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A new optimal estimate for the first stability eigenvalue of closed hypersurfaces in Riemannian space forms

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Abstract In this paper, we obtain a new upper bound for the first eigenvalue λ_1^J of the stability operator J of a closed constant mean curvature hypersurface in a Riemannian space form, in terms of the mean curvature and the length of the total umbilicity operator of Σ^n. When the ambient space is the Euclidean sphere, through the calculus of λ_1^J of the Clifford torus, we also show that our estimate is optimal and that it is a refinement of a previous one due to Alías et al. in Am Math Soc 133:875–884, 2004. As an application, we derive a nonexistence result concerning strongly stable closed hypersurfaces. Furthermore, from the values of λ_1^J of the hyperbolic cylinders, we conclude that our estimate does not hold in general for complete noncompact hypersurfaces with two distinct principal curvatures in the hyperbolic space.

Keywords Riemannian space forms · Closed H-hypersurfaces · Strong stability · First stability eigenvalue · Constant mean curvature · Clifford torus · Circular and hyperbolic cylinders

Mathematics Subject Classification Primary 53C42; Secondary 53A10 · 53C20 · 53C50

1 Introduction and statements of the results

Let us denote by $\mathbb{Q}_c^{n+1}$ the standard model of an $(n+1)$-dimensional Rieamannian space form with constant sectional curvature c, with $c \in \{0, 1, -1\}$. That is, $\mathbb{Q}_c^{n+1}$ denotes the Euclidean

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space $\mathbb{R}^{n+1}$ when $c = 0$, the Euclidean sphere $\mathbb{S}^{n+1}$ when $c = 1$, and the hyperbolic space $\mathbb{H}^{n+1}$ when $c = -1$. Along this paper, we will deal with closed orientable constant mean curvature hypersurfaces $\psi : \Sigma^n \to \mathbb{Q}_c^{n+1}$ immersed into $\mathbb{Q}_c^{n+1}$. In this setting, we denote by $d\Sigma$ the volume element with respect to the metric induced by ψ.

It is well known that minimal hypersurfaces are characterized as critical points of the area functional $\mathcal{A} = \int_\Sigma d\Sigma$, for every variation of Σ^n with compact support and fixed boundary. Whereas any hypersurface Σ^n with constant mean curvature H (shortly, *H-hypersufcace*) is a critical point of $\mathcal{A}$ for volume-preservation variations, by meaning that the variations under consideration preserve a certain volume function (for more details, see [6]).

For these critical points, Proposition 2.5 of [6] asserts that the stability of the corresponding variational problem is given by the second variation of the area functional

$$\delta_f^2 \mathcal{A} = \frac{d^2\mathcal{A}}{dt^2}(0) = -\int_\Sigma f J f \, d\Sigma$$

with $f \in C^\infty(\Sigma)$ satisfying $\int_\Sigma f d\Sigma = 0$ and

$$J = \Delta + |A|^2 + nc,$$

where Δ stands for the Laplacian operator on Σ^n and $|A|$ denotes the length of the shape operator A of Σ^n with respect to N. In this setting, we recall that an H-hypersurface Σ^n is said to be *strongly stable* if $\delta_f^2\mathcal{A} \geq 0$ for every $f \in C^\infty(\Sigma)$ and J is called the *Jacobi* or *stability operator* of Σ^n. We note that J belongs to a class of operators which are usually referred to as Schrödinger operators, that is, operators of the form $\Delta + q$, where q is any continuous function on Σ^n. The *first stability eigenvalue* $\lambda_1^J(\Sigma)$ of Σ^n is defined as been the smallest real number λ which satisfies

$$Jf + \lambda f = 0 \quad \text{in} \quad \Sigma^n,$$

for some nonzero smooth function $f \in C^\infty(\Sigma)$. As is well known, $\lambda_1^J(\Sigma)$ has the following min-max characterization

$$\lambda_1^J(\Sigma) = \min\left\{\frac{-\int_\Sigma fJf\,d\Sigma}{\int_\Sigma f^2\,d\Sigma} \ : \ f \in C^\infty(\Sigma)\,,\, f \neq 0\right\}. \tag{1.1}$$

We observe that, in terms of the first stability eigenvalue, a closed H-hypersurface Σ^n is strongly stable if and only if $\lambda_1^J(\Sigma) \geq 0$.

To carry out the study of the first stability eigenvalue $\lambda_1^J(\Sigma)$ of a closed H-hypersurface Σ^n is more convenient to rewrite the Jacobi operator J in terms of the traceless second fundamental form Φ, which is defined by $\Phi = A - nH$, where I denotes the identity operator on $\mathfrak{X}(\Sigma)$. We note that $|\Phi|^2 = |A|^2 - nH^2$, with $|\Phi| \equiv 0$ if and only if Σ^n is totally umbilical. For this reason Φ is also called the *total umbilicity operator* of Σ^n. From here we get

$$J = \Delta + |\Phi|^2 + n(H^2 + c). \tag{1.2}$$

In his seminal work [10], Simons studied the first stability eigenvalue of a minimal compact hypersurface Σ^n immersed in the Euclidean sphere $\mathbb{S}^{n+1}$. In this setting, he proved that either $\lambda_1^J(\Sigma) = -n$, and Σ^n is a totally geodesic sphere $\mathbb{S}^n \hookrightarrow \mathbb{S}^{n+1}$, or $\lambda_1^J(\Sigma) \leq -2n$, otherwise. Later on, Wu in [11] characterized the equality $\lambda_1^J(\Sigma) = -2n$ by showing that it holds only for the minimal Clifford torus of the form $\mathbb{S}^p(\sqrt{p/n}\,) \times \mathbb{S}^{n-p}(\sqrt{(n-p)/n}\,)$, with $p \in \{1, \ldots, n-1\}$. Shortly thereafter, Perdomo [9] provides a new proof of this spectral characterization by the first stability eigenvalue. Afterwards, Alías, Barros and Brasil Jr. [1] extended these results to the case of H-hypersurfaces in $\mathbb{S}^{n+1}$, characterizing some

Clifford torus of the form $\mathbb{S}^1(r)\times\mathbb{S}^1(\sqrt{1-r^2})$, $r\in(0,\sqrt{1/2})\bigcup(\sqrt{1/2},1)$, and $\mathbb{S}^{n-1}(r)\times\mathbb{S}^1(\sqrt{1-r^2})$, with $r\in(0,\sqrt{(n-1)/n})$, via the value of their first stability eigenvalue. More recently, the second author jointly with Aquino, dos Santos and Velásquez [5] obtained upper bounds for $\lambda_1^J(\Sigma)$ of a closed H-hypersurface Σ^n immersed either in the Euclidean space $\mathbb{R}^{n+1}$ or in the hyperbolic space $\mathbb{H}^{n+1}$ in terms of H and $|\Phi|$. As application, they derived a nonexistence result concerning strong stable hypersurfaces in these ambient spaces.

Here, we will deal with closed hypersurfaces which satisfy the following Okumura type inequality, introduced by Meléndez in [7],

$$|\mathrm{tr}(\Phi^3)|\leq C(n,p)|\Phi|^3, \tag{1.3}$$

where $C(n,p)=\dfrac{(n-2p)}{\sqrt{np(n-p)}}$ for a given integer $1\leq p\leq n/2$. In this setting, we proceed with the picture described above establishing the following result:

Theorem 1 *Let Σ^n be a closed H-hypersurface immersed in $\mathbb{Q}_c^{n+1}$, with $n\geq 2$, and let $\lambda_1^J(\Sigma)$ stand for the first stability eigenvalue of Σ^n. If its total umbilicity operator Φ satisfies (1.3) for some integer $1\leq p\leq n/2$, then*

(i) either $\lambda_1=-n(H^2+c)$, and Σ^n is a totally umbilical hypersurface,
(ii) or

$$\lambda_1^J(\Sigma)\leq -2n(H^2+c)+nC(n,p)|H|\max_\Sigma|\Phi|. \tag{1.4}$$

Moreover, when $c=1$ the equality in (1.4) is attained if and only if Σ^n is either a minimal Clifford torus or a product of the form $\mathbb{S}^{n-p}(r)\times\mathbb{S}^p(\sqrt{1-r^2})$, with $r^2<1-p/n$ if $H\neq 0$; when $c\in\{-1,0\}$, the inequality in (1.4) is strict.

We observe that, taking into account the classical lemma of Okumura [8], inequality (1.3) is automatically true when $p=1$. Furthermore, when $1<p<\frac{n}{2}$ we claim that to suppose that inequality (1.3) holds is weaker than to assume the geometric condition of the hypersurface has two distinct principal curvatures with multiplicities p and $n-p$. Indeed, in this latter case Φ also has two distinct eigenvalues, said μ and ν, with multiplicity p and $n-p$, respectively. In particular, we get $\mu=-\frac{n-p}{p}\nu$ and $|\Phi|^2=p\mu^2+(n-p)\nu^2$, which implies that

$$\mathrm{tr}(\Phi^3)=p\mu^3+(n-p)\nu^3=\pm C(n,p)|\Phi|^3,$$

proving our claim.

The proof of Theorem 1 is given in Sect. 2. In Sect. 3, we discuss on the first stability eigenvalue of circular and hyperbolic cylinders. In particular, we conclude that estimate (1.4) does not hold in general for complete noncompact hypersurfaces satisfying (1.3) in $\mathbb{H}^{n+1}$. We also point out that, since $C(n,p)$ is a decreasing function on p, in the case $c=1$, our estimate (1.4) is a refinement of that in Theorem 2.2 of [1] and, in the case $c\in\{0,-1\}$, our result also generalizes Theorem 1 of [5].

It is well known that there are no strongly stable closed H-hypersurfaces immersed in $\mathbb{S}^{n+1}$ (see, for instance, Section 2 of [2]). Taking into account the nonexistence of minimal closed hypersurfaces in $\mathbb{R}^{n+1}$ and observing that Lemma 8 of [4] guarantees that $H^2>1$ for a closed H-hypersurface in $\mathbb{H}^{n+1}$, from Theorem 1 we obtain an extension of this result when the ambient space is either $\mathbb{R}^{n+1}$ or $\mathbb{H}^{n+1}$. More precisely,

Corollary 1 *There is not exist strongly stable closed H-hypersurface satisfying (1.3) in $\mathbb{Q}_c^{n+1}$, with $c\in\{0,-1\}$, $n\geq 3$, $1\leq p<n/2$ and such that its total umbilicity operator Φ satisfies*

$$|\Phi| \leq \frac{2(H^2+c)}{C(n,p)|H|}.$$

In particular, from Theorem 1 we also obtain the following nonexistence result:

Corollary 2 *There is not exist strongly stable closed* H*-surface with two distinct principal curvatures in* $\mathbb{Q}_c^3$.

2 Proof of Theorem 1

Let us reason as in the proofs of Theorem 2.2 of [1], if $c = 1$, and Theorem 1 of [5], if $c \in \{0, -1\}$. By taking $f = 1$, it follows from (1.1) and (1.2) that

$$\lambda_1^J(\Sigma) \leq -n(H^2+c) - \frac{1}{\mathrm{vol}(\Sigma)}\int_\Sigma |\Phi|^2 d\Sigma \leq -n(H^2+c),$$

with equality $\lambda_1^J(\Sigma) = -n(H^2+c)$ if and only if Σ^n is a totally umbilical hypersurface.

Next, assuming that Σ^n is non-totally umbilical, we can reason as in [1,5] replacing $C(n,1)$ by $C(n,p)$ in order to infer estimate (1.4).

Then, when $c = 1$ and the equality $\lambda_1^J(\Sigma) = -2n(H^2+1) + nC(n,p)|H| \max_\Sigma |\Phi|$ holds, the aforementioned ideas give

$$|\Phi|^2 \equiv \frac{n}{4p(n-p)}\left(\sqrt{n^2H^2 + 4p(n-p)} - (n-2p)|H|\right)^2.$$

Hence, we can apply Theorem 2 of [3] when $n = 2$, Theorem 3 of [3] when $n \geq 3$ and $p = 1$, Theorem 1.4 of [7] when $n \geq 3$ and $1 < p < n/2$, and reason as in the proof of this last result when $p = n/2$ to conclude that Σ^n must be either a minimal Clifford torus or a product of the form $\mathbb{S}^{n-p}(r) \times \mathbb{S}^p(\sqrt{1-r^2})$, with $r^2 < 1 - p/n$ if $H \neq 0$. Reciprocally, supposing that Σ^n is one of these torus and replacing 1 by p in [1], we deduce that

$$\lambda_1^J(\Sigma) = -2n(H^2+1) + nC(n,p)|H||\Phi|.$$

To conclude our proof, we note that the case $c \in \{0, -1\}$ follows in a similar way of the proof of Theorem 1 in [5], changing $C(n,1)$ by $C(n,p)$.

3 The first stability eigenvalue of circular and hyperbolic cylinders

Let Σ^n be a complete hypersurface immersed in $\mathbb{Q}_c^{n+1}$. We recall that the first stability eigenvalue $\lambda_1^J(D)$ for some bounded open domain in Σ^n is defined as the smallest real number λ that satisfies

$$Jf + \lambda f = 0 \ \text{ in } \ D,$$

for some nonzero smooth function $f \in C^\infty(D)$ with $f\ |_{\partial D} = 0$. So, the first stability eigenvalue $\lambda_1^J(\Sigma)$ of Σ^n is defined by

$$\lambda_1^J(\Sigma) = \inf\left\{\lambda_1(D) : D \subset \Sigma^n \text{ is a bounded open domain}\right\}.$$

Let us consider the circular cylinder

$$\mathcal{M}^n(p,r) = \mathbb{S}^{n-p}(r) \times \mathbb{R}^p \hookrightarrow \mathbb{R}^{n+1}$$

and the hyperbolic cylinder

$$\mathcal{N}^n(p,r) = \mathbb{S}^{n-p}(r) \times \mathbb{H}^p\left(-\sqrt{1+r^2}\right) \hookrightarrow \mathbb{H}^{n+1},$$

where $1 \leq p \leq \frac{n}{2}$ and $r > 0$.

We can reason as in Section 4 of [5], replacing 1 by p, to conclude that

$$\lambda_1^J\left(\mathcal{M}^n(p,r)\right) = -2nH^2 + nC(n,p)H|\Phi|$$

and

$$\lambda_1^J\left(\mathcal{N}^n(p,r)\right) \geq -2n(H^2-1) + nC(n,p)H|\Phi|.$$

We note that the last inequality follows from the fact that $\lambda_1^{\Delta}\left(\mathbb{H}^p\left(-\sqrt{1+r^2}\right)\right) = \frac{(p-1)^2}{4(1+r^2)}$. Moreover, the equality holds if, and only if, $p = 1$.

As a consequence of this previous digression, while in $\mathbb{R}^{n+1}$ the estimate (1.4) may be still extended for complete hypersurfaces, we conclude that it does not hold in general for complete noncompact hypersurfaces satisfying (1.3) in $\mathbb{H}^{n+1}$.

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