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Problem of descent spectrum equality

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Abstract Let $\mathcal{B}(X)$ be the algebra of all bounded operators acting on an infinite dimensional complex Banach space X. We say that an operator $T \in \mathcal{B}(X)$ satisfies the problem of descent spectrum equality, if the descent spectrum of T as an operator coincides with the descent spectrum of T as an element of the algebra of all bounded linear operators on X. In this paper we are interested in the problem of descent spectrum equality. Specifically, the problem is to consider the following question: let $T \in \mathcal{B}(X)$ such that $\sigma(T)$ has non empty interior, under which condition on T does $\sigma_{desc}(T) = \sigma_{desc}(T, \mathcal{B}(X))$?

Keywords Descend · Pole · Resolvent · Spectrum

Mathematics Subject Classification 47A10 · 47A11

1 Introduction

In this paper, X denotes a complex Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators on X. Let $T \in \mathcal{B}(X)$, we denote by $R(T)$, $N(T)$, $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$ and $\sigma_{su}(T)$ respectively the range, the kernel, the resolvent set, the spectrum, the point spectrum, the approximate point spectrum and the surjectivity spectrum of T. It is well known that $\sigma(T) = \sigma_{su}(T) \cup \sigma_p(T) = \sigma_{su}(T) \cup \sigma_{ap}(T)$. The ascent of T is defined by $a(T) = \min\{p : N(T^p) = N(T^{p+1})\}$, if no such p exists, we let $a(T) = \infty$. Similarly, the descent of T is $d(T) = \min\{q : R(T^q) = R(T^{q+1})\}$, if no such q exists, we let $d(T) = \infty$ [1,4] and [6]. It is well known that if both $a(T)$ and $d(T)$ are finite then $a(T) = d(T)$ and we have the decomposition $X = R(T^p) \oplus N(T^p)$ where $p = a(T) = d(T)$. The descent and ascent spectrum are defined by:

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$$\sigma_{desc}(T) = \{\lambda \in \mathbb{C} : d(\lambda - T) = \infty\}$$
$$\sigma_{asc}(T) = \{\lambda \in \mathbb{C} : a(\lambda - T) = \infty\}$$

$\mathcal{A}$ will denote a complex Banach algebra with unit. For every $a \in \mathcal{A}$, the left multiplication operator L_a is given by $L_a(x) = ax$ for all $x \in \mathcal{A}$. By definition the descent of an element $a \in \mathcal{A}$ is $d(a) := d(L_a)$, and the descent spectrum of a is the set $\sigma_{desc}(a) := \{\lambda \in \mathbb{C} : d(a - \lambda) = \infty\}$.

In general $\sigma_{desc}(T) \subseteq \sigma_{desc}(T, \mathcal{B}(X))$, and we say that an operator T satisfies the descent spectrum equality whenever, the descent spectrum of T as an operator coincides with the descent spectrum of T as an element of the algebra of all bounded linear operators on X.

The operator $T \in \mathcal{B}(X)$ is said to have the single-valued extension property at $\lambda_0 \in \mathbb{C}$, abbreviated T has the SVEP at λ_0, if for every neighbourhood $\mathcal{U}$ of λ_0 the only analytic function $f : \mathcal{U} \to X$ which satisfies the equation $(\lambda I - T) f(\lambda) = 0$ is the constant function $f \equiv 0$. For an arbitrary operator $T \in B(X)$ let $\mathcal{S}(T) = \{\lambda \in \mathbb{C} : T$ does not have the SVEP at $\lambda\}$. Note that $\mathcal{S}(T)$ is open and is contained in the interior of the point spectrum $\sigma_p(T)$. The operator T is said to have the SVEP if $\mathcal{S}(T)$ is empty. According to [3] we have $\sigma(T) = \sigma_{su}(T) \cup \mathcal{S}(T)$.

For an operator $T \in \mathcal{B}(X)$ we shall denote by $\alpha(T)$ the dimension of the kernel $N(T)$, and by $\beta(T)$ the codimension of the range $R(T)$. We recall that an operator $T \in \mathcal{B}(X)$ is called upper semi-Fredholm if $\alpha(T) < \infty$ and $R(T)$ is closed, while $T \in \mathcal{B}(X)$ is called lower semi-Fredholm if $\beta(T) < \infty$. Let $\Phi_+(X)$ and $\Phi_-(X)$ denote the class of all upper semi-Fredholm operators and the class of all lower semi-Fredholm operators, respectively. The class of all semi-Fredholm operators is defined by $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$, while the class of all Fredholm operators is defined by $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$. If $T \in \Phi_\pm(X)$, the index of T is defined by $\operatorname{ind}(T) := \alpha(T) - \beta(T)$. The class of all upper semi-Browder operators is defined by $B_+(X) := \{T \in \Phi_+(X) : a(T) < \infty\}$, the upper semi-Browder spectrum of $T \in \mathcal{B}(X)$ is defined by $\sigma_{ub}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B_+(X)\}$. The class of all upper semi-Weyl operators is defined by $W_+(X) := \{T \in \Phi_+(X) : \operatorname{ind}(T) \leq 0\}$, the upper semi-Weyl spectrum is defined by $\sigma_{uw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\}$.

Recently, Haily et al. [2] have studied and characterized the Banach spaces verifying property descent spectrum equality, (Banach spaces which are isomorphic to $\ell^1(I)$ or $\ell^2(I)$ for some set I, or the Banach spaces which are not isomorphic to any of its proper quotients...). On the other hand, they have shown that if $T \in \mathcal{B}(X)$ with a spectrum $\sigma(T)$ of empty interior, then $\sigma_{desc}(T) = \sigma_{desc}(T, \mathcal{B}(X))$.
It is easy to construct an operator T satisfying the descent spectrum equality such that the interior of the point spectrum $\sigma(T)$ is nonempty. For example, let T the unilateral right shift on the Hilbert space $\ell^2(\mathbb{N})$, so that $T(x_n)_n = (0, x_0, x_1, \ldots)$ for all $(x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$. It is easily seen that $\sigma(T) = \overline{\mathbb{D}}$ closed unit disk. Since $\ell^2(\mathbb{N})$ is a Hilbert space, then $\sigma_{desc}(T) = \sigma_{desc}(T, \mathcal{B}(X))$. Motivated by the previous Example, our goal is to study the following question:

Question 1 *Let $T \in \mathcal{B}(X)$. If $\sigma(T)$ has non empty interior, under which condition on T does $\sigma_{desc}(T) = \sigma_{desc}(T, \mathcal{B}(X))$?*

2 Main results

We start by the following lemmas.

Lemma 1 [2] *Let T be in $\mathcal{B}(X)$ with finite descent $d = d(T)$. Then there exists $\delta > 0$ such that, for every $\mu \in \mathbb{K}$ with $0 < |\mu| < \delta$, we have:*

1. *$T - \mu$ is surjective,*
2. *$dim N(T - \mu) = dim(N(T) \cap R(T^d))$.*

Lemma 2 [2] *Let X, Y and Z be Banach spaces, and let $F : X \to Z$ and $G : Y \to Z$ be bounded linear operators such that $N(G)$ is complemented in Y, and $R(F) \subseteq R(G)$. Then there exists a bounded linear operator $S : X \to Y$ satisfying $F = GS$.*

We have the following theorem.

Theorem 1 *Let $T \in \mathcal{B}(X)$ and $D \subseteq \mathbb{C}$ be a closed subset such that $\sigma(T) = \sigma_{su}(T) \cup D$, then*

$$\sigma_{desc}(T) \cup int(D) = \sigma_{desc}(T, \mathcal{B}(X)) \cup int(D)$$

Proof Let λ be a complex number such that $T - \lambda$ has finite descent d and $\lambda \notin int(D)$. According to lemma 1, there is $\delta > 0$ such that, for every $\mu \in \mathbb{C}$ with $0 < |\lambda - \mu| < \delta$, the operator $T - \mu$ is surjective and $\dim N(T - \mu) = \dim N(T - \lambda) \cap R(T - \lambda)^d$. Let $D^*(\lambda, \delta) = \{\mu \in \mathbb{C} : 0 < |\lambda - \mu| < \delta\}$. Since $\lambda \notin int(D)$, then $D(\lambda, \delta)\backslash D \neq \emptyset$ is non-empty open subset of $\mathbb{C}$. Let $\lambda_0 \in D^*(\lambda, \delta)\backslash D$, then $T - \lambda_0$ is invertible, hence the continuity of the index ensures that $\text{ind}(T - \mu) = 0$ for all $\mu \in D^*(\lambda, \delta)$. But for $\mu \in D^*(\lambda, \delta)$, $T - \mu$ is surjective, so it follows that $T - \mu$ is invertible. Therefore, λ is isolated in $\sigma(T)$. By [1, Theorem 3.81], we have λ is a pole of the resolvent of T. Using [4, Theorem V.10.1], we obtain $T - \lambda$ has a finite descent and a finite ascent and $X = N((T - \lambda)^d) \oplus R((T - \lambda)^d)$. It follows that $N((T - \lambda)^d)$ is complemented in X. Applying lemma 2, there exists $S \in \mathcal{B}(X)$ satisfying $(T - \lambda)^d = (T - \lambda)^{d+1} S$, which forces that $\lambda \notin \sigma_{desc}(T, \mathcal{B}(X)) \cup int(D)$. □

Corollary 1 *Let $T \in \mathcal{B}(X)$. If T satisfies any of the conditions following:*

1. *$\sigma(T) = \sigma_{su}(T)$,*
2. *int($\sigma_{ap}(T)$) $= \emptyset$,*
3. *int($\sigma_p(T)$) $= \emptyset$,*
4. *int($\sigma_{asc}(T)$) $= \emptyset$,*
5. *int($\sigma_{ub}(T)$) $= \emptyset$,*
6. *int($\sigma_{uw}(T)$) $= \emptyset$,*
7. *$\mathcal{S}(T) = \emptyset$.*

Then

$$\sigma_{desc}(T) = \sigma_{desc}(T, \mathcal{B}(X))$$

Proof The assertions 1, 2, 3, and 7 are obvious.

4. Note that, $\sigma(T) = \sigma_{su}(T) \cup \sigma_{asc}(T)$. Indeed, let $\lambda \notin \sigma_{su}(T) \cup \sigma_{asc}(T)$, then $T - \lambda$ is surjective and $T - \lambda$ has finite ascent, therefore $a(T - \lambda) = d(T - \lambda) = 0$, and hence $\lambda \notin \sigma(T)$. If int($\sigma_{asc}(T)$) $= \emptyset$, by Theorem 1, we have $\sigma_{desc}(T) = \sigma_{desc}(T, \mathcal{B}(X))$.

5. If int($\sigma_{ub}(T)$) $= \emptyset$, then int($\sigma_{asc}(T)$) $= \emptyset$, therefore $\sigma_{desc}(T) = \sigma_{desc}(T, \mathcal{B}(X))$.

6. Note that, $\sigma(T) = \sigma_{su}(T) \cup \sigma_{uw}(T)$. Indeed, let $\lambda \notin \sigma_{su}(T) \cup \sigma_{uw}(T)$, then $T - \lambda$ is surjective and ind$(T - \lambda) \leq 0$, therefore ind$(T - \lambda) = \dim N(T - \lambda) = 0$, and hence $\lambda \notin \sigma(T)$. If int($\sigma_{uw}(T)$) $= \emptyset$, by Theorem 1, we have $\sigma_{desc}(T) = \sigma_{desc}(T, \mathcal{B}(X))$. □

Example 1 We consider the Césaro operator C_p defined on the classical Hardy space $\mathbb{H}_p(\mathbb{D})$, $\mathbb{D}$ the open unit disc and $1 < p < \infty$. The operator C_p is defined by $(C_p f)(\lambda) := \frac{1}{\lambda}\int_0^{\lambda}\frac{f(\mu)}{1-\mu}d\mu$ or all $f \in \mathbb{H}_p(\mathbb{D})$ and $\lambda \in \mathbb{D}$. As noted by Miller et al. [5], the spectrum of the operator C_p is the entire closed disc Γ_p, centered at $p/2$ with radius $p/2$, and $\sigma_{ap}(C_p)$ is the boundary $\partial\Gamma_p$, then int($\sigma_{ap}(C_p)$)=int($\sigma_p(C_p)$) $= \emptyset$. By applying corollary 1, then $\sigma_{desc}(C_p) = \sigma_{desc}(C_p, \mathcal{B}(\mathbb{H}_p(\mathbb{D})))$.

Example 2 Suppose that T is an unilateral weighted right shift on $\ell^p(\mathbb{N})$, $1 \le p < \infty$, with weight sequence $(\omega_n)_{n\in\mathbb{N}}$, T is the operator defined by: $Tx := \sum_{n=1}^{\infty}\omega_n x_n e_{n+1}$ for all $x := (x_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$. If $c(T) = \lim_{n\to+\infty}\inf(\omega_1 \dots \omega_n)^{1/n} = 0$, by [1, Corollary 3.118], we have T has SVEP. By applying corollary 1, then $\sigma_{desc}(T) = \sigma_{desc}(T, \mathcal{B}(X))$.

A mapping $T : \mathcal{A} \to \mathcal{A}$ on a commutative complex Banach algebra $\mathcal{A}$ is said to be a multiplier if:

$$u(Tv) = (Tu)v \quad \text{for all } u, v \in \mathcal{A}.$$

Any element $a \in \mathcal{A}$ provides an example, since, if $L_a : \mathcal{A} \to \mathcal{A}$ denotes the mapping given by $L_a(u) := au$ for all $u \in \mathcal{A}$, then the multiplication operator La is clearly a multiplier on $\mathcal{A}$. The set of all multipliers of $\mathcal{A}$ is denoted by $M(\mathcal{A})$. We recall that an algebra $\mathcal{A}$ is said to be semi-prime if $\{0\}$ is the only two-sided ideal J for which $J^2 = 0$.

Corollary 2 *Let $T \in M(\mathcal{A})$ be a multiplier on a semi-prime commutative Banach algebra $\mathcal{A}$ then:*

$$\sigma_{desc}(T) = \sigma_{desc}(T, \mathcal{B}(X))$$

Proof If $T \in M(\mathcal{A})$, from [1, Proposition 4.2.1], we have $\sigma(T) = \sigma_{su}(T)$. By applying corollary 1, then: $\sigma_{desc}(T) = \sigma_{desc}(T, \mathcal{B}(X))$.

Theorem 2 *Let $T \in \mathcal{B}(X)$. If for every connected component G of $\rho_{desc}(T)$ we have that $G \cap \rho(T) \neq \emptyset$, then*

$$\sigma_{desc}(T) = \sigma_{desc}(T, \mathcal{B}(X))$$

Proof Let λ be a complex number such that $T-\lambda$ has finite descent d. According to lemma 1, there is $\delta > 0$ such that, for every $\mu \in \mathbb{C}$ with $0 < |\lambda-\mu| < \delta$, the operator $T-\mu$ is surjective and $\dim N(T-\mu) = \dim N(T-\lambda) \cap R(T-\lambda)^d$. $D^*(\lambda, \delta) = \{\mu \in \mathbb{C} : 0 < |\lambda - \mu| < \delta\}$ is a connected subset of $\rho_{desc}(T)$, then there exists a connected component G of $\rho_{desc}(T)$ contains $D^*(\lambda, \delta)$. Since $G \cap \rho(T)$ is non-empty hence the continuity of the index ensures that ind$(T-\mu) = 0$ for all $\mu \in D^*(\lambda, \delta)$. But for $\mu \in G$, $T-\mu$ is surjective, so it follows that $T-\mu$ is invertible. Thus $G \subseteq \rho(T)$, therefore, λ is isolated in $\sigma(T)$. Consequently $\lambda \notin \sigma_{desc}(T, \mathcal{B}(X))$, which completes the proof.

Remark 1 We recall that an operator $R \in \mathcal{B}(X)$ is said to be Riesz if $R-\lambda$ is Fredholm for every non-zero complex number λ. From [4], $\sigma_{desc}(R) = \{0\}$, then for every connected component G of $\rho_{desc}(R)$, we have that $G \cap \rho(R) \neq \emptyset$. Consequently $\sigma_{desc}(R, \mathcal{B}(X)) = \{0\}$

Example 3 Consider the unilateral right shift operator T on the space $X := \ell^p$ for some $1 \le p \le \infty$. Because $\sigma(T) = \sigma_{desc}(T)$, then for every G is a connected component of $\rho_{desc}(T)$ we have that $G \cap \rho(T) \neq \emptyset$. Consequently $\sigma_{desc}(T, \mathcal{B}(X)) = \overline{\mathbb{D}}$ closed unit disk.

Theorem 3 *Let $T \in \mathcal{B}(X)$. If for every connected component G of $\rho_{su}(T)$ we have that $G \cap \rho_p(T) \neq \emptyset$, then:*

$$\sigma_{desc}(T) = \sigma_{desc}(T, \mathcal{B}(X))$$

Proof Let λ be a complex number such that $T - \lambda$ has finite descent d. According to lemma 1, there is $\delta > 0$ such that, for every $\mu \in \mathbb{C}$ with $0 < |\lambda - \mu| < \delta$, the operator $T - \mu$ is surjective and $\dim N(T-\mu) = \dim N(T-\lambda) \cap R(T-\lambda)^d$. Therefore $D^*(\lambda, \delta) = \{\mu \in \mathbb{C} : 0 < |\lambda - \mu| < \delta\}$ is a connected subset of $\rho_{su}(T)$, then there exists a connected component G of $\rho_{su}(T)$ contains $D^*(\lambda, \delta)$. Since $G \cap \rho_p(T)$ is non-empty hence the continuity of the index ensures that $\text{ind}(T - \mu) = 0$ for all $\mu \in D^*(\lambda, \delta)$. But for $\mu \in G$, $T - \mu$ is surjective, so it follows that $T - \mu$ is invertible, therefore, λ is isolated in $\sigma(T)$. Consequently $\lambda \notin \sigma_{desc}(T, \mathcal{B}(X))$.

Remark 2 Let $T \in \mathcal{B}(X)$ an operator such that $\sigma(T) = \sigma_{su}(T)$, then for every connected component G of $\rho_{su}(T)$, we have $G \cap \rho_p(T) \neq \emptyset$. Using Theorem 3, we obtain $\sigma_{desc}(T) = \sigma_{desc}(T, \mathcal{B}(X))$.

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References

1. Aiena, P.: Fredholm and Local Spectral Theory with Applications to Multipliers. Kluwer Academic Press, Dordrecht (2004)
2. Haïly, A., Kaidi, A., Rodríguez Palacios, A.: Algebra descent spectrum of operators. Isr. J. Math. **177**, 349–368 (2010)
3. Laursen, K.B., Neumann, M.M.: Introduction to Local Spectral Theory. Clarendon Press, Oxford (2000)
4. Lay, D.C.: Spectral analysis using ascent, descent, nullity and defect. Math. Ann. **184**, 197–214 (1970)
5. Miller, T.L., Miller, V.G., Smith, R.C.: Bishop's property (β) and the Césaro operator. J. Lond. Math. Soc. (2) **58**, 197–207 (1998)
6. Müller, V.: Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras. Operator Theory Advances and Applications, vol. 139, 2nd edn. Birkhäuser, Basel (2007)