1$.

Recall here that $\mathcal{E}$ is globally generated if $h^1(\mathcal{E}(-1)) = 0$ and $\mathcal{E}|_H$ is globally generated for any hyperplane H in $\mathbb{P}^n$ by [15, Lemma 3]. In order to show that $\mathcal{E}$ lies in the case (15) of Theorem 1, it is enough to show that $h^0(\mathcal{E}) = r + 1$ and that $\mathcal{E}$ is globally generated.

Suppose that $n = 3$. As we have seen in §7.1, we have $h^1(\mathcal{E}|_H) \leq 1$.

Suppose furthermore that $h^0(\mathcal{E}(-1)) = 1$. Then we get the case (2) of Lemma 7.10 in case $n = 3$. Note here that we have $c_3 = 27$ in this case. Indeed, the argument above shows that $h^0(\mathcal{E}|_H(-1)) = 1$, and this implies that $h^1(\mathcal{E}|_H) = 0$ as we have also seen in §7.1 and §7.1. It then follows from (3.2) and (3.4) that $h^q(\mathcal{E}(-1)) = 0$ for $q \geq 2$. Now we have

$$1 \geq 1 - h^1(\mathcal{E}(-1)) = \chi(\mathcal{E}(-1)) = (c_3 - 25)/2$$

by (3.20). Note here that $c_3 \geq 27$ by (3.23). Hence we infer that $c_3 = 27$ and that $h^1(\mathcal{E}(-1)) = 0$.

Suppose furthermore that $h^0(\mathcal{E}(-1)) = 0$. It follows from $h^1(\mathcal{E}|_H) \leq 1$ and (3.2) that $h^2(\mathcal{E}(-1)) \leq 1$ and that equality holds if and only if $h^1(\mathcal{E}|_H) = 1$. Moreover we have $h^3(\mathcal{E}(-1)) = 0$ by (3.2) and (3.4). It then follows from (3.20) that

$$1 \geq -h^1(\mathcal{E}(-1)) + h^2(\mathcal{E}(-1)) = \chi(\mathcal{E}(-1)) = (c_3 - 25)/2.$$

Since $c_3 \geq 27$ by (3.23), this implies that $c_3 = 27$, that $h^2(\mathcal{E}(-1)) = 1$, and that $h^1(\mathcal{E}(-1)) = 0$. Hence $h^1(\mathcal{E}|_H) = 1$. As we have seen in §7.3, this implies that $\mathcal{E}|_H$ is globally generated and that $h^0(\mathcal{E}|_H) = r + 1$. Therefore $\mathcal{E}$ is globally generated and $h^0(\mathcal{E}) = r + 1$. This is the case (1) of Lemma 7.10 in case $n = 3$.

Suppose that $n \geq 4$. If $h^0(\mathcal{E}(-1)) = 1$, we get the case (2) of Lemma 7.10. Suppose that $h^0(\mathcal{E}(-1)) = 0$. We shall show that $h^{n-3}(\mathcal{E}(2-n)) \leq 1$ by induction on $n \geq 4$. Note here that $h^q(\mathcal{E}(3-n)) = 0$ for all $q > 0$ by (3.2). Hence $h^{n-3}(\mathcal{E}(2-n)) \leq h^{n-4}(\mathcal{E}|_H(3-n)) \leq 1$ by induction hypothesis if $n \geq 5$ and by what we have shown, i.e., $h^0(\mathcal{E}|_H(-1)) \leq 1$ if $n = 4$. If $h^{n-3}(\mathcal{E}(2-n)) = 1$, we obtain the case (3) of Lemma 7.10.

Suppose that $n \geq 4$, that $h^0(\mathcal{E}(-1)) = 0$, and that $h^{n-3}(\mathcal{E}(2-n)) = 0$. We claim here that $\mathcal{E}$ lies in the case (15) of Theorem 1. We proceed by induction not only on $n \geq 4$ but also on $n \geq 3$; if $n = 3$, then the two conditions $h^0(\mathcal{E}(-1)) = 0$ and $h^{n-3}(\mathcal{E}(2-n)) = 0$ become the same and $\mathcal{E}$ lies in the case (15) of Theorem 1 as we have seen above. Suppose now that $n \geq 4$. The assumption $h^{n-3}(\mathcal{E}(2-n)) = 0$ implies that $h^{n-4}(\mathcal{E}|_H(3-n)) = 0$ by (3.2) if $n \geq 5$ and by the assumption $h^0(\mathcal{E}(-1)) = 0$ if $n = 4$. Hence we infer that $h^0(\mathcal{E}|_{L^3}(-1)) = h^{n-3}(\mathcal{E}(2-n)) = 0$ for any linear subspace L^3 of dimension three in $\mathbb{P}^n$. Therefore we see that $h^0(\mathcal{E}|_H(-1)) \leq h^0(\mathcal{E}|_{L^3}(-1)) = 0$ by (3.1). Now it follows from the induction hypothesis that $\mathcal{E}|_H$ lies in the case (15) of Theorem 1. Thus $h^0(\mathcal{E}|_H) = r + 1$ and $\mathcal{E}|_H$ is globally generated. Moreover $h^1(\mathcal{E}(-1)) = 0$ by (3.2) and $h^0(\mathcal{E}(-1)) = 0$ by assumption. Hence $h^0(\mathcal{E}) = r + 1$ and $\mathcal{E}$ is globally generated. $\square$

8 Several remarks on Theorem 1

Remark 8.1 The exact sequence in the case (7) of Theorem 1 induces the following

$$0 \to T_{\mathbb{P}^3}(-2) \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r+2} \to \mathcal{E} \to 0,$$

and, dualizing this, we obtain the following exact sequence

$$0 \to \mathcal{E}^\vee \to \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus r+2} \to \Omega_{\mathbb{P}^3}(2) \to 0.$$

Note that the injection $H^0(\mathcal{E}^\vee) \to H^0(\mathcal{O}(-1) \oplus \mathcal{O}^{\oplus r+2})$ induces a splitting injection $\mathcal{O} \otimes H^0(\mathcal{E}^\vee) \to \mathcal{O}^{\oplus r+2}$ and that the composite of two splitting injection $\mathcal{O} \otimes H^0(\mathcal{E}^\vee) \to \mathcal{O}^{\oplus r+2}$ and $\mathcal{O}^{\oplus r+2} \to \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus r+2}$ is equal to the composite of $\mathcal{O} \otimes H^0(\mathcal{E}^\vee) \to \mathcal{E}^\vee$ and $\mathcal{E}^\vee \to \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus r+2}$. Thus $\mathcal{O} \otimes H^0(\mathcal{E}^\vee) \to \mathcal{E}^\vee$ is also a splitting injection. Hence $\mathcal{E}^\vee \cong \mathcal{E}_0^\vee \oplus \mathcal{O} \otimes H^0(\mathcal{E}^\vee)$ for some vector bundle $\mathcal{E}_0$ of rank $s = r - h^0(\mathcal{E}^\vee)$. Since $c_3(\mathcal{E}_0) = 2 \neq 0$, we infer that $s \geq 3$. Note that $h^0(\mathcal{E}_0^\vee) = 0$ and that $\mathcal{E}_0^\vee$ fits in an exact sequence

$$0 \to \mathcal{E}_0^\vee \to \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus s+2} \to \Omega_{\mathbb{P}^3}(2) \to 0.$$

Since $h^0(\Omega_{\mathbb{P}^3}(2)) = 6$ by the Bott formula [14, p. 8], we see that $s \leq 4$. Moreover $h^1(\mathcal{E}_0^\vee) = 4 - s$. The image of $\mathcal{E}_0^\vee \to \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus s+2} \to \mathcal{O}(-1)$ is $\mathcal{I}_Z(-1)$ for some closed subscheme Z in $\mathbb{P}^3$.

Suppose that $Z = \emptyset$. Let $\mathcal{F}^\vee$ be the kernel of the surjection $\mathcal{E}_0^\vee \to \mathcal{O}(-1)$. Then $\mathcal{F}$ fits in an exact sequence

$$0 \to T_{\mathbb{P}^3}(-2) \to \mathcal{O}^{\oplus s+2} \to \mathcal{F} \to 0,$$

and $\mathcal{F}$ is a nef vector bundle of rank $s - 1$ with $c_1(\mathcal{F}) = 2$. Moreover, as we have seen in [11, Remark 6.7], $\mathcal{F} \cong \Omega_{\mathbb{P}^3}(2)$ if $s = 4$ and $\mathcal{F} \cong \mathcal{N}(1)$ if $s = 3$ where $\mathcal{N}$ is a null correlation bundle on $\mathbb{P}^3$. Hence $\mathcal{E}$ is either $\mathcal{O}(1) \oplus \Omega_{\mathbb{P}^3}(2) \oplus \mathcal{O}^{\oplus r-4}$ or $\mathcal{O}(1) \oplus \mathcal{N}(1) \oplus \mathcal{O}^{\oplus r-3}$ if $Z = \emptyset$.

Remark 8.2 Suppose that $\mathcal{E}$ is in the case (8) of Theorem 1. Then $\mathcal{E}$ has $\mathcal{O}^{\oplus r-4}$ as a subbundle; let $\mathcal{E}_0$ be the quotient bundle $\mathcal{E}/\mathcal{O}^{\oplus r-4}$ of rank four. In [1, §6 III (a)], it is stated that $\mathcal{E}_0 \cong \Omega_{\mathbb{P}^4}(2)$. (Therefore we see that $\mathcal{E} \cong \Omega_{\mathbb{P}^4}(2) \oplus \mathcal{O}^{\oplus r-4}$.)

For the sake of completeness, we give a different proof of this result in our context. First note that $\mathcal{E}_0$ fits in an exact sequence

$$0 \to T_{\mathbb{P}^4}(-3) \to \mathcal{O}(-1)^{\oplus 10} \to \mathcal{O}^{\oplus 10} \to \mathcal{E}_0 \to 0.$$

Therefore we obtain the following exact sequence

$$0 \to \mathcal{E}_0^\vee(-1) \to \mathcal{O}(-1)^{\oplus 10} \to \mathcal{O}^{\oplus 10} \to \Omega_{\mathbb{P}^4}(2) \to 0.$$

We split this sequence into the following two exact sequences:

$$0 \to \mathcal{E}_0^\vee(-1) \to \mathcal{O}(-1)^{\oplus 10} \to \mathcal{G} \to 0; \tag{8.1}$$

$$0 \to \mathcal{G} \to \mathcal{O}^{\oplus 10} \to \Omega_{\mathbb{P}^4}(2) \to 0. \tag{8.2}$$

We claim here that the induced map $H^0(\mathcal{O}^{\oplus 10}) \to H^0(\Omega_{\mathbb{P}^4}(2))$ is an isomorphism. Since $h^0(\Omega_{\mathbb{P}^4}(2)) = 10$ by the Bott formula [14, p. 8], it is enough to show that the map is injective. Suppose, to the contrary, that there exists a subbundle $\mathcal{O} \to \mathcal{O}^{\oplus 10}$ such that the composite $\mathcal{O} \to \mathcal{O}^{\oplus 10} \to \Omega_{\mathbb{P}^4}(2)$ is zero. Then the subbundle morphism $\mathcal{O} \to \mathcal{O}^{\oplus 10}$ induces a subbundle morphism $\mathcal{O} \to \mathcal{G}$; let $\mathcal{G}_0$ be the quotient bundle $\mathcal{G}/\mathcal{O}$. The composite of the

subbundle morphism $\mathcal{O} \to \mathcal{G}$ and the extension class in $\mathrm{Ext}^1(\mathcal{G}, \mathcal{E}_0^\vee(-1))$ corresponding to (8.1) lies in $H^1(\mathcal{E}_0^\vee(-1))$, and it gives rise to an exact sequence

$$0 \to \mathcal{E}_0^\vee \to \mathcal{F} \to \mathcal{O}(1) \to 0. \tag{8.3}$$

Then $\mathcal{F}$ is a vector bundle, and it also fits in an exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}^{\oplus 10} \to \mathcal{G}_0(1) \to 0.$$

Hence $\mathcal{F}^\vee$ is nef. On the other hand, it follows from (8.3) that $c_3(\mathcal{F}^\vee) = -2$ since $c_2 = 4$ and $c_3 = 2$. This contradicts the non-negativity of the Chern classes of nef vector bundles. Therefore the claim holds. Hence we may assume that the dual of (8.2) is nothing but the exact sequence induced by the two wedge $\wedge^2(\mathcal{O}^{\oplus 5}) \to \wedge^2(T_{\mathbb{P}^4}(-1))$ of the surjection in the Euler exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus 5} \to T_{\mathbb{P}^4}(-1) \to 0. \tag{8.4}$$

In particular, $\mathcal{G} \cong \Omega^2_{\mathbb{P}^4}(2)$. Thus the exact sequence (8.1) implies an exact sequence

$$0 \to \mathcal{E}_0^\vee \to \mathcal{O}^{\oplus 10} \to \Omega^2_{\mathbb{P}^4}(3) \to 0. \tag{8.5}$$

Next we claim that the induced map $H^0(\mathcal{O}^{\oplus 10}) \to H^0(\Omega^2_{\mathbb{P}^4}(3))$ is an isomorphism. Since $h^0(\Omega^2_{\mathbb{P}^4}(3)) = 10$ by the Bott formula, it is enough to show that the map is injective. Suppose, to the contrary, that there exists a subbundle $\mathcal{O} \to \mathcal{O}^{\oplus 10}$ such that the composite $\mathcal{O} \to \mathcal{O}^{\oplus 10} \to \Omega^2_{\mathbb{P}^4}(3)$ is zero. Then the subbundle morphism $\mathcal{O} \to \mathcal{O}^{\oplus 10}$ induces a subbundle morphism $\mathcal{O} \to \mathcal{E}_0^\vee$; let $\mathcal{E}_1^\vee$ be the quotient bundle $\mathcal{E}_0^\vee/\mathcal{O}$. Then $\mathcal{E}_1^\vee$ fits in an exact sequence

$$0 \to \mathcal{E}_1^\vee \to \mathcal{O}^{\oplus 9} \to \Omega^2_{\mathbb{P}^4}(3) \to 0.$$

Hence it follows from the Bott formula that $h^0(\mathcal{E}_1) = 9$. Since $h^0(\mathcal{E}_0) = 10$, this implies that $\mathcal{E}_0 \cong \mathcal{E}_1 \oplus \mathcal{O}$. Thus $c_4(\mathcal{E}_0) = 0$, which however contradicts that $c_4 = 1$. Therefore $H^0(\mathcal{O}^{\oplus 10}) \to H^0(\Omega^2_{\mathbb{P}^4}(3))$ is an isomorphism, and we conclude that the exact sequence (8.5) is nothing but the exact sequence induced by the two wedge $\wedge^2(\mathcal{O}^{\oplus 5}) \to \wedge^2(T_{\mathbb{P}^4}(-1))$ of the surjection in the Euler exact sequence (8.4). Therefore $\mathcal{E}_0 \cong \Omega_{\mathbb{P}^4}(2)$.

Remark 8.3 Suppose that $n = 4$ and that $\mathcal{E}$ fits in an exact sequence in the case (10) of Theorem 1. Then $\mathcal{E}$ is an extension of the Tango bundle by a trivial bundle $\mathcal{O}^{\oplus r-3}$.

The reason is as follows. Since $\mathcal{E}$ is globally generated, $\mathcal{E}$ has $\mathcal{O}^{\oplus r-4}$ as a subbundle; denote by $\mathcal{E}_0$ the quotient bundle $\mathcal{E}/\mathcal{O}^{\oplus r-4}$. Since $\mathcal{E}_0$ is globally generated of rank four with $c_4(\mathcal{E}_0) = 0$, $\mathcal{E}_0$ has also $\mathcal{O}$ as a subbundle; denote by $\mathcal{E}_1$ the quotient bundle $\mathcal{E}_0/\mathcal{O}$. We show that $\mathcal{E}_1$ is the Tango bundle. First note that the dual $\mathcal{E}_1^\vee$ of $\mathcal{E}_1$ fits in an exact sequence

$$0 \to \mathcal{E}_1^\vee \to \mathcal{O}^{\oplus 7} \to \Omega_{\mathbb{P}^4}(2) \to 0.$$

Note also that $h^0(\mathcal{E}_1^\vee) = 0$; indeed, if $h^0(\mathcal{E}_1^\vee) \neq 0$, then $\mathcal{E}_1$ would admit $\mathcal{O}$ as a direct summand, which contradicts the fact that $c_3(\mathcal{E}_1) = 5 \neq 0$ and the rank of $\mathcal{E}_1$ is three. Since $h^0(\Omega_{\mathbb{P}^4}(2)) = 10$ by the Bott formula [14, p. 8], this implies that $h^1(\mathcal{E}_1^\vee) = 3$. Now we have an isomorphism $\mathrm{Ext}^1(\mathrm{Ext}^1(\mathcal{O}, \mathcal{E}_1^\vee) \otimes \mathcal{O}, \mathcal{E}_1^\vee) \cong \mathrm{End}(\mathrm{Ext}^1(\mathcal{O}, \mathcal{E}_1^\vee))$; let ξ be the element in $\mathrm{Ext}^1(\mathrm{Ext}^1(\mathcal{O}, \mathcal{E}_1^\vee) \otimes \mathcal{O}, \mathcal{E}_1^\vee)$ corresponding to the identity in $\mathrm{End}(\mathrm{Ext}^1(\mathcal{O}, \mathcal{E}_1^\vee))$. Consider the extension

$$0 \to \mathcal{E}_1^\vee \to \mathcal{F} \to \mathrm{Ext}^1(\mathcal{O}, \mathcal{E}_1^\vee) \otimes \mathcal{O} \to 0$$

corresponding to ξ; then $H^0(\mathcal{F}) \cong H^0(\mathcal{E}_1^\vee) = 0$ and $H^1(\mathcal{F}) = 0$. Let

$$0 \to \mathcal{O}^{\oplus 7} \to \mathcal{O}^{\oplus 10} \to \mathrm{Ext}^1(\mathcal{O}, \mathcal{E}_1^\vee) \otimes \mathcal{O} \to 0$$

be the extension corresponding to the image of ξ via the map

$$\mathrm{Ext}^1(\mathrm{Ext}^1(\mathcal{O}, \mathcal{E}_1^\vee) \otimes \mathcal{O}, \mathcal{E}_1^\vee) \to \mathrm{Ext}^1(\mathrm{Ext}^1(\mathcal{O}, \mathcal{E}_1^\vee) \otimes \mathcal{O}, \mathcal{O}^{\oplus 7}).$$

Then $\mathcal{F}$ fits in an exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}^{\oplus 10} \to \Omega_{\mathbb{P}^4}(2) \to 0.$$

Since $h^0(\mathcal{F}) = h^1(\mathcal{F}) = 0$, the induced map $H^0(\mathcal{O}^{\oplus 10}) \to H^0(\Omega_{\mathbb{P}^4}(2))$ is an isomorphism. Therefore $\mathcal{F}^\vee \cong \Omega_{\mathbb{P}^4}^2(3)$, and thus $\mathcal{E}_1$ is the Tango bundle.

According to [8, §4], Trautmann [20] and Vetter [21] give an explicit construction of the bundle which is, up to taking duals and twists by $\mathcal{O}(1)$, the Tango bundle.

In the following, we give an example in case (16) of Theorem 1.

Example 1 Let $\psi : \mathcal{O}(-2)^{\oplus 3} \to \mathcal{O}(-1)^{\oplus 3}$ be the morphism defined by a matrix

$$\begin{bmatrix} 0 & y & x - \lambda z \\ x & 0 & y \\ z & z - x & 0 \end{bmatrix},$$

where $(x : y : z)$ are homogeneous coordinates of $\mathbb{P}^2$ and $\lambda \in K \setminus \{0, 1\}$. Then $\mathrm{Coker}(\psi)$ is supported on an elliptic curve $E : y^2 z = x(x - z)(x - \lambda z)$, and its Chern polynomial $c_t(\mathrm{Coker}(\psi)) = 1 + 3t + 9t^2$. Moreover $h^0(\mathrm{Coker}(\psi)) = 0$. Take a sufficiently large integer r (e.g., $r \geq 6$) and a general morphism $\psi' : \mathcal{O}(-2)^{\oplus 3} \to \mathcal{O}^{\oplus r}$, and consider a subbundle morphism $\Psi = {}^t[\psi', \psi] : \mathcal{O}(-2)^{\oplus 3} \to \mathcal{O}^{\oplus r} \oplus \mathcal{O}(-1)^{\oplus 3}$. Let $\mathcal{E}$ be the cokernel of Ψ. Then $\mathcal{E}$ fits in an exact sequence

$$0 \to \mathcal{O}^{\oplus r} \xrightarrow{\varphi} \mathcal{E} \to \mathrm{Coker}(\psi) \to 0,$$

where $c_1(\mathcal{E}) = 3$ and $c_2(\mathcal{E}) = 9$. The degeneracy locus Z of the composite of a general inclusion $\mathcal{O}^{\oplus r-1} \hookrightarrow \mathcal{O}^{\oplus r}$ and φ has codimension two, and we see that the cokernel of the composite is isomorphic to $\mathcal{I}_Z(3)$, where Z is a 0-dimensional closed subscheme of length 9 in $\mathbb{P}^2$. Moreover we have the following exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{I}_Z(3) \to \mathrm{Coker}(\psi) \to 0.$$

Therefore Z lies on the elliptic curve E, and $\mathrm{Coker}(\psi)$ is isomorphic to $\mathcal{O}_E(\mathfrak{d})$, where $\mathfrak{d}$ is a divisor of degree zero on E; thus $\mathcal{E}$ fits in the following exact sequence

$$0 \to \mathcal{O}^{\oplus r} \to \mathcal{E} \to \mathcal{O}_E(\mathfrak{d}) \to 0.$$

This implies that $\mathcal{E}$ is nef. Finally note that $\mathfrak{d} \neq 0$ since $h^0(\mathrm{Coker}(\psi)) = 0$.

We end this section with the following question about some properties of a nef vector bundle $\mathcal{E}$ in case (16) of Theorem 1.

Question 2 Is the support of the evaluation map $H^0(\mathcal{E}) \otimes \mathcal{O} \to \mathcal{E}$ in case (16) of Theorem 1 necessarily reduced, irreducible and nonsingular?

9 Nef but non-globally generated vector bundles

Lemma 9.1 *Let $\mathcal{F}$ be a nef vector bundle on a smooth projective surface X. Let $\mathcal{E}_0$ be a torsion-free quotient of $\mathcal{F}$, i.e., there exists a surjection $\mathcal{F} \rightarrow \mathcal{E}_0$ with $\mathcal{E}_0$ a torsion-free coherent sheaf. Let $\mathcal{E}$ denote the double dual $\mathcal{E}_0^{\vee\vee}$ of $\mathcal{E}_0$. Then $\mathcal{E}$ is a nef vector bundle.*

Proof Since $\mathcal{E}$ is a reflexive sheaf on a smooth surface, $\mathcal{E}$ is a vector bundle. To show that $\mathcal{E}$ is nef, it is enough to show that, for any finite morphism $C \rightarrow X$ from a smooth curve C, every quotient line bundle $\mathcal{L}$ of $\mathcal{E}|_C$ has non-negative degree. Note here that the natural injection $\mathcal{E}_0 \rightarrow \mathcal{E}$ induces a generically injective morphism $\mathcal{E}_0|_C \rightarrow \mathcal{E}|_C$. Now let $\mathcal{M}$ be the image of the composite of the morphism $\mathcal{E}_0|_C \rightarrow \mathcal{E}|_C$ and the surjection $\mathcal{E}|_C \rightarrow \mathcal{L}$. Since the composite of $\mathcal{F}|_C \rightarrow \mathcal{E}_0|_C$ and $\mathcal{E}_0|_C \rightarrow \mathcal{M}$ is surjective and $\mathcal{F}$ is nef, we see that $\mathcal{M}$ has non-negative degree. Hence $\mathcal{L}$ has non-negative degree since there is an injection $\mathcal{M} \rightarrow \mathcal{L}$ of line bundles on the smooth curve C. Therefore $\mathcal{E}$ is nef. □

As is indicated by the statement in Lemma 5.1 (2) (a), we can construct a nef but non-globally generated vector bundle on $\mathbb{P}^2$ if $c_1 = 3$ and $c_2 = 8$. See also Example 3 in §10 (besides Example 1 in the previous section) for an example of a nef but non-globally generated vector bundle on $\mathbb{P}^2$ with $c_1 = 3$ and $c_2 = 9$.

Proof of Proposition 2 Given an integer $r \geq 2$ and a closed point w in $\mathbb{P}^2$, note first that there exists a section s in $H^0(\mathcal{O}(3)^{\oplus r+1})$ such that the zero locus $(s)_0$ of s is $\{w\}$ as closed subschemes. Let $\varphi : \mathcal{O}(-3) \rightarrow \mathcal{O}^{\oplus r+1}$ be the morphism determined by s, and $\mathcal{E}_0$ the cokernel of φ. The dual $\varphi^\vee : \mathcal{O}^{\oplus r+1} \rightarrow \mathcal{O}(3)$ of φ has $\mathcal{I}_w(3)$ as its image, and we obtain an exact sequence

$$0 \rightarrow \mathcal{E}_0^\vee \rightarrow \mathcal{O}^{\oplus r+1} \rightarrow \mathcal{I}_w(3) \rightarrow 0.$$

On the other hand, the ideal sheaf $\mathcal{I}_w$ sits in an exact sequence

$$0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}(-1)^{\oplus 2} \rightarrow \mathcal{I}_w \rightarrow 0. \tag{9.1}$$

Therefore $\mathrm{Tor}_i^{\mathcal{O}_w}(\mathcal{I}_w(3), k(w)) = 0$ for $i > 1$. Hence $\mathcal{E}_0^\vee$ is a vector bundle. The exact sequence (9.1) also implies that $\mathcal{E}xt^1(\mathcal{I}_w(3), \mathcal{O}) \cong k(w)$. Let $\mathcal{E}$ be the double dual $\mathcal{E}_0^{\vee\vee}$ of $\mathcal{E}_0$. Since $\mathcal{H}om(\mathcal{I}_w(3), \mathcal{O}) \cong \mathcal{O}(-3)$, the vector bundle $\mathcal{E}$ fits in the desired exact sequence

$$0 \rightarrow \mathcal{O}(-3) \xrightarrow{\varphi} \mathcal{O}^{\oplus r+1} \rightarrow \mathcal{E} \rightarrow k(w) \rightarrow 0. \tag{9.2}$$

Suppose next that a vector bundle $\mathcal{E}$ fits in the exact sequence (9.2). We split the sequence (9.2) into the following two exact sequences:

$$0 \rightarrow \mathcal{O}(-3) \rightarrow \mathcal{O}^{\oplus r+1} \rightarrow \mathcal{E}_0 \rightarrow 0;$$
$$0 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E} \rightarrow k(w) \rightarrow 0.$$

We see that $\mathcal{E}_0$ is a torsion-free sheaf of rank r with $c_1(\mathcal{E}_0) = 3$, $c_2(\mathcal{E}_0) = 9$, and $h^1(\mathcal{E}_0) = 1$. We have $\mathcal{E} \cong \mathcal{E}_0^{\vee\vee}$, and thus $\mathcal{E}$ is a nef vector bundle by Lemma 9.1. Moreover $c_1 = 3$ and $c_2 = 8$. Since $c_2 < 9$, we obtain $h^1(\mathcal{E}) = 0$ by (3.8). Hence $H^0(\mathcal{E}_0) \cong H^0(\mathcal{E})$. Therefore $\mathcal{E}$ is not globally generated. □

Remark 9.1 If $r = 2$, the exact sequence in Proposition 2 already appears in [9, 3.2.5]. Professor Adrian Langer kindly informed the author of this fact and that he ruled out this case by mistake.

10 Some examples

Let X be a smooth projective variety, and $\mathcal{E}$ a vector bundle on X of rank r. It is well known (see, e.g., [2, Statement(folklore) 4.1]) that if $\mathcal{E}$ is globally generated then $r-1$ general global sections of $\mathcal{E}$ define an injection $\mathcal{O}_X^{\oplus r-1} \to \mathcal{E}$ and this injection extends to an exact sequence

$$0 \to \mathcal{O}_X^{\oplus r-1} \to \mathcal{E} \to \mathcal{I}_Z \otimes \det \mathcal{E} \to 0$$

where Z is a locally complete intersection subscheme of codimension two in X, if not empty. For nef vector bundles, however, analogous results do not hold in general, even if $h^0(\mathcal{E}) \geq r-1$, as the following examples show.

Example 2 Let $\mathcal{E}_0$ be a nef vector bundle of rank $r-2$ on $\mathbb{P}^2$ fitting in the following exact sequence

$$0 \to \mathcal{O}(-4) \to \mathcal{O}^{\oplus r-1} \to \mathcal{E}_0 \to 0.$$

Then $r \geq 4$ and $h^1(\mathcal{E}_0) = 3$. Let ξ_1 and ξ_2 be linearly independent elements in $H^1(\mathcal{E}_0)$, and let

$$0 \to \mathcal{E}_0 \to \mathcal{E} \to \mathcal{O}^{\oplus 2} \to 0$$

be the exact sequence whose extension class in $\mathrm{Ext}^1(\mathcal{O}^{\oplus 2}, \mathcal{E}_0)$ is determined by ξ_1 and ξ_2. Then the connecting homomorphism $H^0(\mathcal{O}^{\oplus 2}) \to H^1(\mathcal{E}_0)$ is injective, and thus $h^0(\mathcal{E}) = h^0(\mathcal{E}_0) = r-1$. Moreover $\mathcal{E}$ is a nef vector bundle of rank r by [10, Theorem 6.2.12 (ii)]. In this example, every morphism $\mathcal{O}^{\oplus r-1} \to \mathcal{E}$ is not injective, since it factors through the bundle $\mathcal{E}_0$ of rank $r-2$.

Example 3 Let $\mathcal{E}_0$ be a nef vector bundle of rank $r-1$ on $\mathbb{P}^2$ fitting in the following exact sequence

$$0 \to \mathcal{O}(-3) \to \mathcal{O}^{\oplus r} \to \mathcal{E}_0 \to 0.$$

Then $r \geq 3$ and $h^1(\mathcal{E}_0) = 1$. Let

$$0 \to \mathcal{E}_0 \to \mathcal{E} \to \mathcal{O} \to 0$$

be a non-split exact sequence; the connecting homomorphism $H^0(\mathcal{O}) \to H^1(\mathcal{E}_0)$ is an isomorphism. Then $h^0(\mathcal{E}) = h^0(\mathcal{E}_0) = r$, and it follows from [10, Theorem 6.2.12 (ii)] that $\mathcal{E}$ is a nef but non-globally generated vector bundle of rank r with $c_1 = 3$ and $c_2 = 9$. In this example, a general morphism $\mathcal{O}^{\oplus r-1} \to \mathcal{E}$ is injective, but its cokernel $\mathcal{C}$ fits in a non-split exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{C} \to \mathcal{O} \to 0$$

where $\mathcal{O}_C$ is the structure sheaf of some curve C of degree 3 in $\mathbb{P}^2$; since $\mathcal{C}$ has a non-zero torsion subsheaf $\mathcal{O}_C$, $\mathcal{C}$ is not isomorphic to a torsion-free coherent sheaf $\mathcal{I}_Z \otimes \det \mathcal{E}$ for any closed subscheme Z of $\mathbb{P}^2$.

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