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On the dynamics of rational maps with two free critical points

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Abstract In this paper we discuss the dynamical structure of the rational family (f_t) given by

$$f_t(z) = tz^m \left(\frac{1-z}{1+z}\right)^n \quad (m \geq 2,\ n \in \mathbb{N},\ t \in \mathbb{C}\backslash\{0\}).$$

Each map f_t has super-attracting immediate basins $\mathscr{A}_t$ and $\mathscr{B}_t$ about $z = 0$ and $z = \infty$, respectively, and two free critical points. We prove that $\mathscr{A}_t$ (for $0 < |t| \leq 1$) and $\mathscr{B}_t$ (for $|t| \geq 1$) are completely invariant, and at least one of the free critical points is inactive. Based on this separation we draw a detailed picture of the structure of the dynamical and the parameter plane.

Keywords Julia set · Bifurcation locus · Escape locus · Basin of attraction · Mandelbrot set · Hyperbolic component

Mathematics Subject Classification 37F10 · 37F45

1 Introduction

Non-trivial rational families (f_t) normally contain specific maps of different character with most interesting and unexpected Julia sets:

- totally disconnected Julia sets (Cantor sets) occur in any family $z \mapsto z^d + t$;

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- Julia sets consisting of uncountably many (a Cantor set of) quasi-circles occur in the McMullen family $z \mapsto z^m + t/z^n$, which was introduced in [8]. The number of papers on various features of this family is legion; [3] marks the preliminary end of a long list of papers.
- Julia sets that are Sierpiński curves (Milnor and Tan Lei [12] were the first to construct examples with this property) occur again in the McMullen family [16], the Morosawa-Pilgrim family $z \mapsto t\left(1 + \frac{(4/27)z^3}{1-z}\right)$ [4, 17], and the family $t \mapsto -\frac{t}{4}\frac{(z^2-2)^2}{z^2-1}$ [7].
- In any reasonable family, copies of the Mandelbrot sets of the families $z \mapsto z^d + t$ are dense in the bifurcation locus—the Mandelbrot set is universal [10].

Each of these families has just one *free* critical point (or several free critical points which have the same dynamical behaviour, this happens, for example, in the McMullen family; the quasi-conjugated family $F_t(z) = z^m(1 + t/z)^d$ has just one free critical point). In contrast to this the rational maps

$$f_t(z) = tz^m\left(\frac{1-z}{1+z}\right)^n \quad (m \geq 2,\ n \in \mathbb{N},\ d = m+n,\ t \neq 0) \tag{1}$$

in the family under consideration have two free critical points. In this paper we will give a complete description of the parameter plane and the various dynamical planes. For basic notations and results the reader is referred to the texts [1, 2, 9, 11, 15].

2 Notation

The rational map (1) has

- two super-attracting fixed points 0 and ∞ with corresponding basins $\mathscr{A}_t$ and $\mathscr{B}_t$, respectively. Then $\mathscr{A}_t$, say, either is completely invariant or else has a single pre-image $\mathscr{A}_t^*$ that is mapped in a $(n{:}1)$-manner onto $\mathscr{A}_t$, which will be written as

$$\mathscr{A}_t^* \xrightarrow{n:1} \mathscr{A}_t;$$

- two free critical points

$$\alpha = -\frac{n}{m} + \sqrt{1 + \left(\frac{n}{m}\right)^2} \quad \text{and} \quad \beta = -\frac{n}{m} - \sqrt{1 + \left(\frac{n}{m}\right)^2}$$

  and critical values

$$v_t^\alpha = f_t(\alpha) = tv_1^\alpha \quad \text{and} \quad v_t^\beta = f_t(\beta) = tv_1^\beta;$$

- two *escape loci* Ω^α and Ω^β, with $t \in \Omega^\alpha$ and $t \in \Omega^\beta$ if and only if $f_t^k(\alpha) \to 0$ and $f_t^k(\beta) \to \infty$, respectively, as $k \to \infty$;
- two *residual sets* $\Omega^\alpha_{\rm res}$ and $\Omega^\beta_{\rm res}$, with $t \in \Omega^\alpha_{\rm res}$ and $t \in \Omega^\beta_{\rm res}$ if and only if $v_t^\beta \in \mathscr{A}_t$ and $v_t^\alpha \in \mathscr{B}_t$, respectively.

The notation of the residual sets indicates that $\Omega^\alpha_{\rm res}$ is related to Ω^α rather than Ω^β. The open sets Ω^α and Ω^β are in a natural way sub-divided into

- Ω^α_0 resp. Ω^β_0: $v_t^\alpha \in \mathscr{A}_t$ resp. $v_t^\beta \in \mathscr{B}_t$, and
- Ω^α_k resp. Ω^β_k: $f_t^k\left(v_t^\alpha\right) \in \mathscr{A}_t$, but $f_t^{k-1}\left(v_t^\alpha\right) \notin \mathscr{A}_t$ resp.
  $f_t^k\left(v_t^\beta\right) \in \mathscr{B}_t$, but $f_t^{k-1}\left(v_t^\beta\right) \notin \mathscr{B}_t$ $(k \geq 1)$.

Hitherto, f_t is hyperbolic and the Fatou set of f_t consists of the basins $\mathscr{A}_t$ and $\mathscr{B}_t$, and their pre-images, if any. However, there may and will be also other hyperbolic components. By $\mathbf{W}_k^\alpha$ and $\mathbf{W}_k^\beta$ we denote the open sets such that α and β belongs to some (super-)attracting cycle of Fatou domains $U_1, \ldots, U_k$, respectively, not containing 0 and ∞.

The *bifurcation* locus $\mathbf{B}$ of the family $(f_t)_{0<|t|<\infty}$ is the set of t such that the Julia set $\mathscr{J}_t$ does not move continuously over any neighbourhood of t, see McMullen [9]. In order that $t \in \mathbf{B}$ it is necessary and sufficient that at least one of the free critical points is *active*. Thus $\mathbf{B} = \mathbf{B}^\alpha \cup \mathbf{B}^\beta$, where $t \in \mathbf{B}^\alpha$ resp. $t \in \mathbf{B}^\beta$ means that α resp. β is active. It is *a priori* not excluded that $\mathbf{B}^\alpha$ and $\mathbf{B}^\beta$ overlap. Although there is just one parameter plane, each point of this plane carries at least two pieces of information, so one could also speak of the v_t^α- and v_t^β-plane.

We also set

$$Q_0(t) = v_t^\alpha = t v_1^\alpha \quad \text{and} \quad Q_k(t) = f_t^k(v_t^\alpha) = f_t(Q_{k-1}(t)) \quad (k \geq 1)$$

and note that Q_k is a rational function of degree $1 + d + \cdots + d^k = \frac{d^{k+1}-1}{d-1}$ with a zero of order $\frac{m^{k+1}-1}{m-1}$ at the origin.

From

$$-1/f_t(-1/z) = f_{(-1)^{d+1}/t}(z) \quad (d = m + n)$$

it follows that f_t is conjugated to $f_{1/t}$ if d is odd, and to $f_{-1/t}$ if d is even, hence $t \in \Omega^\alpha$ if and only if $(-1)^{d+1}/t \in \Omega^\beta$, and this is also true for Ω_k^α and Ω_k^β, $\Omega_{\text{res}}^\alpha$ and $\Omega_{\text{res}}^\beta$, $\mathbf{W}_k^\alpha$ and $\mathbf{W}_k^\beta$, and $\mathbf{B}^\alpha$ and $\mathbf{B}^\beta$. This also indicates that the circle $|t| = 1$ plays a distinguished role with strong impact on what follows.

Lemma 1 *For every $m \geq 2$, $n \geq 1$ there exists some $r > 0$, such that for $0 < |t| \leq 1$ the disc $\triangle_{r|t|} : |z| < r|t|$ contains $f_t(\overline{\triangle}_{r|t|} \cup [0, 1])$, but does not contain v_t^β.*

Proof We will first consider f_1 and show that there exists some disc $\triangle_r : |z| < r$ such that $f_1(\overline{\triangle}_r \cup [0, 1]) \subset \triangle_r$ holds. This is easy to show if $n < m$ for $r = \frac{1}{3}$:

$$|f_1(z)| \leq 3^{-m} 2^n < \tfrac{1}{3}$$

holds if $|z| \leq \frac{1}{3}$ and $m > n \geq 1$, and from

$$0 \leq f_1(x) \leq x^2 \frac{1-x}{1+x} \leq \tfrac{1}{2}\left(5\sqrt{5} - 11\right) < \tfrac{1}{10} \quad (0 \leq x \leq 1)$$

the assertion follows.

We now consider the case $n \geq m$. Then $f_1(\overline{\triangle}_r) \subset \triangle_r$ holds as long as

$$g(r) = r^{m-1} \left(\frac{1+r}{1-r}\right)^n < 1,$$

and f_1 maps $[0, 1]$ into $\triangle_r$ provided

$$v_1^\alpha = \max_{0 \leq x \leq 1} x^m \left(\frac{1-x}{1+x}\right)^n < r.$$

Since g is increasing this may be achieved if $g(v_1^\alpha) < 1$ holds. To prove this we note that $\sqrt{1+\tau} - 1 = \frac{\tau}{2\sqrt{1+\theta\tau}}$ $(0 < \theta < 1,\ \tau = \frac{m^2}{n^2} \leq 1)$ implies $\frac{m}{2\sqrt{2}n} < \alpha < \frac{m}{2n}$, while from

$\log \frac{1-x}{1+x} < -2x$ $(0 < x < 1)$ it follows that

$$v_1^\alpha < \left(\frac{m}{2n}\right)^m e^{-2\frac{m}{2\sqrt{2}}} = \left(\frac{m}{2e^{\frac{1}{\sqrt{2}}}n}\right)^m < \left(\frac{m}{4n}\right)^m = \mu^m.$$

Moreover, from

$$\log\frac{1+x}{1-x} = 2x\left(1+\frac{1}{3}x^2+\frac{1}{5}x^4+\cdots\right) \le 2x\left(1+\frac{x^2}{3}\frac{1}{1-x^2}\right) \le 2x\left(1+\tfrac{1}{45}\right),$$

which holds for $x = \left(\frac{m}{4n}\right)^{m-1} \le \frac{1}{4}$, we obtain

$$\left(\frac{1+\mu^m}{1-\mu^m}\right)^n = \left(\frac{1+\frac{m}{4}\frac{\mu^{m-1}}{n}}{1-\frac{m}{4}\frac{\mu^{m-1}}{n}}\right)^n \le e^{\frac{23}{45}m\mu^{m-1}} < \left(e^{\left(\frac{m}{4n}\right)^{m-1}}\right)^m.$$

Thus $g(v_1^\alpha) < 1$ follows from $\left(\frac{m}{4n}\right)^{m-1} e^{\left(\frac{m}{4n}\right)^{m-1}} \le \frac{1}{4}e^{\frac{1}{4}} < 1$.

With this choice of $r \in (0,1)$ it is clear that v_t^β belongs to $\triangle_r$ if $|t|$ is small. For individual $0 < |t| \le 1$, $f_t(z) = tf_1(z)$ maps $\overline{\triangle}_{r|t|} \cup [0,1]$ into $\triangle_{r|t|}$, while $v_t^\beta \notin \triangle_{r|t|}$ follows from $|v_t^\beta| = |t|/v_1^\alpha > |t| > r|t|$. □

3 The escape loci

The purpose of Lemma 1 is twofold. First of all it shows that the critical points α and β cannot be simultaneously active, and the bifurcation sets $\mathbf{B}^\alpha$ and $\mathbf{B}^\beta$ are separated by the unit circle $|t| = 1$. Secondly, the condition $v_t^\beta \notin \triangle_{r|t|}$ $(0 < |t| \le 1)$ ensures that in an exhaustion (D_κ) of $\mathscr{A}_t$ starting with $D_0 = \triangle_{r|t|}$, D_κ is simply connected as long as $\beta \notin D_\kappa$, and $f_t : D_\kappa \xrightarrow{d:1} D_{\kappa-1}$ has degree $d = m+n$. In particular, for $t \in \Omega^\alpha_{\mathrm{res}}$ there exists some simply connected and forward invariant domain $D_\kappa \subset \mathscr{A}_t$ that contains v_t^β (Figs. 1, 2).

We note some more simple consequences of Lemma 1; our focus is on the critical point α and the $^\alpha$-sets.

- $\{t : 0 < |t| \le 1\} \subset \Omega_0^\alpha$;
- $\overline{\Omega^\alpha_{\mathrm{res}}} \subset \mathbb{D}$;
- α is inactive on $0 < |t| \le 1$;
- $\overline{\bigcup_{k\ge1}(\Omega_k^\alpha \cup \mathbf{W}_k^\alpha)} \subset \{t : 1 < |t| < T\}$ for some $T = T_{mn} > 1$;
- $\mathbf{B}^\alpha \subset \{t : 1 < |t| < T\}$ for some $T = T_{mn} > 1$.

The consequences for the dynamical planes are as follows.

Theorem 1 *For $t \in \Omega_0^\alpha$, the basin $\mathscr{A}_t$ is completely invariant, and any other Fatou component is simply connected. Moreover,*

- *for $t \in \Omega_0^\alpha \cap \Omega_0^\beta$ also $\mathscr{B}_t$ is completely invariant, the Julia set $\mathscr{J}_t = \partial\mathscr{A}_t = \partial\mathscr{B}_t$ is a quasi-circle, and f_t is quasi-conformally conjugated to $z \mapsto z^d$;*
- *for $t \in \Omega^\alpha_{\mathrm{res}}$, $\mathscr{A}_t$ is infinitely connected and the Fatou set consists of $\mathscr{A}_t$, $\mathscr{B}_t$, and the predecessors of $\mathscr{B}_t$ of any order.*

Proof To prove complete invariance of $\mathscr{A}_t$ we first assume $0 < |t| \le 1$. Then $\mathscr{A}_t$ contains the interval $[0, 1]$ by Lemma 1, hence is completely invariant. If, however, $|t| > 1$, then

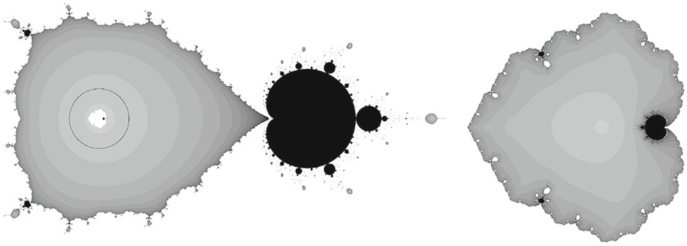

Fig. 1 *Left* the α-parameter plane for $f_t(z) = tz^2 \dfrac{1-z}{1+z}$ displaying the *unit circle*, Ω^α (*gray*), $\Omega^\alpha_{\text{res}}$ and $\Omega^\beta_{\text{res}}$ (*white*, in and outside the *unit circle*), and $\mathbf{W}^\alpha$ (*black*). *Right* a neighbourhood of the origin displaying $\Omega^\alpha_{\text{res}}$ (*gray*) surrounded by points of Ω^α_0 (*white*), Ω^β_k ($k \ge 1$, *white*, small), and $\mathbf{W}^\beta$ (*black*)

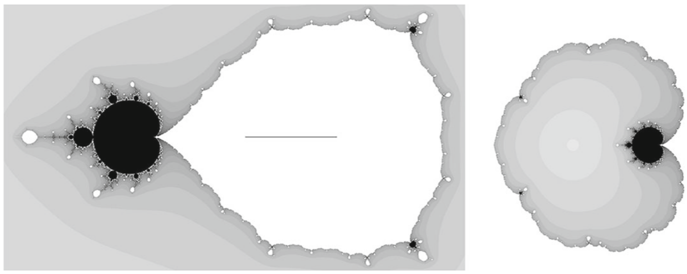

Fig. 2 *Left* the parameter plane of $P_c(z) = cz^2(z+1)$. The escape region for P_c (*gray*), the *white* region with slit, and the *black* regions correspond to $\Omega^\alpha_{\text{res}}$, $\Omega^\beta \cap \mathbb{D}$, and $\mathbf{W}^\beta$, in case of $m = 2$, $n = 1$, respectively. The punctured disc $0 < |t| < 1$ corresponds to $\mathbb{C}\backslash[-2, 0]$ in the c-plane. *Right* the parameter plane of $P_{-\frac{1}{2}(t+2+\frac{1}{t})}(z)$ in $-0.2 < \operatorname{Re} t < 0.25$, $-0.25 < \operatorname{Im} t < 0.25$ (see also Fig. 1 *right*)

$\mathscr{B}_t$ is completely invariant, and any other Fatou component is simply connected. Assuming $1 \notin \mathscr{A}_t$ ($t \in \Omega^\alpha_0$, $|t| > 1$) we obtain either $f_t : \mathscr{A}^*_t \xrightarrow{n:1} \mathscr{A}_t$ with $n = (n-1) + 1$ critical points if $\alpha \in \mathscr{A}^*_t$ or else $f_t : \mathscr{A}_t \xrightarrow{m:1} \mathscr{A}_t$ with $m = (m-1) + 1$ critical points if $\alpha \in \mathscr{A}_t$, this contradicting simple connectivity of both domains $\mathscr{A}_t$ and $\mathscr{A}^*_t$ by the Riemann–Hurwitz formula.

The first assertion is obvious since $\mathscr{B}_t$ shares the properties of $\mathscr{A}_t$ and f_t is hyperbolic.

The second assertion follows from the Riemann-Hurwitz formula, since $f_t : \mathscr{A}_t \xrightarrow{d:1} \mathscr{A}_t$ has degree d and $r = (m-1) + (n-1) + 1 + 1 = d$ critical points 0, 1 (if $n > 1$), α, and β. $\square$

Theorem 2 $\Omega^\alpha_0 \cup \{0\}$, $\Omega^\alpha_{\text{res}} \cup \{0\}$, *and the connected components of* Ω^α_k ($k \ge 1$) *are simply connected domains. Riemann maps onto* $\mathbb{D}$ *are given by any branch of* $\sqrt[m]{E_0(t)}$, $\sqrt[m]{E_{\text{res}}(t)}$, *and* $\sqrt[n]{E_k(t)}$, *respectively.*

For the proof we need two auxiliary results on the maps

$$\begin{aligned} E_0(t) &= t\left(\Phi_t(v^\alpha_t)\right)^{m-1} \quad \left(t \in \Omega^\alpha_0\right), \\ E_{\text{res}}(t) &= t\left(\Phi_t(v^\beta_t)\right)^{m-1} \quad \left(t \in \Omega^\alpha_{\text{res}}\right), \text{ and} \\ E_k(t) &= t^{\frac{1}{m-1}}\Phi_t\left(f^k(v^\alpha_t)\right) \left(t \in \Omega^\alpha_k,\ k \ge 1\right), \end{aligned} \tag{2}$$

where Φ_t denotes the Böttcher function to the fixed point $z = 0$. In the first step (Lemma 2) of the proof of Theorem 2 we will show that the functions (2) provide proper maps on $\mathbb{D}\setminus\{0\}$ and $\mathbb{D}$, respectively, which are only ramified over the origin. In the second step (Lemma 3) this will be used to show that the corresponding domains (with 0 included, if necessary) are simply connected.

The solution to Böttcher's functional equation

$$\Phi_t(f_t(z)) = t\Phi_t(z)^m \quad (\Phi_t(z) \sim z \text{ as } z \to 0) \tag{3}$$

is locally given by

$$\Phi_t(z) = \lim_{k\to\infty} \sqrt[m^k]{f_t^k(z)/t^{1+m+\cdots+m^{k-1}}} = t^{-\frac{1}{m-1}} \lim_{k\to\infty} \sqrt[m^k]{f_t^k(z)};$$

it conjugates f_t to $\zeta \mapsto \zeta^m$. This conjugation holds throughout $\mathscr{A}_t$ in the third case, when Φ_t maps $\mathscr{A}_t$ conformally onto the disc $|z| < |t|^{-\frac{1}{m-1}}$; the maps E_k are analytic and well-defined on the components of Ω_k^α, $k \ge 1$.

In the first case the conjugation holds on some simply connected neighbourhood of $z = 0$ that contains $z = 0$ and $z = v_t^\alpha$, but does not contain $z = 1$. The analytic continuation of Φ_t causes singularities at $z = 1$ and its preimages under f_t^k, nevertheless $|\Phi_t(z)|$ is well-defined on $\mathscr{A}_t$ and $|\Phi_t(z)| \to |t|^{-\frac{1}{m-1}}$ as $z \to \partial\mathscr{A}_t$ holds anyway. Thus $E_0(t) = t\Phi_t(v_t^\alpha)^{m-1}$ is holomorphic on Ω_0^α and zero-free, with $E_0(t) \sim t(v_t^\alpha)^{m-1} = f_1(\alpha)^{m-1}t^m$ as $t \to 0$.

In the second case we construct an exhaustion (D_κ) of $\mathscr{A}_t$ such that $f_t : D_\kappa \xrightarrow{d:1} D_{\kappa-1}$ has degree d and D_κ is simply connected for $\kappa \le \kappa_0$ with $v_t^\beta \in D_{\kappa_0}$ and $\beta \in D_{\kappa_0+1}\setminus D_{\kappa_0}$. This is possible by Lemma 1, and the procedure applied to $t^{-\frac{1}{m-1}}\Phi_t(v_t^\alpha)$ on Ω_0^α also applies to $t^{-\frac{1}{m-1}}\Phi_t(v_t^\beta)$ on $\Omega_{\text{res}}^\alpha$.

Lemma 2 *The functions in* (2) *are well-defined and provide proper maps from* $\Omega_0^\alpha \cup \{0\}$, $\Omega_{\text{res}}^\alpha \cup \{0\}$, *and the connected components of* Ω_k^α *with* $k \ge 1$, *respectively, onto the unit disc* $\mathbb{D}$.

Proof To prove that $|E_0(t)| \to 1$ as $t \in \Omega_0^\alpha$ tends to $\partial\Omega_0^\alpha\setminus\{0\}$ we choose any disc $\triangle_r : |z| < r$ that is invariant under f_t for every $t \in \Omega_0^\alpha$. This is possible since Ω_0^α is contained in some disc $|t| < T$, hence we may choose $r < 1$ such that $Tr^{m-1}\left(\frac{1+r}{1-r}\right)^n = 1$ holds. By $k = k(t)$ we denote the largest integer such that $f_t^k(v_t^\alpha) \notin \triangle_r$. Then $k(t) \to \infty$ as $t \to \partial\Omega_0^\alpha\setminus\{0\}$, and $|f_t^{k(t)}(v_t^\alpha)| \ge r$ implies

$$\liminf_{t\to\Omega_0^\alpha\setminus\{0\}} |\Phi_t(v_t^\alpha)| \ge \lim_{t\to\Omega_0^\alpha\setminus\{0\}} |t|^{-\frac{1}{m-1}} \sqrt[m^{k(t)}]{r} = |t|^{-\frac{1}{m-1}},$$

while $|\Phi_t(z)| < |t|^{-\frac{1}{m-1}}$ is always true. Thus E_0 maps each connected component of Ω_0^α properly onto $\mathbb{D}\setminus\{0\}$. It follows that the origin is removable for (a zero of) E_0, and $\Omega_0^\alpha \cup \{0\}$ is a domain which is mapped by E_0 properly with degree m onto the unit disc $\mathbb{D}$.

If $t \in \Omega_k^\alpha$ for some $k \ge 1$, then again $|E_k(t)|$ tends to 1 as $t \to \partial\Omega$, where Ω is any component of Ω_k^α. Thus E_k is a proper map of Ω onto $\mathbb{D}$. We will prove that E_k is ramified only over zero even for $k \ge 0$, that is $E_k'(t) = 0$ implies $E_k(t) = 0$. This is a well-known procedure, the idea of which is due to Roesch [13], and outlined in detail for the Morosawa-Pilgrim family $z \mapsto t\left(1 + \frac{(4/27)z^3}{1-z}\right)$ in [17, Lemma 2].

We take any $t_0 \in \Omega_k^\alpha$ and choose $\varepsilon > 0$ such that for t sufficiently close to t_0, the closed disc $\triangle_{3\epsilon} : |w - v_{t_0}^\alpha| \le 3\varepsilon$ belongs to the Fatou component D_{t_0} of f_{t_0} containing

$v_{t_0}^\alpha$ (D_{t_0} is a predecessor of $\mathscr{A}_{t_0}$ of order $\ell \ge 0$). Furthermore let $\eta_t : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be any diffeomorphism such that $\eta_t(w)$ depends analytically on t, $\eta_t(w) = w$ holds on $|w - v_{t_0}^\alpha| \ge 3\varepsilon$ and $\eta_t(w) = w + (v_t^\alpha - v_{t_0}^\alpha)$ on $|w - v_{t_0}^\alpha| < \varepsilon$. Then $g_t = \eta_t \circ f_{t_0} : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is a quasi-regular map which equals f_{t_0} on $\widehat{\mathbb{C}} \setminus f_{t_0}^{-1}(\triangle_{3\epsilon})$, and is analytic on $\widehat{\mathbb{C}} \setminus f_{t_0}^{-1}(A)$ with $A = \{w : \varepsilon \le |w - v_{t_0}^\alpha| \le 3\varepsilon\}$. To apply Shishikura's qc-lemma [14] we need to know that g_t is uniformly K-quasi-regular, that is, all iterates g_t^p are K-quasi-regular with one and the same K. This is obviously true if the sets $f_{t_0}^{-p}(A)$ ($p = 1, 2, \ldots$) are visited at most once by any iterate of g_t. This is trivial if $k \ge 1$: the sets $f_{t_0}^{-p}(A)$ belong to different Fatou components, namely predecessors of D_{t_0} of order p. If $k = 0$ the argument is different. Let $\triangle_0 : |z| < \delta$ be such that $f_{t_0}(\overline{\triangle_0}) \subset \triangle_0$ and set $\triangle_\nu = f_{t_0}^{-1}(\triangle_{\nu-1})$. Then choosing $\epsilon > 0$ sufficiently small we have $A \subset \triangle_\ell \setminus \overline{\triangle_{\ell-1}}$ for some ℓ and $f_{t_0}^{-p}(A) \subset \triangle_{\ell+p} \setminus \overline{\triangle_{\ell+p-1}}$. By the above mentioned qc-lemma, g_t is quasi-conformally conjugated to some rational function

$$R_t = h_t \circ g_t \circ h_t^{-1}.$$

The quasi-conformal mapping $h_t : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is uniquely determined by the normalisation $h_t(z) = z$ for $z = 0, \alpha, 1$, and depends analytically on the parameter t. Also h_t is analytic on $\widehat{\mathbb{C}} \setminus \overline{\bigcup_{p \ge 0} f_{t_0}^{-p}(A)}$, which, in particular, contains the points 0, v_t^α, and $v_{t_0}^\alpha$. We set $z_0 = h_t(-1)$ to obtain $R_t(z) = a(t) z^m \left(\frac{1-z}{z-z_0}\right)^n$. Since $h_t(\alpha) = \alpha$, R_t has a critical point at $z = \alpha$, and solving $R_t'(\alpha) = 0$ for z_0 yields $z_0 = -1$, thus

$$R_t(z) = a(t) z^m \left(\frac{1-z}{1+z}\right)^n.$$

From $R_t = h_t \circ \eta_t \circ f_{t_0}$ and $h_t(\alpha) = \alpha$, however, it follows that

$$a(t) v_1^\alpha = R_t(\alpha) = h_t \circ \eta_t \circ f_{t_0}(\alpha) = h_t \circ \eta_t \left(v_{t_0}^\alpha\right) = h_t \left(v_t^\alpha\right),$$

hence $R_t(z) = f_\tau(z)$ with $\tau = \tau(t) = h_t(v_t^\alpha)/v_1^\alpha$ and $v_\tau^\alpha = h_t(v_t^\alpha)$; in particular, τ depends analytically on t. On some neighbourhood of $z = 0$ we have

$$\begin{aligned}
(t_0/\tau)^{\frac{1}{m-1}} \Phi_{t_0} \circ h_t^{-1} \circ f_\tau &= (t_0/\tau)^{\frac{1}{m-1}} \Phi_{t_0} \circ g_t \circ h_t^{-1} \\
&= (t_0/\tau)^{\frac{1}{m-1}} \Phi_{t_0} \circ \eta_t \circ f_{t_0} \circ h_t^{-1} \\
&= (t_0/\tau)^{\frac{1}{m-1}} \Phi_{t_0} \circ f_{t_0} \circ h_t^{-1} \\
&= (t_0/\tau)^{\frac{1}{m-1}} t_0 \left(\Phi_{t_0} \circ h_t^{-1}\right)^m \\
&= \tau \left((t_0/\tau)^{\frac{1}{m-1}} \Phi_{t_0} \circ h_t^{-1}\right)^m,
\end{aligned}$$

hence $\phi_\tau = (t_0/\tau)^{\frac{1}{m-1}} \Phi_{t_0} \circ h_t^{-1}$ solves Böttcher's functional equation

$$\phi_\tau \circ f_\tau(z) = \tau(\phi_\tau(z))^m.$$

Since τ and h_t depend analytically on t, this is also true for h_t^{-1}, which is not self-evident. Also from $h_t(g_t(z)) = f_\tau(h_t(z)) \sim \tau h_t(z)^m$ and $g_t(z) = f_{t_0}(z) \sim t_0 z^m$ as $z \to 0$ it follows that $h_t(t_0 z^m) \sim \tau h_t(z)^m$, hence $h_t(z) \sim \sqrt[m-1]{t_0/\tau}\, z$, $h_t^{-1}(z) \sim \sqrt[m-1]{\tau/t_0}\, z$ and $\phi_\tau(z) \sim \lambda z$ as $z \to 0$, with $\lambda^{m-1} = 1$. This implies $\phi_\tau = \lambda \Phi_\tau$ by uniqueness of the Böttcher coordinate, and from $\tau(t_0) = t_0$ and analytic dependence on t it follows that $\lambda = 1$ and $\phi_\tau = \Phi_\tau$, hence

$$\begin{aligned} E_k(\tau) &= \tau^{\frac{1}{m-1}} \Phi_\tau(Q_k(\tau)) = \tau^{\frac{1}{m-1}} \Phi_\tau(f_\tau^k(v_\tau^\alpha)) \\ &= t_0^{\frac{1}{m-1}} \Phi_{t_0} \circ h_t^{-1}(f_\tau^k(v_\tau^\alpha)) = t_0^{\frac{1}{m-1}} \Phi_{t_0} \circ f_{t_0}^k \circ h_t^{-1}(v_\tau^\alpha) \\ &= t_0^{\frac{1}{m-1}} \Phi_{t_0}(f_{t_0}^k(v_t^\alpha)) \quad \text{if } k \geq 1, \quad \text{and} \\ E_0(\tau) &= t_0(\Phi_{t_0}(v_t^\alpha))^m. \end{aligned}$$

Since $t \mapsto \tau$ is locally univalent, E_k is univalent at t_0 if and only if the map $t \mapsto t_0^{\frac{1}{m-1}} \Phi_{t_0}(f_{t_0}^k(v_t^\alpha))$ is univalent on some neighbourhood of t_0. If $k \geq 1$, Φ_{t_0} is univalent on $\mathscr{A}_{t_0}$, and $f_{t_0}^k$ is univalent on $|z - v_{t_0}^\alpha| < \delta$ provided $Q_k(t_0) = f_{t_0}^k(v_{t_0}^\alpha) \neq 0$, while $f_{t_0}^k$ is n-valent at $v_{t_0}^\alpha$ if $Q_k(t_0) = 0$. In case of $k = 0$ we note that Φ_{t_0} is locally univalent on some forward invariant domain D that contains 0 and $v_{t_0}^\alpha$, and $v_t^\alpha = t v_1^\alpha \neq 0$ is trivially univalent. □

The proof of Theorem 2 will be finished by

Lemma 3 *Let h be a proper map of degree m of the domain D onto the unit disc $\mathbb{D}$, and assume that h is ramified exactly over zero, that is, $h'(z) = 0$ implies $h(z) = 0$. Then D is simply connected and h has a single zero on D.*

Proof Assume that h has zeros with multiplicities m_ν $(1 \leq \nu \leq n)$. Then h has degree $d = m_1 + \cdots + m_n$ and $r = d - n$ critical points. The Riemann-Hurwitz formula then yields $\#D - 2 = -d + r = -n$, hence $\#D = 2 - n$, which only is possible if $n = 1$ and $\#D = 1$. □

Remark Each connected component of Ω_k^α contains a zero of $Q_k(t) = t f_1(Q_{k-1}(t))$ which is not a zero of Q_{k-1}, hence is a zero of $Q_{k-1}(t) - 1$. Thus Ω_k^α consists of at most $\frac{d^k - 1}{d-1}$ connected components.

4 The hyperbolic loci

The bifurcation locus $\mathbf{B}^\beta$ is contained in some annulus $\delta < |t| < 1$, and this also holds for $\mathbf{W}^\beta$. Hence (super-)attracting cycles $U_1, \ldots, U_k$ that contain the critical point β may occur only if $\delta < |t| < 1$.

Theorem 3 *For $0 < |t| < 1$, f_t is quasi-conformally conjugated to some polynomial*

$$P_c(z) = cz^m(z+1)^n \quad (c = c_t \neq 0)$$

with free critical point $-\frac{m}{m+n}$. The basin $\mathscr{A}_t$ is completely invariant, and simply connected if and only if $t \notin \Omega_{\text{res}}^\alpha$. For $t \notin \Omega_0^\beta$, the Fatou set consists of $\mathscr{A}_t$, the simply connected basin $\mathscr{B}_t$ and its pre-images and, additionally, of some (super-)attracting cycle of Fatou components and their pre-images if $t \in \mathbf{W}^\beta$; the cycle absorbs the critical point β.

Proof To prove the second assertion we note that by Lemma 1 the pre-image D of the disc $\triangle = \triangle_{r|t|}$ is a simply connected Jordan domain that contains $\overline{\triangle} \cup [0, 1]$, but does not contain v_t^β. Then $D_2 = \widehat{\mathbb{C}} \setminus \overline{\triangle}$ is a backward invariant domain, and

$$f_t : D_1 \xrightarrow{d:1} D_2 \quad \big(D_1 = f_t^{-1}(D_2)\big)$$

is a polynomial-like mapping in the sense of [6], of degree $d = m + n$, hence is hybrid equivalent to some polynomial P of degree d. We may assume that the quasi-conformal conjugation ψ_t with

$$\psi_t \circ f_t = P \circ \psi_t$$

maps ∞, 0, and -1 onto 0, ∞, and -1, respectively. Thus P is given by $P(z) = P_c(z) = cz^m(z+1)^n$, and ψ_t, hence also $c = c_t$ depends analytically on t. $\square$

Remark We note that $D_2 = D_2(|t|) = \{z : |z| > r|t|\}$ increases if $|t|$ decreases, while $D_1 = f_t^{-1}(\widehat{\mathbb{C}} \setminus \overline{\Delta}_{r|t|}) = f_1^{-1}(\widehat{\mathbb{C}} \setminus \overline{\Delta}_r)$ is independent of t. Thus the conformal modulus $\mu(|t|)$ of $D_2(|t|) \setminus \overline{D_1}$ satisfies $\mu(1) \le \mu(|t|) - \log \frac{1}{|t|} \le \log \frac{\inf_{z \in D_1} |z|}{r}$. The bifurcation locus of P_c corresponds conformally to the bifurcation locus $\mathbf{B}^\beta$, and the hyperbolic components are just quasi-conformal images of the hyperbolic components of the quadratic family $z \mapsto z^2 + \xi$.

For $t \in \mathbf{W}_k$, the multiplier map $t \mapsto \lambda_t$ is an algebraic function of t. This is easily seen by writing the equations $f_t^k(z) = z$ and $\lambda = (f_t^k)'(z)$ as polynomial equations $q_1(z, t) = 0$ and $q_2(z, t, \lambda) = 0$, and computing the resultant $R_f(t, \lambda)$ of q_1 and q_2 with respect to z. For example, in case of $k = 1$, $m = 2$, and $n = 1$ we obtain

$$R_f(t, \lambda) = \left[-2 + 14t - 2t^2\right] + \left[1 - 10t + t^2\right]\lambda + 2t\lambda^2 = 0.$$

For $P_c(z) = cz^2(z+1)$ we obtain in the same manner (multiplier μ)

$$R_P(c, \mu) = 9 + 2c - (c+6)\mu + \mu^2 = 0.$$

Since the quasi-conformal conjugation respects multipliers ($\lambda_t = \mu_{c_t}$), c_t is an algebraic function of t by the identity theorem; in the present case we obtain $(1 + 2t + t^2 + 2tc)^2 = 0$ by computing the resultant of $R_f(t, \lambda)$ and $R_P(c, \lambda)$ with respect to λ, hence

$$t \mapsto c = c_t = -\tfrac{1}{2}\left(t + 2 + \frac{1}{t}\right) \quad \left(c = -\tfrac{9}{2} \leftrightarrow t = \tfrac{1}{2}\left(\sqrt{49} - \sqrt{45}\right)\right)$$

maps $0 < |t| < 1$ conformally onto $\mathbb{C} \setminus [-2, 0]$, see Fig. 2.

The following result was not explicitly stated but proved in [17]. The proof is an adaption of the procedure due to Douady [5], applied to the hyperbolic components of the quadratic family $R_t(z) = z^2 + t$ with one free critical point. The occurrence of several critical points requirers a slightly more sophisticated argument. The present version applies to a wider class of functions like $R_t(z) = z^d + t$, $R_t(z) = z^m + t/z^n$, $R_t(z) = t\left(1 + \frac{((d-1)^{d-1}/d^d)z^d}{1-z}\right)$ $(d \ge 3)$, $R_t(z) = -\frac{t}{4}\frac{(z^2-2)^2}{z^2-1}$, the present family, and many others, to show that the hyperbolic components are simply connected and are mapped properly onto the unit disc by the multiplier map $t \mapsto \lambda_t$.

Theorem 4 *Let $(R_t)_{t \in T}$ be any family of rational maps that is analytically parametrised over some domain T. Suppose that each R_t has a (super-)attracting cycle $U_0 \xrightarrow{m_1:1} U_1 \xrightarrow{m_2:1} \cdots \xrightarrow{m_{n-1}:1} U_{n-1} \xrightarrow{m_n:1} U_n = U_0$, such that R_t^n has a single critical point $c_t \in U_0$ of multiplicity $m-1$, where $m = m_1 \cdots m_n$ is the degree of $R_t^n : U_0 \xrightarrow{m:1} U_0$. Assume also that the multiplier λ_t satisfies $|\lambda_t| \to 1$ as $t \to \partial T$. Then the multiplier map $t \mapsto \lambda_t$ provides a proper map $T \xrightarrow{(m-1):1} \mathbb{D}$ which is ramified just over $w = 0$, and T is simply connected.*

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