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A characterization for Meir–Keeler contractions

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Abstract In this manuscript the notion of S-operators is introduced and as a result a new characterization of Meir–Keeler contractions is presented. Also it is shown that the set of S-operators includes the set of continuous R-contractions, and by providing an example it is justified that this inclusion is proper. Then Edelstein's theorem for contractive mappings on compact metric spaces is generalized to S_0-operators. Finally the set of S-operators is extended to the set of orbitally S-operators that includes Matkowski contractions.

Keywords R-contractions · Meir–Keeler contraction · S-operator · Simulation function · Asymptotically regular sequence

Mathematics Subject Classification 54H25 · 47H10

1 Introduction

Throughout this context (X, d) is a complete metric space, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $\mathbb{R}$ stand for the set of nonnegative integers and the field of Real numbers respectively.

Banach contraction principle is a powerful classical result in nonlinear analysis that has been extended in many directions [1]. As there have been given lots of generalizations for Banach contraction principle, therefore unifying different generalizations of this result has been really very important. Recently Khojasteh et al. [10] introduced the notion of simulation functions to study different kinds of contractions in a unified way and defined the set of $\mathcal{Z}$-contractions. Then Gavruta et al. [7] showed that each $\mathcal{Z}$-contraction is indeed a Meir–Keeler contraction. Therefore finding a true generalization of Meir–Keeler contraction motivated De Hierro et al. [12] to introduce the set of R-contractions. They proved that the set of R-contractions not only includes the set of Meir–Keeler contractions but contains a large

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family of contractions such as Geraghty contractions and contractions defined by simulation or manageable functions [4,8,10]. The main obstacle to verify a particular operator as a $\mathcal{Z}$-contraction (or an R-contraction) is to define so-called a simulation function or an auxiliary function on which these notions are depended on. It is worth mentioning that in comparison with $\mathcal{Z}$-contractions or R-contractions [10,12] the notion of S-operator is dependent upon no simulation or auxiliary function which was the main motivation for doing this research. In this paper at first the set of S-operators along with the set of S_1 and S_0 operators are introduced. Then it is verified that each of these sets of operators contains the next one respectively and by some examples it is confirmed that all of these inclusions are proper. Then a characterization for Edelstein contractions via S_0-operators is presented (Theorem 11) which shows that S_0-operators are the extension of Edelstein contractions on compact metric spaces (Remark 4). Then imposing a simple condition on S-operators, a characterization for Meir–Keeler contractions is provided. This enables us to see if a given contraction is Meir–Keeler or not and extend some earlier results (Sect. 3; Theorems 15 and 16). For other characterizations of Meir–Keeler or Meir–Keeler type results [9,11]. Then it is shown that the set of S-operators contains the set of continuous R-contractions and by an example it is verified that this inclusion is proper. Finally the set of orbitally S-operators is introduced. Meir–Keeler and Matkowski contractions are among the most important generalizations of Banach contractions and it is proved that these sets of contractions are included in the set of orbitally S-operators properly.

For the sake of completeness we present here some basic definitions and results that will be needed in the sequel.

Definition 1 Let (X, d) be a metric space and $T : X \to X$ be a mapping. For $x_0 \in X$ the Picard sequence of T based at the point x_0 is defined by $x_{n+1} = T(x_n)$ for all $n \geq 1$ and is denoted by $O\{T, x_0\}$. An arbitrary sequence $\{x_n\} \subseteq X$ is called asymptotically regular if $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. T is called weakly Picard operator if for each $x_0 \in X$ the Picard sequence of T based at x_0 be convergent to a fixed point of T. Moreover T is called a Picard operator if T is weakly Picard operator and has a unique fixed point [12]. T is called a Banach contraction if there exists $k \in [0, 1)$ such that:

$$d(T(x), T(y)) \leq kd(x, y) \quad \text{for all } x, y \in X, \tag{1}$$

T is called a nonexpansive mapping if

$$d(T(x), T(y)) \leq d(x, y) \quad \text{for all } x, y \in X, \tag{2}$$

T is called a contractive mapping if

$$d(T(x), T(y)) < d(x, y) \quad \text{for all } x, y \in X \text{ with } x \neq y. \tag{3}$$

Theorem 1 (Banach contraction principle [1]) *Every Banach contraction on a complete metric space has a unique fixed point.*

Theorem 2 (Edelstein [6]) *Let (X, d) be a complete metric space and $T : X \to X$ be a contractive self mapping and*

$$\{f^{n(i)}x\} \subseteq \{f^{(n)}x\} \quad \textit{for some } x \in X \textit{ with } z = \lim_{i\to\infty} f^{n(i)}x \in X. \tag{4}$$

Then z is a unique fixed point of T.

When X is compact, then Eq. (4) hold and the following theorem is deduced.

Theorem 3 (Edelstein [6]) *Let (X, d) be a compact metric space and $T : X \to X$ be a contractive self mapping, then T has a unique fixed point.*

Theorem 4 (Meir and Keeler [13]) *Let (X, d) be a metric space and $T : X \to X$ be a self mapping. T is a Meir–Keeler contraction if for each $\epsilon > 0$ there exists $\delta > 0$ such that if $x, y \in X$ and $\epsilon \leq d(x, y) < \epsilon + \delta$, then $d(T(x), T(y)) < \epsilon$.*
Every Meir–Keeler contraction $T : X \to X$ is contractive and has a unique fixed point.

Theorem 5 (Geraghty [8]) *Let (X, d) be a complete metric space, let $T : X \to X$, and suppose that for each $x, y \in X$:*

$$d(T(x), T(y)) \leq \alpha(d(x, y))d(x, y).$$

Then T has a unique fixed point $z \in X$, and $\{T^n(x)\}$ converges to z, for each $x \in X$. Where $\alpha : \mathbb{R}_+ \to [0, 1)$ is a function such that $\alpha(t_n) \to 1$ implies that $t_n \to 0$, for each sequence $\{t_n\} \subset [0, +\infty)$.

The following contraction is introduced by Matkowski which generalizes the Banach contractions [1].

Theorem 6 *Let (X, d) be a metric space and $T : X \to X$ be a self mapping such that:*

$$d(T(x), T(y)) \leq \psi(d(x, y)) \quad \text{for all } x, y \in X, \tag{5}$$

where $\psi : (0, \infty) \to (0, \infty)$ is monotone nondecreasing mapping such that $lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$. Then T has a unique fixed point.

Definition 2 (Khojasteh et al. [10]) Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping, then ζ is called a simulation function provided the following conditions hold:

($\zeta 1$) $\zeta(0, 0) = 0$;
($\zeta 2$) $\zeta(t, s) < s - t$ for all $t, s > 0$;
($\zeta 3$) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Definition 3 (Khojasteh et al. [10]) Let (X, d) be a metric space and $T : X \to X$ be a mapping. T is a $\mathcal{Z}$-contraction if there exists a simulation function ζ such that:

$$\zeta(d(T(x), T(y)), d(x, y)) \geq 0 \quad \text{for all } x, y \in X \text{ such that } x \neq y. \tag{6}$$

Definition 4 (López De Hierro and Shahzad [12]) Let A be a nonempty subset of $\mathbb{R}$ and $\rho : A \times A \to \mathbb{R}$ be a function such that:

(ρ_1) If $\{a_n\} \subset A \cap [0, \infty)$ be a sequence such that $\rho(a_{n+1}, a_n) > 0$ for all $n \in \mathbb{N}_0$, then $a_n \to 0$.
(ρ_2) For any two sequences $\{a_n\}, \{b_n\} \subset (0, \infty) \cap A$ converging to the same *limit* $L \geq 0$ with $L < a_n$ and $\rho(a_n, b_n) > 0$ for all $n \in \mathbb{N}_0$, then L=0.

Then ρ is called R-function which is simply denoted by $\rho \in R_A$.

In some cases the following condition is also considered for an R-function:

(ρ_3) For any two sequences $\{a_n\}, \{b_n\} \subset (0, \infty) \cap A$, if $b_n \to 0$ and $\rho(a_n, b_n) > 0$ for all $n \in \mathbb{N}_0$, then $a_n \to 0$.

Definition 5 (López De Hierro and Shahzad [12]) Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is an R-contraction if there exists an R-function $\rho : A \times A \to \mathbb{R}$ that $d(x, y) \in A$ for all $x, y \in X$ and

$$\rho(d(T(x), T(y)), d(x, y)) > 0 \quad \text{for all } x, y \in X \text{ such that } x \neq y. \tag{7}$$

Theorem 7 (López De Hierro and Shahzad [12]) *Let (X, d) be a complete metric space and $T : X \to X$ be an R-contraction with respect to $\rho \in R_A$. Assume that one of the following conditions holds.*

(a) *T is continuous.*
(b) *The function ρ satisfies the condition (ρ_3).*
(c) *$\rho(t, s) \leq s - t$ for all $t, s \in A \cap (0, \infty)$.*

Then T is a Picard operator. In particular, It has a unique fixed point.

2 Main results

In this section we introduce the set of S-operators. Then a characterization for Edelstein contractions on compact metric spaces is provided and we also characterize Meir–Keeler contractions via S_1-operators. As the final result of this section we show that the set of S-operators contains the set of continuous R-contractions.

Definition 6 Let (X, d) be a metric space. $T : X \to X$ is called an S-operator if the following conditions hold:

s(i) there exists $x_0 \in X$ such that the Picard sequence of T based at x_0 is asymptotically regular;
s(ii) for any sequences $\{x_n\}$ and $\{y_n\}$, if $d(x_n, y_n)$ and $d(T(x_n), T(y_n))$ converge to the same *limit* $L \geq 0$ with $d(T(x_n), T(y_n)) > L$ for all $n \in \mathbb{N}_0$, then L=0;
s(iii) T is continuous.

Theorem 8 *Let (X, d) be a complete metric space. Then each S-operator $T : X \to X$ has a fixed point in X.*

Proof Suppose that $x_0 \in X$ and $\{x_n\}$ be the Picard sequence of T based at x_0, which by assumption is asymptotically regular. Assume $\{x_n\}$ is not a Cauchy sequence. Then using exactly the same argument as given in [12] there exists $\epsilon_0 > 0$ and two sequences $\{n(k)\}$ and $\{m(k)\}$ of nonnegative integers with $k \leq n(k) < m(k)$ for each $k \in \mathbb{N}_0$ and

$$d(x_{n(k)}, x_{m(k)-1}) \leq \epsilon_0 < d(x_{n(k)}, x_{m(k)}),$$

we have

$$d(x_{n(k)-1}, x_{m(k)-1}) \leq d(x_{n(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)-1}) \leq \epsilon_0 + d(x_{n(k)-1}, x_{n(k)}), \tag{8}$$

and

$$\epsilon_0 < d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}), \tag{9}$$

but $\{x_n\}$ is an asymptotically regular sequence, consequently we get:

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = \epsilon_0.$$

Now condition $s(ii)$ implies that $\epsilon_0 = 0$, this contradiction shows that $\{x_n\}$ is a Cauchy sequence and since (X, d) is complete, it converges to some point $z \in X$. As T is continuous, we get:

$$d(z, T(z)) = \lim_{n\to\infty} d(x_{n+1}, T(z)) = \lim_{n\to\infty} d(T(x_n), T(z)) = 0, \tag{10}$$

and this shows that z is a fixed point of T which completes the proof.

Replacing the condition $d(T(x_n), T(y_n)) > L$ with $d(T(x_n), T(y_n)) \geq L$ in the condition s(ii) of Definition 6 or remove it, the S_1-operators and S_0-operators are defined as follow.

Definition 7 Let (X, d) be a metric space. $T : X \to X$ is called an S_1-operator if the following conditions hold:

s(i) there exists $x_0 \in X$ that the Picard sequence of T based at x_0 is asymptotically regular;
s(ii) for any two sequences $\{x_n\}$ and $\{y_n\}$, if $d(x_n, y_n)$ and $d(T(x_n), T(y_n))$ converge to the same *limit* $L \geq 0$ with $d(T(x_n), T(y_n)) \geq L$ for all $n \in \mathbb{N}_0$, then L=0;
s(iii) T is continuous.

Definition 8 Let (X, d) be a metric space. $T : X \to X$ is called an S_0-operator if the following conditions hold:

s(i) there exists $x_0 \in X$ that the Picard sequence of T based on x_0 is asymptotically regular;
s(ii) for any two sequences $\{x_n\}$ and $\{y_n\}$, if $d(x_n, y_n)$ and $d(T(x_n), T(y_n))$ converge to the same *limit* $L \geq 0$, then L=0;
s(iii) T is continuous.

Corollary 1 *Every S_1-operator on a complete metric space is an S-operator. Furthermore every S_1-operator has a unique fixed point.*

Proof The first part is clear. To justify the uniqueness of fixed points, suppose that x, y be fixed points of T. Let $x_n = x$, $y_n = y$ for all nonnegative integers n. Then $d(x_n, y_n)$ and $d(T(x_n), T(y_n))$ converge to $L = d(x, y)$ and $d(T(x_n), T(y_n)) \geq L$ for all $n \in \mathbb{N}_0$, and by condition $s(ii)$ in Definition 7 we get $d(x, y) = 0$. Hence $x = y$ and this shows that T has a unique fixed point.

Remark 1 See Example 1 for an S-operator that has two fixed points. Therefore the set of S-operators includes the set of S_1-operators properly.

Corollary 2 *Let (X, d) be a complete metric space, then every S_0-operator on X is an S_1-operator (and thus has a unique fixed point). The converse is also true when (X, d) is a compact metric space.*

Proof The first part is clear. To prove the second part suppose that (X, d) be a compact metric space and $T : X \to X$ is an S_1-operator. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $d(x_n, y_n)$ and $d(T(x_n), T(y_n))$ converge to the same $L \geq 0$. Since X is compact, without loss of generality we may suppose that $\{x_n\}$ and $\{y_n\}$ converge to $x, y \in X$ respectively. Using the continuity of T we get:

$$d(x, y) = d(T(x), T(y)), \tag{11}$$

now define $x_n = x$ and $y_n = y$ for all $n \in \mathbb{N}_0$ and put $L = d(x, y)$. Then we have:

$$d(x_n, y_n), d(T(x_n), T(y_n)) \to L \text{ and } d(T(x_n), T(y_n)) \geq L \quad \text{for all } n \in \mathbb{N}_0, \tag{12}$$

since T is an S_1-operator, $L = 0$, and the proof is complete.

Remark 2 See Example 2 for an S_1-operator which is not an S_0-operator. Therefore the set of S_1-operators contains the set of S_o-operators properly.

The following theorem shows that the set of S_0-operators is big enough to include all Banach contractions.

Theorem 9 *Let (X, d) be complete metric space, then every Banach contraction mapping $T : X \to X$ is an S_0-operator.*

Proof T is continuous and has a fixed point, then the conditions $s(i)$, $s(iii)$ in Definition 8 hold. To verify that T satisfies the condition $s(ii)$, let $T : X \to X$ be a Banach contraction such that $d(T(x), T(y)) \leq kd(x, y)$ for all $x, y \in X$, where $k \in (0, 1)$ is a constant number. Now suppose that $\{x_n\}, \{y_n\}$ are two sequences such that $d(x_n, y_n)$ and $d(T(x_n), T(y_n))$ converge to some $L \geq 0$. Then

$$d(T(x_n), T(y_n)) \leq kd(x_n, y_n) \quad \text{for all } n \in \mathbb{N}_0, \tag{13}$$

taking *limit* as $k \to \infty$ we get $L \leq kL$. Since $L > 0$, this shows that $L = 0$ which completes the proof.

See Example 4 for an S_0-operator which is not a Banach contraction. The following theorem shows that in Definition 8, condition $s(ii)$ implies $s(i)$ for nonexpansive mappings. By Example 3 we see that the converse is not true when T is not nonexpansive.

Theorem 10 *Every nonexpansive mapping on a complete metric space satisfying the condition $s(ii)$ of Definition 8 is an S_0-operator and hence has a unique fixed point.*

Proof First suppose that T is a nonexpansive mapping. Let $x_0 \in X$ and $\{x_n\}$ be the Picard sequence based at x_0. Then

$$d(x_{n+2}, x_{n+1}) = d(T(x_{n+1}), T(x_n)) \leq d(x_{n+1}, x_n) \quad \text{for all } n \in \mathbb{N}_0. \tag{14}$$

Then $\{d(x_{n+1}, x_n)\}$ is a decreasing sequence of nonnegative real numbers, so converges to some $L \geq 0$. Now put $y_n = x_{n+1}$ for all $n \in \mathbb{N}_0$. We get that $d(T(y_n), T(x_n)), d(y_n, x_n) \to L$ as $n \to \infty$. Since T satisfies the condition $s(ii)$ of Definition 8, then $L = 0$.
Therefore $\{x_n\}$ is asymptotically regular. Since T is continuous, T is an S_0-operator and by Corollary 2 has a unique fixed point.

Lemma 1 *Let (X, d) be a compact metric space and $T : X \to X$ be a continuous mapping.*

(i) *If*

$$d(T(x), T(y)) \neq d(x, y) \quad \textit{for all } x, y \in X \textit{ with } x \neq y, \tag{15}$$

then T satisfies the condition $s(ii)$ of Definition 8,

(ii) *suppose that*

$$d(T(x), T(y)) > d(x, y) \quad \textit{for all } x, y \in X \textit{ with } x \neq y, \tag{16}$$

then T is an S_0-operator.

Proof (i) Suppose that $\{x_n\}, \{y_n\}$ are two sequences such that $d(x_n, y_n)$ and $d(T(x_n), T(y_n))$ converge to some $L \geq 0$. Assume that $L > 0$. Since X is compact, there exists a subsequence $\{n(k)\} \subseteq \mathbb{N}_0$ such that the sequences $\{x_{n(k)}\}, \{y_{n(k)}\}$ converge to some points $x, y \in X$ respectively. As T is continuous, we have:

$$d(x, y) = d(T(x), T(y))) = L. \tag{17}$$

Since $L > 0$, this follows that $x \neq y$. Therefore $d(T(x), T(y)) \neq d(x, y)$ by (15), which contradicts with (17), this contradiction shows that $L = 0$.

(ii) By part (i) the mapping T satisfies the condition $s(ii)$ in Definition 8. Let $\{x_n\}$ be the Picard sequence based at $x_0 \in X$. Then sequence $\{d(x_{n+1}, x_n)\}$ is increasing. Since X is compact and T is continuous, this sequence is also bounded and converges to some $L \geq 0$. Now as the proof of Theorem 10 $L = 0$. Hence T is an S_0-operator and has a unique fixed point.

Remark 3 Example 3 show that there are operators that satisfy condition (i) of Lemma 1 and still have no fixed points.

The following theorem characterizes Edelstein's contractions on compact metric spaces.

Theorem 11 *Let (X, d) be a compact metric space and $T : X \to X$ be a nonexpansive mapping. Then T is a contractive mapping iff T is an S_0-operator.*

Proof First suppose that T is a contractive mapping. By part (i) of Lemma 1, T satisfies the condition $s(ii)$ of Definition 8. Now T is an S_0-operator by Theorem 10.
Conversely, suppose that T is nonexpansive S_0-operator. Suppose that $x, y \in X$ and $d(T(x), T(y)) = d(x, y) = L$ with $x \neq y$. Let $x_n = x, y_n = y$ for all $n \in N_0$, then $d(T(x_n), T(y_n)), d(x_n, y_n) \to L$ and $L > 0$ which is a contradiction. Therefore T is contractive mapping and the proof is complete.

Remark 4 Notice that each compact metric space is complete and by Theorem 11 every contractive operator on a compact metric space is an S_0-operator, therefore Theorem 11 shows that S_0-operators are the generalization of Edelstein contractions on compact metric spaces (see Theorem 3).

Now we characterize the Meir–Keeler contractions via S_1-operators as follow.

Theorem 12 *Let (X, d) be a complete metric space and $T : X \to X$ be a self mapping. Then T is a Meir–Keeler contraction iff T is an S_1-operator and*

$$d(T(x), T(y)) \leq d(x, y) \quad \text{for all } x, y \in X. \tag{18}$$

Proof Suppose that T be a Meir–Keeler contraction. T is a contractive mapping, that is: $d(T(x), T(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$ (see Theorem 4). Therefore $d(T(x), T(y)) \leq d(x, y)$ for all $x, y \in X$. Now in three steps we show that T is an S_1-operator.

(i) Let $x_0 \in X$ and $\{x_n\}$ be the Picard sequence of T based at x_0. It is known that $\{x_n\}$ is an asymptotically regular sequence [13], but for the sake of completeness we write the proof:
For all $n \in \mathbb{N}_0$, $d(x_{n+2}, x_{n+1}) = d(T(x_{n+1}), T(x_n)) \leq d(x_{n+1}, x_n)$, hence the sequence $d_n = d(T(x_{n+1}), T(x_n))$ is a nonincreasing sequence of nonnegative real numbers, so converges to some $\epsilon_0 \geq 0$. Suppose that $\epsilon_0 > 0$. Since T is a Meir–Keeler contraction and $\epsilon_0 > 0$, there is $\delta > 0$ such that

$$x, y \in X \text{ and } \epsilon_0 \leq d(T(x), T(y)) < \epsilon_0 + \delta \text{ implies } d(T(x), T(y)) < \epsilon_0. \tag{19}$$

Since $d(x_{n+1}, x_n) \downarrow \epsilon_0$, there exists a negative integer m such that $d(x_{m+1}, x_m) < \epsilon_0 + \delta$. Now by (19) we deduce that $d(x_{m+2}, x_{m+1}) = d(T(x_{m+1}), T(x_m) < \epsilon_0$, a contradiction. Therefore $\{x_n\}$ is an asymptotically regular sequence.

(ii) Now suppose that $\{x_n\}, \{y_n\}$ are two sequences in X such that $d(x_n, y_n)$ and $d(T(x_n), T(y_n))$ converge to some $L \geq 0$. Assume $L > 0$. So there exists $\delta > 0$ such that:

$$L \leq d(x, y) < L + \delta \to d(T(x), T(y)) < L \quad \text{for all x,y} \in X. \tag{20}$$

Since $d(x_n, y_n) \geq d(T(x_n), T(y_n)) \geq L$, there exists $n \in \mathbb{N}_0$ such that $L \leq d(x_n, y_n) < L + \delta$. Consequently $d(T(x_n), T(y_n)) < L$.This contradiction shows that $L = 0$.

(iii) every Meir–Keeler contraction is continuous, therefore T satisfies the condition $s(iii)$ in Definition 7.

Therefore T is an S_1-operator.
Conversely let T be a nonexpansive S_1-operator. Suppose that T is not a Meir–Keeler contraction. There exists $\epsilon_0 >$ such that for each $\delta > 0$ there are $x, y \in X$ such that:

$$\epsilon_0 \leq d(x, y) < \epsilon_0 + \delta, \quad \text{and} \quad d(T(x), T(y)) \geq \epsilon_0. \tag{21}$$

Then for each $\delta = \frac{1}{n}$ where $0 \neq n \in \mathbb{N}_0$, there exist $x_n, y_n \in X$ such that:

$$\epsilon_0 \leq d(x_n, y_n) < \epsilon_0 + \frac{1}{n} \quad \text{and} \quad d(T(x_n), T(y_n)) \geq \epsilon_0. \tag{22}$$

And since T is nonexpansive we get:

$$\epsilon_0 \leq d(T(x_n), T(y_n)) \leq d(x_n, y_n) < \epsilon_0 + \frac{1}{n}, \text{ for all } n \in \mathbb{N}_0, \tag{23}$$

this follows that:

$$d(T(x_n), T(y_n)), d(x_n, y_n) \to \epsilon_0 \text{ as } n \to \infty, \quad \text{and} \quad d(T(x_n), T(y_n)) \geq \epsilon_0 \quad \text{for all } n \in \mathbb{N}_0. \tag{24}$$

Since T is an S_1-operator, this implies that $\epsilon_0 = 0$. This contradiction completes the proof.

Lemma 2 *Every R-contraction with respect to an R-function ρ satisfying the condition (ρ_3) of Definition 4 is a continuous function.*

Proof Let T be an operator that satisfies the condition (ρ_3) in Definition 4. To show that T is continuous, let $\{x_n\}$ be a sequence in X which converges to the point $x \in X$. Define $b_n = d(x_n, x)$ and $a_n = d(T(x_n), T(x))$ for each $n \in \mathbb{N}_0$. If $x_n = x$, $T(x_n) = T(x)$. So without loss of generality suppose that $x_n \neq x$ for all $n \in \mathbb{N}_0$. Let $A = \{n \in \mathbb{N}_0 : a_n \neq 0\}$. If A be a finite set, then $T(x_n) \to T(x)$ as $n \to \infty$. Therefore suppose that A is infinite. Notice that there exists a subsequence $\{n(k)\}$ of $\mathbb{N}_0$ such that $A = \{n(k)\}$. Now we have $a_{n(i)}, b_{n(i)} \in (0, +\infty)$ for all $i \in \mathbb{N}_0$ and

$$\rho(a_{n(i)}, b_{n(i)}) = \rho(d(T(x_{n(i)}), T(x)), d(x_{n(i)}, x) > 0 \quad \text{for all } i \in \mathbb{N}_0, \tag{25}$$

now by condition (ρ_3), $a_{n(i)} \to 0$ as $i \to 0$. Consequently $a_n \to 0$ as $n \to 0$ which completes the proof.

Remark 5 In Theorem 7, (c) implies (b) (see Proposition 14 in [12]), and above lemma shows that the condition (b) implies the condition (a). Therefore in this theorem conditions (a) which is the continuity of T is weaker than the other conditions.

Theorem 13 *Every continuous R-contraction is an S-operator.*

Proof Let ρ be an R-function as in Definition 4 and

$$\rho(d(T(x), T(y)), d(x, y)) > 0 \quad \text{for all } x, y \in X \text{ such that } x \neq y. \tag{26}$$

In three steps we show that T is an S-operator.

(i) To verify that T satisfies the condition $s(i)$ in Definition 6, Let $x_0 \in X$ and $\{x_n\}$ be the Picard sequence of T based at x_0. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}_0$, then $\{x_n\}$ is asymptotically regular. Therefore suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}_0$, define $a_n = d(x_{n+1}, x_n)$ for all $n \in \mathbb{N}_0$. Now $a_n \in (0, \infty)$ for each $n \in \mathbb{N}_0$ and

$$\rho(a_{n+1}, a_n) = \rho(d(T(x_{n+1}), T(x_n)), d(x_{n+1}, x_n)) > 0 \quad \text{for all } n \in \mathbb{N}_0, \tag{27}$$

now by condition (ρ_1) in Definition 4, $a_n \to 0$ as $n \to 0$. This shows that the sequence $\{x_n\}$ is asymptotically regular.

(ii) Now we show that T satisfies the condition $s(ii)$ of Definition 6. To do this suppose that $\{x_n\}, \{y_n\}$ are two sequences in X such that $d(x_n, y_n)$ and $d(T(x_n), T(y_n))$ converge to some $L \geq 0$ and $d(T(x_n), T(y_n)) > L$ for all $n \in \mathbb{N}_0$. Suppose that $L > 0$. Since $d(x_n, y_n)$ converges to L and $L > 0$, there exists $k \in \mathbb{N}_0$ such that $d(x_n, y_n) > 0$ for all $n \geq k$. Put $a_n = d(T(x_{n+k}), T(y_{n+k}))$ and $b_n = d(x_{n+k}, y_{n+k})$ for all $n \in \mathbb{N}_0$. Since T is a R-contraction with respect to ρ, we have:

$$\rho(a_n, b_n) = \rho(d(T(x_{n+k}), T(y_{n+k})), d(x_{n+k}, y_{n+k})) > 0 \ (n \in \mathbb{N}_0). \tag{28}$$

Now we have:

$$\rho(a_n, b_n) > 0 \quad \text{and} \quad a_n > L \quad \text{for all } n \in \mathbb{N}_0. \tag{29}$$

Consequently by condition (ρ_2) in Definition 4 we deduce that $L = 0$. This contradiction completes the proof of this part.

(iii) By assumption T is continuous.

3 Applications

In this section we apply our method to some fixed point results. In this way we see that how this method can be used as a criteria to confirm that if a given contraction is a Meir–Keeler one or not. Some earlier results are also extended in the sequel.

Theorem 14 *Every Geraghty contraction (see Theorem 5) is a Meir–Keeler contraction.*

Proof Let T be a Geraghty contraction as in Theorem 5. Suppose that, for the two sequences $\{x_n\}$ and $\{y_n\}$ in X, $d(x_n, y_n)$ and $d(T(x_n), T(y_n))$ converge to the same *limit* $L \geq 0$. We confirm that $L = 0$.
Notice that:

$$d(T(x_n), T(y_n)) \leq \alpha(d(x_n, y_n))d(x_n, y_n),$$

for all positive integer n. Taking *liminf* of above inequality implies that:

$$L \leq \liminf_{n\to\infty} \alpha(d(x_n, y_n))L.$$

Suppose that $L > 0$ which follows that $\liminf_{n\to\infty} \alpha(d(x_n, y_n)) \geq 1$. On the other hand by definition of α, $\limsup_{n\to\infty} \alpha(d(x_n, y_n)) \leq 1$. Therefore $\lim_{n\to\infty} \alpha(d(x_n, y_n)) = 1$. Using definition of α again follows that

$$L = \lim_{n\to\infty} d(x_n, y_n)) = 0,$$

a contradiction. Therefore T satisfies the condition $s(ii)$ of S_0-operators and since T is a nonexpansive mapping, T is an S_0-operator by Theorem 10. Now T is a Meir–Keeler contraction by Corollary 2 and Theorem 12.

Corollary 3 is inspired by Mizoguchi and Takahashi [14]. At first we prove the following theorem and then deduce Corollary 3 as a result.

Theorem 15 *Let (X, d) be complete metric space and $T : X \to X$ be a map with the following property:*

$$d(T(x), T(y)) \le \alpha(d(x, y))d(x, y) \quad \textit{for all } x, y \in X, \tag{30}$$

where $\alpha : (0, \infty) \to [0, 1)$ is a mapping such that:

$$\lim_{t \to r+} \alpha(t) = a, r > 0 \to a < 1. \tag{31}$$

Where $t \to r+$ means that t tends to r from the right. Then T is a Meir–Keeler contraction.

Proof To verify that T satisfies the condition $s(ii)$ of Definition 7 suppose that $\{x_n\}, \{y_n\} \subset X$ and $d(x_n, y_n)$ and $d(T(x_n), T(y_n))$ converge to the same *limit* $L \ge 0$ with $d(T(x_n), T(y_n)) \ge L$ for all $n \in \mathbb{N}_0$. We show that $L = 0$. We have:

$$d(T(x_n), T(y_n)) \le \alpha(d(x_n, y_n))d(x_n, y_n),$$

for all positive integer n. Taking *liminf* of above inequality we get:

$$L \le \liminf_{n \to \infty} \alpha(d(x_n, y_n))L. \tag{32}$$

Suppose that $L > 0$. Dividing both sides of above inequality by L implies that $1 \le \liminf_{n \to \infty}(\alpha(d(x_n, y_n)))$ and by definition of α we get that:

$$\limsup_{n \to \infty}(\alpha(d(x_n, y_n))) \le 1, \tag{33}$$

hence

$$\lim_{n \to \infty}(\alpha(d(x_n, y_n))) = 1,$$

since $d(x_n, y_n) \ge d(T(x_n), T(y_n)) \ge L$ for all $n \in \mathbb{N}_0$, we deduce that

$$\lim_{n \to +\infty}(\alpha(t_n))) = 1,$$

where $t_n = d(x_n, y_n)$ for all $n \in \mathbb{N}_0$. By condition 31 this is a contradiction, which proves our claim. It is easy to see that T is contractive. Therefore by the same argument as given in the proof of Theorem 14, we deduce that T is a Meir–Keeler contraction.

Corollary 3 *Let (X, d) be complete metric space and $T : X \to X$ be a map with the following property:*

$$d(T(x), T(y)) \le \alpha(d(x, y))d(x, y) \quad \textit{for all } x, y \in X, \tag{34}$$

where $\alpha : (0, \infty) \to [0, 1)$ is a mapping such that satisfies one of the following conditions:

(a) $\limsup_{t \to r+} \alpha(t) < 1$, *for all* $r > 0$.
(b) $\liminf_{t \to r+} \alpha(t) < 1$, *for all* $r > 0$.

Then T is a Meir–Keeler contraction.

Proof Both of these conditions imply that if $r > 0$ and $\lim_{t\to r+}\alpha(t) = a$, then $a < 1$. Now by Theorem 15 the proof is complete.

Now we extend Theorem 5 of [7] as follow.

Theorem 16 *Let (X, d) be a complete metric space and $T : X \to X$ be a nonexpansive map such that:*

$$\psi(d(T(x), T(y))) \le \alpha(d(x, y)) - \beta(d(x, y)) \quad \textit{for all } x, y \in X, \tag{35}$$

where $\alpha, \beta : [0, \infty) \to [0, \infty)$ are continuous, lower-semicontinuous maps respectively and $\psi : [0, \infty) \to [0, \infty)$ is a lower-semicontinuous map on the right (that is $\psi(r) \le \liminf_{t\to r+}\psi(t)$ for all $r \ge 0$) and

$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \textit{for all } t > 0, \tag{36}$$

then T is a Meir–Keeler contraction.

Proof Suppose that for the two sequences $\{x_n\}, \{y_n\} \subset X$, $d(x_n, y_n)$ and $d(T(x_n), T(y_n))$ converge to the same *limit* $L \ge 0$ and $d(T(x_n), T(y_n)) \ge L$ for all $n \in \mathbb{N}_0$. We show that $L = 0$. We have:

$$\psi(d(T(x_n), T(y_n))) \le \alpha(d(x_n, y_n)) - \beta(d(x_n, y_n)) \quad \text{for all } n \in X, \tag{37}$$

and taking *limsup* as $n \to \infty$ we get:

$$\begin{aligned} \psi(L)) \le \liminf_{n\to\infty} \psi(d(T(x_n), T(y_n))) &\le \limsup_{n\to\infty} \psi(d(T(x_n), T(y_n))) \\ &\le \limsup_{n\to\infty} \alpha(d(x_n, y_n)) - \liminf_{n\to\infty} \beta(d(x_n, y_n)) \\ &\le \alpha(L) - \beta(L), \end{aligned} \tag{38}$$

and using condition (36) implies that $L = 0$. But T is nonexpansive, consequently by the same argument as given in the proof of Theorem 14, we deduce that T is a Meir–Keeler contraction.

Now we deduce Theorem 5 of [7] as the following corollary. It is worth mentioning that Gavruta et al. [7] provided this theorem as an extension of some results in [5,15], see also [2].

Corollary 4 (Gavruta et al. [7, Theorem 5]) *Let (X, d) be a complete metric space and $T : X \to X$ be a nonexpansive map such that:*

$$\psi(d(T(x), T(y))) \le \alpha(d(x, y)) - \beta(d(x, y)) \quad \textit{for all } x, y \in X, \tag{39}$$

where $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ are nondecreasing, continuous and lower-semicontinuous maps respectively and

$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \textit{for all } t > 0, \tag{40}$$

then T is a Meir–Keeler contraction.

Proof Conditions (39) and (40) when ψ is a nondecreasing map follow that T is nonexpansive, since otherwise there exist $x, y \in X$ with $x \ne y$ such that:

$$\psi(d(x, y)) \le \psi(d(T(x), T(y))) \le \alpha(d(x, y)) - \beta(d(x, y)) \quad \text{for all } x, y \in X. \tag{41}$$

Now put $L = d(x, y) > 0$ and using (40) we have a contradiction. It is easy to see that every nondecreasing mapping is lower-semicontinuous on the right. Therefore T satisfies the conditions of Theorem 16 and hence is a Meir–Keeler contraction.

In 1971, Ćirić [3] introduced orbitally continuous maps on metric spaces as follow.

Definition 9 Let (X, d) be a metric space. A mapping T on X is orbitally continuous if $\lim_{i\to\infty} T^{n_i}(x) = u$ implies $\lim_{i\to\infty} TT^{n_i}(x) = T(u)$ for each $x \in X$.

Now the set of orbitally S-operators is defined as follow and as we see in the sequel, this set of operators includes the set of S-operators properly (see Theorem 17).

Definition 10 (*orbitally S-operator*) Let (X, d) be a complete metric space, the mapping $T : X \to X$ is called orbitally S-operator if the following conditions hold.

- s(i) The Picard sequence $\{x_n\}$ based at x_0 is asymptotically regular for some $x_0 \in X$,
- s(ii) for any two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ if $d(x_{n(k)}, x_{m(k)})$ converges to some *limit* $L \geq 0$ and $d(x_{n(k)}, x_{m(k)}) > L$ for all $k \in \mathbb{N}_0$, then L=0. Where $\{x_n\}$ is the Picard sequence of T based at $x_0 \in X$.
- s(iii) T is orbitally continuous,

Theorem 17 *Every orbitally S-operator on a complete metric space has a fixed point. The set of orbitally S-operators contains the set of S-operators properly.*

Proof The proof of the first part is similar to the proof of Theorem 8, the inclusion is evident. Example 5 shows that this inclusion is proper.

Theorem 18 *Every Matkowski contraction (see Theorem 6) on a complete metric space is an orbitally S-operator.*

Proof Let $x_0 \in X$ and $\{x_n\}$ be the Picard sequence of T based at x_0. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}_0$, then $\{x_n\}$ is asymptotically regular. So suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}_0$. For each $n \geq 1$ we have:

$$d(x_{n+1}, x_n) \leq \psi^n(d(x_0, T(x_0))), \tag{42}$$

since $d(x_0, T(x_0)) > 0$ and $\psi^n(t) \to 0$ for any $t > 0$, this follows that the Picard sequence of T based at any point x_0 is asymptotically regular.
Let $\{x_n\}$ the Picard sequence of T based at x_0 and $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ be any two subsequences of it such that

$$\lim_{k\to\infty} d(T^{n(k)}(x_0), T^{m(k)}(x_0)) = L,$$

we show that L $= 0$. Let $r > 0$, since $\{x_n\}$ is asymptotically regular, there exists m_0 such that $d(T^{m_0}x_0, T^{m_0+1}x_0) < r - \psi(r)$ and put $y_0 = T^{m_0}(x_0)$. Therefore $d(y_0, T(y_0)) < r - \psi(r)$. Suppose that $n \in \mathbb{N}_0$ and $d(y_0, T^n(y_0)) < r$, then

$$\begin{aligned} d(y_0, T^{n+1}(y_0)) &\leq d(y_0, T(y_0)) + d(T(y_0), T^{n+1}(y_0)) \\ &< r - \psi(r) + \psi(d(y_0, T^n(y_0))) \\ &\leq r - \psi(r) + \psi(r) = r, \end{aligned} \tag{43}$$

so by induction $d(y_0, T^n(y_0)) < r$ for all $n \in \mathbb{N}_0$. Let $t \in \mathbb{N}_0$ with $t \geq 1$, choose $k \geq 1$ such that $m(k), n(k) > t$, consequently

$$\begin{aligned} d(T^{n(k)}(y_0), T^{m(k)}(y_0)) &\leq \psi^t(d(T^{n(k)-t}(y_0), T^{m(k)-t}(y_0)) \\ &= \psi^t(d(T^{n(k)-t}(y_0), y_0) + d(y_0, T^{m(k)-t}(y_0))), \\ &\leq \psi^t(2r), \end{aligned} \tag{44}$$

since $\psi^t(2r) \to 0$ as $t \to \infty$, we deduce that $L = 0$.
As T is contractive, so it is continuous. Therefore T is orbitally S-operator.

Remark 6 The set of Matkowski and Meir–Keeler contractions are incomparable [9]. By Theorem 12, Theorem 17 and Theorem 18 the set of orbitally S-operators contains both these set of contractions. By Example 5 the set of orbitally S-operators contains noncontinuous operators, therefore these inclusions are proper.

4 Examples

Let the set of S_0-operators, S_1-operators and S-operators on X be denoted by $\mathfrak{S}_0, \mathfrak{S}_1, \mathfrak{S}$ respectively and the set of orbitally S-operators, continuous R-contractions and Banach contractions on X be denoted by $\mathfrak{O}S$, $\mathfrak{C}R$ and $\mathfrak{B}$ respectively.
The results of the previous sections can be briefed as follow:

$$\mathfrak{B} \subset \mathfrak{S}_0 \subseteq \mathfrak{S}_1 \subseteq \mathfrak{S} \quad \text{and} \quad \mathfrak{C}R \subseteq \mathfrak{S} \subseteq \mathfrak{O}S. \tag{45}$$

In this section some different examples of S-operators are presented to illustrate that all inclusions in (45) are proper.
The following example is an S-operator that has two fixed points. Therefore it is not an R-contraction.

Example 1 Let $X = \{0, 1\}$ and $T : X \to X$ defined by

$$T(x) = \begin{cases} 0 & x = 0 \\ 1 & x = 1 \end{cases}$$

Proof Suppose that $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $d(x_n, y_n)$ and $d(T(x_n), T(y_n))$ converge to the same *limit* $L \geq 0$ with $d(T(x_n), T(y_n)) > L$ for all $n \in \mathbb{N}_0$, then we prove that $L = 0$. Since $d(x_n, y_n) = 0$ or 1 for all $n \in \mathbb{N}_0$ and $d(x_n, y_n) \to L$, we deduce that $L = 0$ or 1. Now $d(T(x_n), T(y_n)) = 0$ or 1 and $d(T(x_n), T(y_n)) > L$ for all $n \in \mathbb{N}_0$. This follows that $L \neq 1$. Therefore $L = 0$ and this confirms that T is an S-operator. It is worth noticing that T is a nonexpansive S-operator which is not a Meir–Keeler contraction, this shows that Theorem 12 dose not hold for S-operators.

Example 2 Define , $x_0, x_1 = 0$ and $x_{n+1} = x_n + 2 - \frac{1}{n}$ for all $n \in \mathbb{N}_0$ with $n \geq 1$. Put $X = \{x_0, x_1, x_2, \ldots\}$ and define $T : X \to X$ by $T(x_n) = x_{n+1}$ for all $n \in \mathbb{N}_0$.
Then (X, d) is a complete metric space with Euclidean metric and $T : X \to X$ is an S_1-operator which is not an S_0-operator.

Proof X is a discrete subset of $\mathbb{R}$ and has no *limit* point and it is evident that (X, d) is a complete metric space and T is continuous. Suppose that $\{a_n\}$ and $\{b_n\}$ be two sequences in X and $d(a_n, b_n)$ and $d(T(a_n), T(b_n))$ converge to the same *limit* $L \geq 0$ with $d(T(a_n), T(b_n)) \geq L$ for all $n \in \mathbb{N}_0$, then we show that $L = 0$.
Since $X = \{x_n : n \in \mathbb{N}_0\}$, then $\{a_n\} = \{x_{n(k)}\}$, and $\{b_n\} = \{x_{m(k)}\}$ for some subsequence $\{n(k)\}, \{m(k)\} \subset \mathbb{N}_0$. Put

$$M = max\{| n(k) - m(k) |: k \in \mathbb{N}_0\}.$$

notice that:

$$x_{n+i} = x_n + 2i - \left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+i-1}\right) \geq x_n + i. \tag{46}$$

In other words if $m > n$, then

$$d(x_m, x_n) = x_m - x_n \geq m - n, \tag{47}$$

now by assumption $d(x_{n(k)}, x_{m(k)}) \to Ł < \infty$, which by (47) follows that $M < \infty$. As there are finite number of choices for $\mid n(k) - m(k) \mid$ for all $k \in \mathbb{N}_0$, we may assume that $\mid n(k) - m(k) \mid$ is constant for a infinite numbers of k. So suppose that $\mid n(k) - m(k) \mid = i$ for all $k \in \mathbb{N}_0$ where i is a nonnegative integer. Now by (46) we deduce that:

$$d(x_{n(k)}, x_{m(k)}) \to Ł = 2i \quad \text{as} \quad k \to \infty, \tag{48}$$

and by our assumption we have:

$$\begin{aligned} &d(x_{n(k)}, x_{m(k)}), d(x_{n(k)+1}, x_{m(k)+1}) \to Ł = 2i, \\ &\text{and } d(x_{n(k)+1}, x_{m(k)+1}) \geq 2i \quad \text{for all } k \in \mathbb{N}_0, \end{aligned} \tag{49}$$

without loss of generality we suppose that $n(k) < m(k)$ for all $k \in \mathbb{N}_0$, therefore $m_k = n_k + i$.

$$\begin{aligned} x_{m(k)+1} &= x_{n(k)+1+i} \\ &= x_{n(k)+1} + 2i - \left(\frac{1}{n(k)+1} + \frac{1}{n(k)+2} + \cdots + \frac{1}{n(k)+i} \right), \end{aligned} \tag{50}$$

this follows that $L = 2i \leq d(T(x_{m(k)}), T(x_{n(k)})) = x_{m(k)+1} - x_{n(k)+1} < 2i$, a contradiction. This contradiction shows that T is an S_1-operator.
Now notice that $d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1}) \to 2$ as $n \to \infty$, putting $a_n = x_{n+1}, b_n = x_n$ for all $n \in \mathbb{N}_0$ we have:

$$d(a_n, b_n), d(T(a_n), T(b_n)) \to 2, \tag{51}$$

which follows that T is not an S_0-operator and the proof is complete.

The following example shows that the condition $s(ii)$ of Definition 7 dose not imply the condition $s(i)$ even when X is compact and T is continuous.

Example 3 Let $X = \{1, 3, 4\}$ define $T : X \to X$ by $T(1) = 3, T(3) = 4, T(4) = 1$. Then X is a compact metric space with Euclidean metric and T is continuous map with no fixed point.

Proof Suppose that $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $d(x_n, y_n)$ and $d(T(x_n), T(y_n))$ converge to the same *limit* $L \geq 0$, then we show that $L = 0$. Since X is finite, so we can choose a constant subsequences $\{x_{n(k)}\}$ of $\{x_n\}$ in X. If $\{y_{n(k)}\}$ is not a constant sequence, a constant subsequence $\{y_{m(k)}\}$ of it can be obtained and the sequences $\{x_{n(k)}\}$ and $\{y_{n(k)}\}$ are replaced with constant subsequences $\{x_{m(k)}\}$ and $\{y_{m(k)}\}$ respectively. So without loss of generality we assume that $\{x_{n(k)}\}, \{y_{n(k)}\}$ are constant subsequences in X and therefore $x_{n(k)} = r$, $y_{m(k)} = s$ for some positive integers $r, s \in X$. From $d(T(x_{n(k)}), T(y_n)), d(x_{n(k)}, y_{n(k)}) \to L$ as $k \to \infty$, we get:

$$d(r, s) = d(T(r), T(s)) = L, \tag{52}$$

the above equation follows that $L = 0$. Since (X, d) is a discrete metric space, T is continuous. But for each $x_0 \in X$ the Picard sequence on x_0 is not asymptotically regular.

See the following example for an S_0-operator which is not a Banach contraction.

Example 4 Let $X = [1, \infty)$ be the metric space with Euclidean metric and $T : X \to X$ be a self mapping defined by $T(x) = x^2$ for all $x \in X$. Then T is an S_0-operator.

Proof We show that T is an S_0-operator.

(i) X is a Banach space and $T(1) = 1$, so the Picard sequence based at $x_0 = 1$ is asymptotically regular.

(ii) Suppose that $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $d(x_n, y_n)$ and $d(T(x_n), T(y_n))$ converge to the same $L \geq 0$, then we show that $L = 0$. Suppose that $L > 0$, then

$$\lim_{n\to\infty} \mid x_n - y_n \mid .(x_n + y_n) = \lim_{n\to\infty} \mid x_n^2 - y_n^2 \mid = L. \tag{53}$$

Since $\mid x_n - y_n \mid .(x_n + y_n)$ and$\mid x_n - y_n \mid$ converge to $L > 0$, $x_n + y_n$ is a convergent sequence. As $x_n, y_n \geq 1$ for all $n \in \mathbb{N}_0$, then $\lim_{n\to}(x_n + y_n) = t \geq 2$. Now taking *limit* of Eq. (53) we get $t.L = L$, hence $t = 1$. This contradiction implies that $L = 0$.

(iii) It is trivial that T is continuous.

Therefore T is an S_0-operator.

Example 5 Let $X = [1, \infty)$ be the metric space with Euclidean metric and $T : X \to X$ be a self mapping defined by:

$$T(x) = \begin{cases} x^2 & x \in [1, 2] \\ 5 & x \in (2, +\infty) \end{cases}$$

Then T is an orbitally S-operators which is not an S-operator.

Proof The Picard sequence of T based at the point $x = 2$ is asymptotically regular. T is not continuous at the point $x = 2$, but it is easy to see that T is an orbitally continuous operator(see Definition 10). With the same argument given in Example 4 we see that T also satisfies the condition $s(ii)$ of Definition 10. Therefore T is an orbitally S-operator which is not an S-operator.

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