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On a theorem of Bishop and commutants of Toeplitz operators in $\mathbb{C}^n$

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Abstract We prove an approximation theorem on a class of domains in $\mathbb{C}^n$ on which the $\overline{\partial}$-problem is solvable in L^∞. Furthermore, as a corollary, we obtain a version of the Axler–Čučković–Rao theorem in higher dimensions.

Keywords Bishop's theorem · Pseudoconvex domain · Toeplitz operator

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Let Ω be a domain in $\mathbb{C}^n$ and ϕ be a complex-valued function on Ω. Let $H^\infty(\Omega)$ and $H^\infty(\Omega)[\phi]$ denote the set of bounded holomorphic functions on Ω and the algebra generated by ϕ over $H^\infty(\Omega)$, respectively. In 1989, Christopher Bishop proved the following approximation theorem (see [6, Theorem 1.2]).

Theorem (Bishop) *Let Ω be an open set in $\mathbb{C}$ and f be a bounded holomorphic function on Ω that is non-constant on every connected component of Ω. Then $H^\infty(\Omega)[\overline{f}]$ is dense in $C(\overline{\Omega})$ in the uniform topology.*

In the same paper, Christophe Bishop also proved a stronger approximation result, [6, Theorem 1.1], on a more restrictive class of domains on which $\overline{f}$ is only assumed to be a non-holomorphic harmonic function. Such a result for the unit disc goes back to Sheldon Axler and Allen Shields [4]. Recently, Guangfu Cao gave an incorrect statement [8, Theorem 5] in an attempt to give a higher dimensional version of Bishop's Theorem. Alexander Izzo and Bo Li [14, pg 246] noticed that the statement is incorrect. Håkan Samuelsson and Erlend

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Wold in [24, Theorem 1.3] proved a partial extension of Bishop's Theorem for pluriharmonic functions and C^1-smooth polynomially convex domains in $\mathbb{C}^n$.

This article is motivated by these papers and is an attempt to contribute an approximation theorem akin to Bishop's Theorem on domains in $\mathbb{C}^n$. We are not able to generalize Bishop's theorem to $\mathbb{C}^n$ and this is still an open problem. However, we prove approximation results under some restrictions on the functions and the domains. Furthermore, we apply our results to prove a version of the Axler–Čučković–Rao Theorem [2] in higher dimensions.

To present our first result we need to make some definitions. Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain and $CL^{\infty}_{(0,q)}(\Omega)$ denote the set of $(0, q)$-forms with coefficient functions that are C^{∞}-smooth and bounded on Ω. That is, $CL^{\infty}_{(0,q)}(\Omega) = L^{\infty}_{(0,q)}(\Omega) \cap C^{\infty}_{(0,q)}(\Omega)$. We call Ω a L^{∞}*-pseudoconvex* domain if for $1 \leq q \leq n$, and $f \in CL^{\infty}_{(0,q)}(\Omega)$ such that $\overline{\partial} f = 0$ there exists $g \in L^{\infty}_{(0,q-1)}(\Omega)$ such that $\overline{\partial} g = f$.

The class of L^{∞}-pseudoconvex domains include the products of C^2-smooth bounded strongly pseudoconvex domains [23], smooth bounded pseudoconvex finite type domains in $\mathbb{C}^2$ [22], smooth bounded finite type convex domains in $\mathbb{C}^n$ [12], and some infinite type smooth bounded convex domains in $\mathbb{C}^2$ [13].

Given a holomorphic mapping $f : \Omega \to \mathbb{C}^m$ (where $\Omega \subset \mathbb{C}^n$) and $\lambda \in \mathbb{C}^m$, we denote the union of all non-isolated points of $f^{-1}(\lambda)$ by $\Omega_{f,\lambda}$. Since $f^{-1}(\lambda)$ is a complex subvariety of Ω (for λ in the range of f), it follows that $\Omega_{f,\lambda}$ is the union of all positive dimensional connected components of $f^{-1}(\lambda)$. In the case f extends smoothly up to the boundary of Ω, we define $\Omega'_{f,\lambda}$ to be the union of all non-isolated points of $f^{-1}(\lambda)$ within $\overline{\Omega}$. Clearly $\Omega'_{f,\lambda} \subset \Omega_{f,\lambda} \cup b\Omega$ where $b\Omega$ denotes the boundary of Ω. We define

$$\Omega_f = \bigcup_{\lambda \in \mathbb{C}^m} \Omega_{f,\lambda}.$$

It is clear that Ω_f is a subset of the set where the Jacobian of f has rank strictly less than n.

Now we are ready to present our first approximation result.

Theorem 1 *Let Ω be a bounded L^{∞}-pseudoconvex domain in $\mathbb{C}^n$ and $f_j \in H^{\infty}(\Omega)$ for $j = 1, \ldots, m$. Assume that $g \in C(\overline{\Omega})$ such that $g|_{b\Omega \cup \Omega_f} = 0$ where $f = (f_1, \ldots, f_m)$. Then g belongs to the closure of $H^{\infty}(\Omega)[\overline{f_1}, \ldots, \overline{f_m}]$ in $L^{\infty}(\Omega)$.*

Theorem 1 and [14, Theorem 4.2] lead to the following corollary.

Corollary 1 *Let Ω be a bounded L^{∞}-pseudoconvex domain in $\mathbb{C}^n$ and $f_j \in H^{\infty}(\Omega)$ for $j = 1, \ldots, m$ and $n \leq m$. Then the following are equivalent.*

i. *$H^{\infty}(\Omega)[\overline{f_1}, \ldots, \overline{f_m}]$ is dense in $L^p(\Omega)$ for all $0 < p < \infty$,*
ii. *$H^{\infty}(\Omega)[\overline{f_1}, \ldots, \overline{f_m}]$ is dense in $L^p(\Omega)$ for some $1 \leq p < \infty$,*
iii. *the Jacobian of $f = (f_1, \ldots, f_m)$ has rank n for some $z \in \Omega$.*

To formulate our next result we will need the following notation. The set of holomorphic functions on Ω that have smooth extensions up to the boundary is denoted by $A^{\infty}(\Omega)$. Given a compact set $K \subset \overline{\Omega}$, we will denote by $A_{\overline{\Omega}}(K)$ the norm closed subalgebra of continuous functions on K spanned by restrictions of $A^{\infty}(U \cap \Omega)$ onto K, where U runs through open neighborhoods of K.

Theorem 2 *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ and $f_j \in A^{\infty}(\Omega)$ for $j = 1, \ldots, m$. Then $g \in C(\overline{\Omega})$ belongs to the closure of $A^{\infty}(\Omega)[\overline{f_1}, \ldots, \overline{f_m}]$ in $L^{\infty}(\Omega)$ if and only if for any λ in the range of $f = (f_1, \ldots, f_m)$ we have $g|_{\Omega'_{f,\lambda}} \in A_{\overline{\Omega}}(\Omega'_{f,\lambda})$.*

Alexander Izzo in [15, Theorem 1.3] proved (among other things) the following interesting result.

Theorem (Izzo) *Let A be a uniform algebra on a compact Hausdorff space X whose maximal ideal space is X and $E \subset X$ be a closed subset such that $X \setminus E$ is an m-dimensional manifold. Assume that*

i. *for any $p \in X \setminus E$ there exists $f_1, \dots, f_m \in A$ that are C^1-smooth on $X \setminus E$ and $df_1 \wedge \dots \wedge df_m(p) \neq 0$,*
ii. *the functions in A that are C^1-smooth on $X \setminus E$ separate points on X.*

Then $A = \{g \in C(X) : g|E \in A|E\}$.

As pointed out to us by Alexander Izzo, a result along the lines of Theorem 1 (for a similar class of domains) can be obtained from [15] as follows. Let us take X to be the maximal ideal space (spectrum) of $H^\infty(\Omega)$ and $X \setminus E$ to be the set of points in Ω where the Jacobian of f has rank n with A being the closure of $H^\infty(\Omega)[\overline{f_1}, \dots, \overline{f_m}]$. Then one obtains Theorem 1 if the set Ω_f is replaced by the set of points where J_f, the Jacobian of f, has rank strictly less than n (usually a larger set than Ω_f).

Next we will present our generalization of the Axler–Čučković–Rao Theorem to $\mathbb{C}^n$, but first we will state the commuting problem for Toeplitz operators.

Let $A^2(\Omega)$ denote the space of square integrable holomorphic functions on Ω and $P : L^2(\Omega) \to A^2(\Omega)$ be the Bergman projection, the orthogonal projection onto $A^2(\Omega)$. For $g \in L^\infty(\Omega)$, the Toeplitz operator $T_g : A^2(\Omega) \to A^2(\Omega)$ is defined as $T_g f = P(gf)$ for all $f \in A^2(\Omega)$.

The *commuting problem* can be stated as follows: Let ϕ be a non-constant bounded function on Ω. Determine all $\psi \in L^\infty(\Omega)$ such that $[T_\phi, T_\psi] = 0$.

The commuting problem was solved by Arlen Brown and Paul Halmos on the Hardy space of the unit disc in a famous paper [5]. However, on the Bergman space, the problem is still open. Many partial answers has been obtained over the years. To list a few, we refer the reader to [1,2,9,20] for results over the unit disc; to [18,19,25] for results over the ball in $\mathbb{C}^n$; and to [3,7,11] for results on Fock spaces.

In this paper, we want to highlight the following result of Sheldon Axler, Željko Čučković, and Nagisetti Rao (see [2]).

Theorem (Axler–Čučković–Rao) *Let Ω be a bounded domain in $\mathbb{C}$ and ϕ be a nonconstant bounded holomorphic function on Ω. Assume that ψ is a bounded measurable function on Ω such that T_ϕ and T_ψ commute. Then ψ is holomorphic.*

As an application of our results, we get the following generalization of the Axler–Čučković–Rao Theorem.

Corollary 2 *Let Ω be a bounded L^∞-pseudoconvex domain in $\mathbb{C}^n$, $g \in L^\infty(\Omega)$, and $f_j \in H^\infty(\Omega)$ for $j = 1, \dots, m$ and $n \leq m$. Assume that the Jacobian of the function $f = (f_1, \dots, f_m) : \Omega \to \mathbb{C}^m$ has rank n for some $z \in \Omega$ and T_g commutes with T_{f_j} for $1 \leq j \leq m$. Then g is holomorphic.*

This paper is organized as follows: The next section contains relevant basic facts and results about $\overline{\partial}$-Koszul complex. Then we will present the proofs of Theorems 1 and 2. We will finish the paper with the proof of Corollaries 1 and 2.

The $\overline{\partial}$-Koszul Complex

Let Ω be a domain in $\mathbb{C}^n$ and V be a vector space of dimension m with a basis $\{e_1, e_2, \ldots, e_m\}$. We define

$$\wedge^r V = \text{span}\left\{e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_r} : j_1 < j_2 < \cdots < j_r\right\}$$

and $\Gamma^{\infty}_{(r,s)} = \wedge^r V \otimes CL^{\infty}_{(0,s)}(\Omega)$ where r and s are nonnegative integers. We note that throughout the paper we use the convention that $\Gamma^{\infty}_{(r,s)} = \{0\}$ if $r \geq m+1$ or $s \geq n+1$. Finally, $CL^{\infty}_{(0,0)}(\Omega) = CL^{\infty}(\Omega)$.

We define the unbounded operator $\overline{\partial} : \Gamma^{\infty}_{(r,s)} \to \Gamma^{\infty}_{(r,s+1)}$ as $\overline{\partial}(e_J \otimes W) = e_J \otimes \overline{\partial} W$ where $e_J \in \wedge^r V$ and $W \in CL^{\infty}_{(0,s)}(\Omega)$. The operator $\overline{\partial}$ is defined on

$$Dom_{\infty}(\overline{\partial}) = \left\{ f \in \Gamma^{\infty}_{(r,s)} : \overline{\partial} f \in \Gamma^{\infty}_{(r,s+1)} \right\}.$$

Let $f = (f_1, \ldots, f_m) : \Omega \to \mathbb{C}^m$ be a bounded holomorphic mapping. Then for $0 \leq s \leq n$ and $0 \leq r \leq m$ we define the operator

$$\mathcal{T}_f : \Gamma^{\infty}_{(r+1,s)} \to \Gamma^{\infty}_{(r,s)}$$

with the following properties:

(1) $\mathcal{T}_f(e_j \otimes W) = f_j W$,
(2) $\mathcal{T}_f(A \wedge B) = \mathcal{T}_f(A) \wedge B + (-1)^{|A|_1} A \wedge \mathcal{T}_f B$ (here $|.|_1$ is the order of A in $\cup_{r=0}^{m} \Lambda^r V$),
(3) $\mathcal{T}_f \overline{\partial} = \overline{\partial} \mathcal{T}_f$ on $Dom_{\infty}(\overline{\partial})$ for $0 \leq s \leq n$ and $0 \leq r \leq m$,
(4) $\mathcal{T}_f \mathcal{T}_f = 0$ and $\overline{\partial}\overline{\partial} = 0$.

We note that $\mathcal{T}_f W = 0$ for $W \in \Gamma^{\infty}_{(0,s)}$ and $0 \leq s \leq n$.

Lemma 1 *Let Ω be a bounded domain in $\mathbb{C}^n, 0 \leq s \leq n, 0 \leq r \leq m$, and $f = (f_1, \ldots, f_m) : \Omega \to \mathbb{C}^m$ be a bounded holomorphic mapping. Assume that $W \in \Gamma^{\infty}_{(r,s)}$ such that supp$(W) \subset \Omega$ and supp$(W) \cap f^{-1}(0) = \varnothing$.*

i. *If $\mathcal{T}_f W = 0$, then there exists $Y \in \Gamma^{\infty}_{(r+1,s)}$ such that*

   a. *$\mathcal{T}_f Y = W$,*
   b. *supp$(Y) \subset \Omega$ and supp$(Y) \cap f^{-1}(0) = \varnothing$.*

ii. *If $\mathcal{T}_f W = 0$ and $\overline{\partial} W \in \Gamma^{\infty}_{(r,s+1)}$, then there exists $Y \in \Gamma^{\infty}_{(r+1,s)}$ such that*

   a. *$\overline{\partial} Y \in \Gamma^{\infty}_{(r+1,s+1)}$ and $\mathcal{T}_f Y = W$,*
   b. *supp$(Y) \subset \Omega$ and supp$(Y) \cap f^{-1}(0) = \varnothing$.*

Proof First let us prove the lemma in case $r = m$. In this case one can show that $\mathcal{T}_f W = 0$ and supp$(W) \cap f^{-1}(0) = \varnothing$ imply that $W = 0$. So we can choose $Y = 0 \in \Gamma^{\infty}_{(m+1,s)}$. For the rest of the proof we will assume that $0 \leq r \leq m-1$.

Now let us prove i. Let $\chi \in C^{\infty}_0(\Omega)$ be a smooth compactly supported cut-off function such that $\chi = 1$ on a neighborhood of supp(W) and supp$(\chi) \cap f^{-1}(0) = \varnothing$. We define

$$g_j = \frac{\chi \overline{f_j}}{\sum_{l=1}^{m} |f_l|^2}$$

and

$$X = \sum_{j=1}^{m} e_j \otimes g_j \in \Gamma^{\infty}_{(1,0)}.$$

Then $g_j \in C_0^\infty(\Omega)$ for $j = 1, 2, \ldots, m$ and $\mathcal{T}_f X = 1 \in \Gamma^\infty_{(0,0)}$ on the support of W because $\chi = 1$ on a neighborhood of $\mathrm{supp}(W)$ and $\sum_{j=1}^m f_j(z)g_j(z) = 1$ whenever $\chi(z) = 1$.

Let us define $Y = X \wedge W \in \Gamma^\infty_{(r+1,s)}$. Then $\mathrm{supp}(Y)$ is a compact subset of Ω and $\mathrm{supp}(Y) \cap f^{-1}(0) = \varnothing$. Furthermore, $\mathcal{T}_f X = 1$ on the support of W and

$$\mathcal{T}_f Y = \mathcal{T}_f(X) \wedge W - X \wedge \mathcal{T}_f W = 1 \wedge W = W$$

because $\mathcal{T}_f W = 0$.

To prove ii. we observe that, in the proof of i. above, X is smooth compactly supported in Ω. Therefore, if $\overline{\partial} W$ is bounded then so is $\overline{\partial} Y$ as $Y = X \wedge W$. □

If $f_j \in A^\infty(\Omega)$ for $j = 1, 2, \ldots, m$ in the lemma above, we have the following lemma.

Lemma 2 *Let Ω be a bounded domain in $\mathbb{C}^n$, V be an m-dimensional vector space, and $f_j \in A^\infty(\Omega)$ for $j = 1, 2, \ldots, m$. Assume that $W \in \wedge^r V \otimes C^\infty_{(0,s)}(\overline{\Omega})$ for $0 \le r \le m$, $0 \le s \le n$, and $\mathrm{supp}(W) \cap f^{-1}(0) = \varnothing$ where $f = (f_1, \ldots, f_m)$. If $\mathcal{T}_f W = 0$ then there exists $Y \in \wedge^{r+1} V \otimes C^\infty_{(0,s)}(\overline{\Omega})$ such that $\mathrm{supp}(Y) \cap f^{-1}(0) = \varnothing$ and $\mathcal{T}_f Y = W$.*

Proof The proof of this lemma is very similar to the proof of Lemma 1. The only difference is that we choose $\chi \in C^\infty(\overline{\Omega})$ be a smooth function such that $\chi = 1$ on a neighborhood of $\mathrm{supp}(W)$ and $\mathrm{supp}(\chi) \cap f^{-1}(0) = \varnothing$. □

Lemma 3 *Let Ω be a bounded L^∞-pseudoconvex domain in $\mathbb{C}^n$, $f = (f_1, \ldots, f_m) : \Omega \to \mathbb{C}^m$ be a bounded holomorphic mapping, and $W \in \Gamma^\infty_{(r,s)}$ for $0 \le r \le m$ and $1 \le s \le n$ such that*

i. $\mathrm{supp}(W) \subset \Omega$ and $\mathrm{supp}(W) \cap f^{-1}(0) = \varnothing$,
ii. $\overline{\partial} W = 0$ and $\mathcal{T}_f W = 0$.

Then there exists $Y \in \Gamma^\infty_{(r+1,s-1)}$ such that $Y \in Dom_\infty(\overline{\partial})$ and $\mathcal{T}_f \overline{\partial} Y = W$.

Proof In case $r = m$, as in the proof of Lemma 1, one can show that if W satisfies the conditions of the lemma then $W = 0$. So we can choose $Y = 0$. For the rest of the proof we will assume that $0 \le r \le m - 1$.

First we will assume that Ω is a bounded L^∞-pseudoconvex domain. We will use a descending induction on s to prove this lemma. So let $s = n$, $0 \le r \le m - 1$, and $W \in \Gamma^\infty_{(r,n)}$ such that $\mathrm{supp}(W) \subset \Omega$, $\mathrm{supp}(W) \cap f^{-1}(0) = \varnothing$, and $\mathcal{T}_f W = 0$ ($\overline{\partial} W = 0$ as any $(0, n)$-form is $\overline{\partial}$-closed). Then i. in Lemma 1 implies that there exists $Y_1 \in \Gamma^\infty_{(r+1,n)}$ with the following properties:

i. $\mathrm{supp}(Y_1) \subset \Omega$ and $\mathrm{supp}(Y_1) \cap f^{-1}(0) = \varnothing$,
ii. $\mathcal{T}_f Y_1 = W$.

Furthermore, since $Y_1 \in \Gamma^\infty_{(r+1,n)}$ it is $\overline{\partial}$-closed. Then (since Ω is L^∞-pseudoconvex) there exists $Y \in \Gamma^\infty_{(r+1,n-1)}$ such that $\overline{\partial} Y = Y_1$. That is, $\mathcal{T}_f \overline{\partial} Y = W$.

Now we will assume that the lemma is true for $s = k + 1, k + 2, \ldots, n$ and $r = 0, 1, \ldots, m - 1$. Let $0 \le r \le m - 1$ and assume that $W \in \Gamma^\infty_{(r,k)}$ with the following properties:

i. $\mathrm{supp}(W) \subset \Omega$ and $\mathrm{supp}(W) \cap f^{-1}(0) = \varnothing$,
ii. $\overline{\partial} W = 0$ and $\mathcal{T}_f W = 0$.

Then ii. in Lemma 1 implies that there exists $Y_1 \in \Gamma^\infty_{(r+1,k)}$ such that

i. $\overline{\partial} Y_1 \in \Gamma^\infty_{(r+1,k+1)}$ and $W = \mathcal{T}_f Y_1$,
ii. $\mathrm{supp}(Y_1) \subset \Omega$ and $\mathrm{supp}(Y_1) \cap f^{-1}(0) = \varnothing$.

Then

$$\mathcal{T}_f \overline{\partial} Y_1 = \overline{\partial} \mathcal{T}_f Y_1 = \overline{\partial} W = 0.$$

So $\overline{\partial} Y_1$ satisfies the conditions in the lemma for $s = k+1$. That is, $\overline{\partial} Y_1 \in \Gamma^\infty_{(r+1,k+1)}$ such that

i. $\mathrm{supp}(\overline{\partial} Y_1) \subset \Omega$ and $\mathrm{supp}(\overline{\partial} Y_1) \cap f^{-1}(0) = \varnothing$,
ii. $\overline{\partial}\,\overline{\partial} Y_1 = 0$ and $\mathcal{T}_f \overline{\partial} Y_1 = \overline{\partial} W = 0$.

By the induction hypothesis, there exists $Y_2 \in \Gamma^\infty_{(r+2,k)}$ such that $\overline{\partial} Y_2 \in \Gamma^\infty_{(r+2,k+1)}$ and $\mathcal{T}_f \overline{\partial} Y_2 = \overline{\partial} Y_1$. Then

$$\overline{\partial} \mathcal{T}_f Y_2 = \mathcal{T}_f \overline{\partial} Y_2 = \overline{\partial} Y_1.$$

We define $Y_3 = Y_1 - \mathcal{T}_f Y_2 \in \Gamma^\infty_{(r+1,k)}$. Then the equality above implies that

$$\mathcal{T}_f Y_3 = \mathcal{T}_f Y_1 - \mathcal{T}_f \mathcal{T}_f Y_2 = W$$

and $\overline{\partial} Y_3 = \overline{\partial} Y_1 - \overline{\partial} \mathcal{T}_f Y_2 = 0$. Since Ω is L^∞-pseudoconvex domain we conclude that there exists $Y \in \Gamma^\infty_{(r+1,k-1)}$ such that $\overline{\partial} Y = Y_3$. That is, $\mathcal{T}_f \overline{\partial} Y = W$. Hence the proof of Lemma 3 is complete. □

Lemma 4 *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, V be an m-dimensional vector space, and $f_i \in A^\infty(\Omega)$ for $i = 1, \ldots, m$. Assume that $W \in \wedge^r V \otimes C^\infty_{(0,s)}(\overline{\Omega})$ for $0 \le r \le m$ and $1 \le s \le n$ such that $\mathrm{supp}(W) \cap f^{-1}(0) = \varnothing$, $\overline{\partial} W = 0$, and $\mathcal{T}_f W = 0$. Then there exists $Y \in \wedge^{r+1} V \otimes C^\infty_{(0,s-1)}(\overline{\Omega})$ such that $\mathcal{T}_f \overline{\partial} Y = W$.*

Proof This proof is similar to the proof of Lemma 3 with the following changes: Instead of Lemma 1 we use Lemma 2 and, at the last step (since and $f_j \in A^\infty(\Omega)$), we use the following result of Joseph Kohn [16] (see also [10, Theorem 6.1.1]): Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $1 \le q \le n$, and $u \in C^\infty_{(0,q)}(\overline{\Omega})$ with $\overline{\partial} u = 0$. Then there exists $f \in C^\infty_{(0,q-1)}(\overline{\Omega})$ such that $\overline{\partial} f = u$. □

Lemma 5 *Let Ω be a bounded domain in $\mathbb{C}^n$ and $f_j \in H^\infty(\Omega)$ for $j = 1, \ldots, m$ such that $\sum_{j=1}^m |f_j|^2 > \varepsilon$ on Ω for some $\varepsilon > 0$ and $\partial f_j \in L^\infty_{(1,0)}(\Omega)$ for $j = 1, \ldots, m$. Assume that $W \in \Gamma^\infty_{(r,s)}$ for $0 \le r \le m$ and $0 \le s \le n$ such that $\mathcal{T}_f W = 0$ and $\overline{\partial} W \in \Gamma^\infty_{(r,s+1)}$. Then there exists $Y \in \Gamma^\infty_{(r+1,s)}$ such that $\overline{\partial} Y \in \Gamma^\infty_{(r+1,s+1)}$ and $\mathcal{T}_f Y = W$.*

Proof The proof will be similar to the proof of Lemma 1. Let V be a vector space of dimension m and $\{e_1, e_2, \ldots, e_m\}$ be a basis for V. We define

$$g_j = \frac{\overline{f_j}}{\sum_{l=1}^m |f_l|^2}$$

and $X = \sum_{j=1}^m e_j \otimes g_j \in \Gamma^\infty_{(1,0)}$. Then $g_j \in L^\infty(\Omega)$ and

$$\overline{\partial} g_j = \frac{\overline{\partial f_j}}{\sum_{l=1}^m |f_l|^2} - \frac{\overline{f_j} \sum_{l=1}^m f_l \overline{\partial f_l}}{\left(\sum_{l=1}^m |f_l|^2\right)^2} \in L^\infty_{(0,1)}(\Omega).$$

Furthermore, $\overline{\partial} X = \sum_{j=1}^{m} e_j \otimes \overline{\partial} g_j \in \Gamma^{\infty}_{(1,1)}$. Then $Y = X \wedge W \in \Gamma^{\infty}_{(r+1,s)}$ satisfies the following properties: $\overline{\partial} Y = \overline{\partial} X \wedge W + X \wedge \overline{\partial} W \in \Gamma^{\infty}_{(r+1,s+1)}$ and

$$\mathcal{T}_f Y = \mathcal{T}_f(X) \wedge W - X \wedge \mathcal{T}_f W = 1 \wedge W = W$$

as $\mathcal{T}_f W = 0$. □

Proposition 1 *Let Ω be a bounded L^{∞}-pseudoconvex domain in $\mathbb{C}^n$ and $f_j \in H^{\infty}(\Omega)$ for $j = 1, \dots, m$ such that $\sum_{j=1}^{m} |f_j|^2 > \varepsilon$ on Ω for some $\varepsilon > 0$ and $\partial f_j \in L^{\infty}_{(1,0)}(\Omega)$ for $j = 1, \dots, m$. Assume that $W \in \Gamma^{\infty}_{(r,s)}$ for $0 \leq r \leq m$ and $0 \leq s \leq n$ such that $\overline{\partial} W = 0$ and $\mathcal{T}_f W = 0$. Then there exists $Y \in \Gamma^{\infty}_{(r+1,s)}$ such that $\overline{\partial} Y = 0$ and $\mathcal{T}_f Y = W$.*

Proof We will use a descending induction on s as in the proof of Proposition 1. Let $s = n$. Any form of type (r, n) for $0 \leq r \leq m$ is $\overline{\partial}$-closed. Then $\overline{\partial} Y = 0$ and Lemma 5 implies that there exists $Y \in \Gamma^{\infty}_{(r+1,n)}$ such that $\mathcal{T}_f Y = W$.

Now we will assume that the lemma is true for $s = l+1, l+2, \dots, n$ and $r = 0, 1, \dots, m$ to prove that it is also true for $s = l \leq n - 1$ and $0 \leq r \leq m$.

Assume that $W \in \Gamma^{\infty}_{(r,l)}$ such that $\overline{\partial} W = 0$ and $\mathcal{T}_f W = 0$. Then Lemma 5 implies that there exists $\widetilde{Y} \in \Gamma^{\infty}_{(r+1,l)}$ such that $\overline{\partial} \widetilde{Y} \in \Gamma^{\infty}_{(r+1,l+1)}$ and $W = \mathcal{T}_f \widetilde{Y}$. Then

$$\mathcal{T}_f \overline{\partial} \widetilde{Y} = \overline{\partial} \mathcal{T}_f \widetilde{Y} = \overline{\partial} W = 0.$$

So $\overline{\partial} \widetilde{Y}$ satisfies the conditions in the lemma for $s = l+1$. That is, $\overline{\partial} \widetilde{Y} \in \Gamma^{\infty}_{(r+1,l+1)}$, $\overline{\partial}\overline{\partial} \widetilde{Y} = 0$ and $\mathcal{T}_f \overline{\partial} \widetilde{Y} = \overline{\partial} W = 0$. Then, by the induction hypothesis, there exists $Y_1 \in \Gamma^{\infty}_{(r+2,l+1)}$ such that $\overline{\partial} Y_1 = 0$ and $\mathcal{T}_f Y_1 = \overline{\partial} \widetilde{Y}$. Then since Ω is a L^{∞}-pseudoconvex domain there exists $Y_2 \in \Gamma^{\infty}_{(r+2,l)}$ such that $\overline{\partial} Y_2 = Y_1$. Then

$$\overline{\partial} \mathcal{T}_f Y_2 = \mathcal{T}_f \overline{\partial} Y_2 = \mathcal{T}_f Y_1 = \overline{\partial} \widetilde{Y}.$$

We define $Y = \widetilde{Y} - \mathcal{T}_f Y_2 \in \Gamma^{\infty}_{(r+1,l)}$. Then the equality above implies that $\overline{\partial} Y = \overline{\partial} \widetilde{Y} - \overline{\partial} \mathcal{T}_f Y_2 = 0$ and

$$\mathcal{T}_f Y = \mathcal{T}_f \widetilde{Y} - \mathcal{T}_f \mathcal{T}_f Y_2 = W.$$

Hence the proof of Proposition 1 is complete. □

As a corollary to the previous proposition (with $W = 1$ and $r = s = 0$) we get the following Corona type result. We refer the reader to [17] and the references therein for more information about Corona problem on domains in $\mathbb{C}^n$.

Corollary 3 *Let Ω be a bounded L^{∞}-pseudoconvex domain in $\mathbb{C}^n$ and $f_j \in H^{\infty}(\Omega)$ for $j = 1, \dots, m$ such that $\sum_{j=1}^{m} |f_j|^2 > \varepsilon$ on Ω for some $\varepsilon > 0$ and $\partial f_j \in L^{\infty}_{(1,0)}(\Omega)$ for $j = 1, \dots, m$. Then there exists $g_i \in H^{\infty}(\Omega)$ for $j = 1, \dots, m$ such that $\sum_{j=1}^{m} f_j g_j = 1$.*

Proofs of results

The proofs of the theorems are mainly inspired by the proof in Christopher Bishop's paper [6].

Proofs of Theorems 1 and 2 The proofs of both theorems are very similar. So we will present the proof of Theorem 1 and comment on how the proof of Theorem 2 differs as we go along.

Let $\epsilon > 0$ and $\lambda \in \mathbb{C}^m$. Since $g \in C(\overline{\Omega})$ and $g|_{b\Omega \cup \Omega_f} = 0$, there exist $g^{\lambda} \in C^{\infty}(\overline{\Omega})$ such that

i. $\sup\{|g(z)-g^{\lambda}(z)| : z \in \overline{\Omega}\} < \varepsilon$,
ii $\operatorname{supp}(\overline{\partial} g^{\lambda}) \cap (b\Omega \cup f^{-1}(\lambda)) = \varnothing$.

In the proof Theorem 2 the second condition above is replaced by $\operatorname{supp}(\overline{\partial} g^{\lambda}) \cap f^{-1}(\lambda) = \varnothing$. This can be seen as follows: We choose an open set U_{ε} in $\mathbb{C}^n$ containing $f^{-1}(\lambda)$ and $g_{\varepsilon} \in A^{\infty}(U_{\varepsilon} \cap \Omega)$ such that $|g-g_{\varepsilon}| < \varepsilon/2$ on $f^{-1}(\lambda)$. Then we choose $\chi_{\varepsilon} \in C_0^{\infty}(U_{\varepsilon})$ such that, $0 \leq \chi_{\varepsilon} \leq 1$, $\chi_{\varepsilon} = 1$ on a neighborhood of $f^{-1}(\lambda)$, and

$$\operatorname{supp}(\chi_{\varepsilon}) \cap \overline{\Omega} \subset \left\{ z \in U_{\varepsilon} \cap \overline{\Omega} : |g(z) - g_{\varepsilon}(z)| < \varepsilon \right\}.$$

Then we define $g^{\lambda} = (1-\chi_{\varepsilon})g + \chi_{\varepsilon} g_{\varepsilon}$. Since g^{λ} is holomorphic on a neighborhood of $f^{-1}(\lambda)$ we have $\overline{\partial} g^{\lambda} = 0$ on the same neighborhood. Furthermore, $|g^{\lambda}(z) - g(z)| = \chi_{\varepsilon}(z)|g_{\varepsilon}(z) - g(z)| < \varepsilon$ for all $z \in \overline{\Omega}$.

Using Lemma 3 with $r = 0, s = 1$, and $W = \overline{\partial} g^{\lambda}$ we get $Y = \sum_{l=1}^{m} e_l \otimes H_l \in \Gamma^{\infty}_{(1,0)}$ such that

$$\overline{\partial} g^{\lambda} = \mathcal{T}_{f-\lambda} \overline{\partial} Y = \sum_{l=1}^{m} (f_l - \lambda_l) \overline{\partial} H_l^{\lambda}. \tag{1}$$

The above equality implies that

$$G_{\lambda} = g^{\lambda} - \sum_{l=1}^{m} (f_l - \lambda_l) H_l^{\lambda}$$

is a bounded holomorphic function.

In the proof of Theorem 2, we use Lemma 4 and get $H_l^{\lambda} \in C^{\infty}(\overline{\Omega})$ for $l = 1, \dots, m$ in the equation (1) and G_{λ} is smooth up to the boundary. Therefore, for $z \in \Omega$ we have

$$|G_{\lambda}(z) - g^{\lambda}(z)| \leq \sum_{l=1}^{m} |f_l(z) - \lambda_l| \sum_{s=1}^{m} |H_s^{\lambda}(z)|.$$

Then the above inequality implies that for $M_{\lambda} = \sum_{s=1}^{m} \|H_s^{\lambda}\|_{L^{\infty}(\Omega)} < \infty$ we have

$$|G_{\lambda}(z) - g^{\lambda}(z)| \leq M_{\lambda} |f(z) - \lambda| \tag{2}$$

for $z \in \Omega$.

Compactness of $\overline{f(\Omega)}$ implies that we can choose a finite collection of points $\{\lambda_j\}_{j=1}^{k} \subset \overline{f(\Omega)}$ such that $\{B(\lambda^j, \epsilon M_{\lambda^j}^{-1})\}_{j=1}^{k}$ forms a finite open cover for $\overline{f(\Omega)}$. Let $\{\chi_j\}_{j=1}^{k}$ be a smooth partition of unity on $\overline{f(\Omega)}$ such that $0 \leq \chi_j \leq 1$ and $\operatorname{supp}(\chi_j) \subset U_j$. Then $\{f^{-1}(B(\lambda^j, \epsilon M_{\lambda^j}^{-1}))\}_{j=1}^{k}$ is an cover for Ω and $|f(z) - \lambda^j| < \epsilon M_{\lambda^j}^{-1}$ for $z \in f^{-1}(B(\lambda^j, \epsilon M_{\lambda^j}^{-1}))$. Then for $z \in \Omega$ we have

$$\begin{aligned}
\left| \sum_{j=1}^{k} G_{\lambda^j}(z) \chi_j(f)(z) - g(z) \right| &\leq \sum_{j=1}^{k} |G_{\lambda^j}(z) - g(z)| \chi_j(f(z)) \\
&\leq \sum_{j=1}^{k} |G_{\lambda^j}(z) - g^{\lambda^j}| \chi_j(f(z)) + \sum_{j=1}^{k} |g^{\lambda^j}(z) - g(z)| \chi_j(f(z)) \\
&\leq \sum_{j=1}^{k} M_{\lambda^j} |f(z) - \lambda^j| \chi_j(f(z)) + \varepsilon \sum_{j=1}^{k} \chi_j(f(z)) \\
&\leq 2\epsilon.
\end{aligned}$$

Finally, the Stone-Weierstrass Theorem implies that $\chi_j(f)$ can be approximated uniformly on $\overline{\Omega}$ by elements of $\mathbb{C}[f_1, \dots, f_m, \overline{f_1}, \dots, \overline{f_m}]$. Hence the proofs of Theorems 1 and 2 are complete. □

Hartogs Extension Theorem together Theorem 2 lead to the following corollary.

Corollary 4 *Let Ω be a bounded L^∞-pseudoconvex domain in $\mathbb{C}^n$. Assume that $f = (f_1, \dots, f_m) : \Omega \to \mathbb{C}^m$ be a bounded holomorphic mapping and $g \in C(\overline{\Omega})$ such that $\overline{\partial} g$ is supported away from $b\Omega$ and the set of points at which the Jacobian of f has rank strictly less than n. Then g belongs to the closure of $H^\infty(\Omega)[\overline{f_1}, \dots, \overline{f_m}]$ in $L^\infty(\Omega)$.*

Proof Since $\overline{\partial} g$ vanishes near the boundary of Ω, Hartogs Extension Theorem implies that there exists $g_1 \in H^\infty(\Omega)$ such that $g = g_1$ near the boundary of Ω. Then $g_2 = g - g_1 \in C(\overline{\Omega})$ and g_2 is compactly supported in Ω. Furthermore, g_2 is holomorphic on a neighborhood of the set where the Jacobian of f has rank strictly less than n. Therefore, Theorem 2 implies that g_2 can be approximated in the sup-norm by functions in $H^\infty(\Omega)[\overline{f_1}, \dots, \overline{f_m}]$. This completes the proof of the corollary. □

Next we provide the proof of Corollary 1.

Proof of Corollary 1 Obviously i. implies ii. So to prove that ii. implies iii., let us assume that $H^\infty(\Omega)[\overline{f_1}, \dots, \overline{f_m}]$ is dense in $L^p(\Omega)$ for some $1 \le p < \infty$. Let $B \subset \Omega$ be a ball such that $\overline{B} \subset \Omega$. Then, the algebra $H^\infty(B)[\overline{f_1}, \dots, \overline{f_m}]$ is dense in $L^p(B)$ for some $1 \le p < \infty$. Moreover, the algebra generated by $\{z_1, \dots, z_n\}$ is dense in $H^\infty(B)$ and $f_1, \dots, f_m$ are holomorphic on a neighborhood of $\overline{B}$. Next we adopt [14, Theorem 4.2] to our set-up. Namely, [14, Theorem 4.2] implies that if the algebra generated by $\{z_1, \dots, z_n, \overline{f}_1, \dots, \overline{f}_m\} \subset C^\infty(B)$ is dense in $L^p(B)$ for some $1 \le p < \infty$ then the real Jacobian of $\{z_1, \dots, z_n, \overline{f}_1, \dots, \overline{f}_m\}$ is of full rank on a dense open set in B. Hence the rank of J_f is n on a dense open subset in B and (by identity principle) in Ω. Hence, we have iii.

Finally, to prove iii. implies i. we assume that the rank of J_f is n for some $z \in \Omega$. Then, the set of points at which J_f has rank strictly less than n is a closed set of measure 0 (see [21, Theorem 3.7]). One can show that X_f, the set of smooth functions with compact support in Ω and vanish where J_f has rank strictly less than n, is dense in $L^p(\Omega)$ for all $0 < p < \infty$. On the other hand, Theorem 1 implies that any function in X_f is in the closure of $H^\infty(\Omega)[\overline{f_1}, \dots, \overline{f_m}]$ in $L^\infty(\Omega)$. Therefore, $H^\infty(\Omega)[\overline{f_1}, \dots, \overline{f_m}]$ is dense in $L^p(\Omega)$. Hence, we have i. □

We finally end the paper with the proof of Corollary 2.

Proof of Corollary 2 We will use the fact that T_g can be defined by the following formula

$$\langle T_g \phi, \psi \rangle_{A^2(\Omega)} = \langle g\phi, \psi \rangle_{L^2(\Omega)}$$

for all $\phi, \psi \in A^2(\Omega)$. Since T_g commutes with $T_{P(f)}$, for any holomorphic polynomial P, we have

$$\langle gP(f), \psi \rangle = \langle T_g T_{P(f)}(1), \psi \rangle = \langle P(f) T_g(1), \psi \rangle$$

for all $\psi \in A^2(\Omega)$. Then $\langle T_g(1) - g, \overline{P(f)}\psi \rangle = 0$ for all $\psi \in A^2(\Omega)$. Since, by Corollary 1, the subspace generated by $\{\overline{P(f)}\psi : \psi \in A^2(\Omega)\}$ is dense in $L^2(\Omega)$, we conclude that $T_g(1) = g$. That is, g is holomorphic. □

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