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Property (k) and commuting Riesz-type perturbations

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Abstract We study the stability of the spectral property (k) introduced and studied by Kaushik and Kashyap (Int J Math Arch 12:167–171, 2014), under commuting perturbations by Riesz operators, and we give generalizations of some known results.

Keywords Property (k) · Riesz perturbation · Direct sum

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1 Introduction and basic definitions

For T in the Banach algebra $\mathcal{B}(X)$ of bounded linear operators acting on a Banach space X, we will denote by $\sigma(T)$ the spectrum of T, by $\sigma_a(T)$ the approximate point spectrum of T, by $\mathcal{N}(T)$ the null space of T, by $n(T)$ the nullity of T, by $\mathcal{R}(T)$ the range of T and by $d(T)$ its defect. If $n(T) < \infty$ and $d(T) < \infty$, then T is called a *Fredholm* operator and its index is defined by $\text{ind}(T) = n(T) - d(T)$. A *Weyl* operator $T \in \mathcal{B}(X)$ is a Fredholm operator of index zero and the *Weyl spectrum* is defined by $\sigma_w(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not a Weyl operator}\}$. $T \in \mathcal{B}(X)$ is called an *upper* (resp., *a lower*) *semi-Fredholm* if $\mathcal{R}(T)$ is closed and $n(T) < \infty$ (resp., $d(T) < \infty$). The respective *semi-Fredholm spectrum* and *semi-Weyl spectrum* of T are defined respectively, by $\sigma_{sf}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not a semi-Fedholm operator}\}$, and $\sigma_{sf_+^-}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not an upper semi-Fredholm operator with index less or equal than zero}\}$.

For a bounded linear operator T and $n \in \mathbb{N}$, let $T_{[n]} : \mathcal{R}(T^n) \to \mathcal{R}(T^n)$ be the restriction of T to $\mathcal{R}(T^n)$. $T \in \mathcal{B}(X)$ is said to be a *semi B-Fredholm* if for some integer $n \geq 0$, the

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range $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is a semi-Fredholm; its index is defined as the index of the semi-Fredholm operator $T_{[n]}$. The respective *semi B-Fredholm spectrum* of T is defined by $\sigma_{sbf}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I$ is not a semi B-Fredholm operator$\}$.

The *ascent* of an operator T is defined by $a(T) = \inf\{n \in \mathbb{N} \mid \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$, and the *descent* of T is defined by $\delta(T) = \inf\{n \in \mathbb{N} \mid \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$, with $\inf \emptyset = \infty$. According to Heuser [8], a complex number $\lambda \in \sigma(T)$ is a *pole* of the resolvent of T if $T - \lambda I$ has finite ascent and finite descent, and in this case they are equal. An operator $T \in \mathcal{B}(X)$ is said to be *Browder* operator if it is a Fredholm with finite ascent and descent, and is said to be an *upper Browder* operator if it is an upper semi-Fredholm operator with finite ascent. The respective *Browder spectrum* and *upper Browder spectrum* of T are defined respectively, by $\sigma_b(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I$ is not a Browder operator$\}$, and $\sigma_{ub}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I$ is not an upper Browder operator$\}$.

In the following, we recall the definition of a property which has a relevant role in local spectral theory. For more details about this property see the monographs of Laursen and Neumann [10] and Aiena [1].

Definition 1.1 An operator $T \in \mathcal{B}(X)$ is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open neighborhood $\mathcal{U}$ of λ_0, the only analytic function $f : \mathcal{U} \longrightarrow X$ which satisfies the equation $(T - \lambda I) f(\lambda) = 0$ for all $\lambda \in \mathcal{U}$ is the function $f \equiv 0$. An operator $T \in \mathcal{B}(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Evidently, $T \in \mathcal{B}(X)$ has SVEP at every isolated point of the spectrum. We summarize in the following list the usual notations and symbols needed later.

Notations and symbols

$\mathcal{F}(X)$	The ideal of finite rank operators in $\mathcal{B}(X)$,
$\mathcal{K}(X)$	The ideal of compact operators in $\mathcal{B}(X)$,
$\mathcal{N}(X)$	The class of nilpotent operators on X,
$\mathcal{Q}(X)$	The class of quasi-nilpotent operators on X,
iso A	Isolated points of a subset $A \subset \mathbb{C}$,
acc A	Accumulations points of a subset $A \subset \mathbb{C}$,
$D(0, 1)$	The closed unit disc in $\mathbb{C}$,
$C(0, 1)$	The unit circle of $\mathbb{C}$,
$\Pi(T)$	poles of T,
$\Pi^0(T)$	Poles of T of finite rank,
$\Pi_a(T)$	Left poles of T,
$\Pi_a^0(T)$	Left poles of T of finite rank,
$\sigma_p(T)$	Eigenvalues of T,
$\sigma_p^f(T)$	Eigenvalues of T of finite multiplicity,
$E^0(T)$	$\text{iso}\, \sigma(T) \cap \sigma_p^f(T)$,
$E(T)$	$\text{iso}\, \sigma(T) \cap \sigma_p(T)$,
$E_a^0(T)$	$\text{iso}\, \sigma_a(T) \cap \sigma_p^f(T)$,
$\sigma_b(T) = \sigma(T) \setminus \Pi^0(T)$	Browder spectrum of T,
$\sigma_{ub}(T) = \sigma_a(T) \setminus \Pi_a^0(T)$	Upper-Browder spectrum of T,
$\sigma_w(T)$	Weyl spectrum of T,
$\sigma_{sf_+^-}(T)$	Semi-Weyl spectrum of T,
$\sigma_{sbf}(T)$	Semi B-Fredholm spectrum of T.

Definition 1.2 [2,6,9,12] Let $T \in \mathcal{B}(X)$. T is said to satisfy

(i) a-Browder's theorem if $\sigma(T)\backslash\sigma_{sf_+^-}(T) = \Pi_a^0(T)$, or equivalently $\sigma_{ub}(T) = \sigma_{sf_+^-}(T)$.
(ii) Browder's theorem if $\sigma(T)\backslash\sigma_w(T) = \Pi^0(T)$, or equivalently $\sigma_b(T) = \sigma_w(T)$.
(iii) Weyl's theorem if $\sigma(T)\backslash\sigma_w(T) = E^0(T)$.
(iv) Property (k) if $\sigma(T)\backslash\sigma_w(T) = E(T)$.

Definition 1.3 Let $T \in \mathcal{B}(X)$ and $S \in \mathcal{B}(X)$. We will say that T and S have *a shared stable sign index* if for each $\lambda \notin \sigma_{sbf}(T)$ and $\mu \notin \sigma_{sbf}(S)$, $\text{ind}(T - \lambda I)$ and $\text{ind}(S - \mu I)$ have the same sign.

For examples we have:

1. It is easily verified that if $T \in \mathcal{B}(X)$ has SVEP then $\text{ind}(T - \mu I) \leq 0$ for every $\mu \notin \sigma_{sbf}(T)$. So if S and T have SVEP, then they have a shared stable sign index.
2. Here and elsewhere, $\mathcal{H}$ denotes a Hilbert space. It is well known that every hyponormal operator T acting on $\mathcal{H}$ has property (H_1) (for the definition of (H_1), see the end of the second section) and hence has SVEP. As a consequence of the first point, every two hyponormal operators have a shared stable sign index. Recall that $T \in \mathcal{B}(\mathcal{H})$ is said to be *hyponormal* if $T^*T - TT^* \geq 0$ (or equivalently $\|T^*x\| \leq \|Tx\|$ for all $x \in \mathcal{H}$). The class of hyponormal operators includes also *subnormal* operators and *quasinormal* operators, see Conway [4].

After giving an introduction and some preliminaries in the first section, we study in the second section the preservation of property (k) introduced and studied by Kaushik and Kashyap [9], under several commuting Riesz-type perturbations. We prove in particular that if T is an isoloid operator acting on a Banach space and satisfies property (k), then $T + S$ satisfies property (k) for every finite rank power operator S which commutes with T. Moreover, we give generalization of some perturbation results to commuting Riesz operators such as Theorems 2.3 and 2.8 of Berkani and Zariouh [3]. In the end of this paper, we prove that if S and T are isoloid bounded operators acting on Banach spaces and satisfy property (k), then $S \oplus T$ satisfies property (k) if and only if $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$, extending a result of Kaushik and Kashyap [9]. Some crucial examples are also given.

2 Property (k) and perturbations

We recall that an operator $R \in \mathcal{B}(X)$ is said to be *Riesz* if $R - \mu I$ is Fredholm for every non-zero complex μ, that is, $\pi(R)$ is quasi-nilpotent in the Calkin algebra $C(X) = \mathcal{B}(X)/\mathcal{K}(X)$ where π is the canonical mapping of $\mathcal{B}(X)$ into $C(X)$. We denote by $\mathcal{R}(X)$ the class of Riesz operators and by $\mathcal{F}^0(X)$, the class of finite rank power operators as follows:

$$\mathcal{F}^0(X) = \left\{S \in \mathcal{B}(X) : S^n \in \mathcal{F}(X) \text{ for some } n \in \mathbb{N}\right\}.$$

Clearly, $\mathcal{F}(X) \cup \mathcal{N}(X) \subset \mathcal{F}^0(X) \subset \mathcal{R}(X)$, and $\mathcal{K}(X) \cup \mathcal{Q}(X) \subset \mathcal{R}(X)$.

According to Oberai [11], Rakočević [14] and Tylli [15], we know that for every $T \in \mathcal{B}(X)$ and $R \in \mathcal{R}(X)$ such that $TR = RT$, $\sigma_*(T + R) = \sigma_*(T)$; where $\sigma_* \in \{\sigma_{sf_+^-}, \sigma_w, \sigma_b, \sigma_{ub}\}$. From this, we give the following known lemma which we need in the proof of the next main results.

Lemma 2.1 *Let $T \in \mathcal{B}(X)$ and let $R \in \mathcal{R}(X)$ be a commuting operator with T. The following statements hold:*

(i) *T satisfies Browder's theorem if and only if $T+R$ satisfies Browder's theorem.*
(ii) *T satisfies a-Browder's theorem if and only if $T+R$ satisfies a-Browder's theorem.*

We start this section by the following nilpotent perturbation result.

Proposition 2.2 *Let $T \in \mathcal{B}(X)$ and let $N \in \mathcal{N}(X)$ which commutes with T. Then T satisfies property (k) if and only if $T+N$ satisfies property (k).*

Proof Since N is nilpotent and commutes with T, we know that $\sigma(T+N)=\sigma(T)$, and it is easily seen that $0<n(T+N) \Longleftrightarrow 0<n(T)$. Since $\sigma_w(T)=\sigma_w(T+N)$, it follows that T satisfies property (k) if and only if $T+N$ satisfies property (k). □

- Note that the assumption of commutativity in the Proposition 2.2 is crucial. Let T and N be defined on $\ell^2(\mathbb{N})$ by $T(x_1,x_2,\ldots)=(0,\frac{x_1}{2},\frac{x_2}{3},\ldots)$ and $N(x_1,x_2,\ldots)=(0,\frac{-x_1}{2},0,0,\ldots)$. Clearly N is nilpotent and does not commute with T. The property (k) is satisfied by T, since $\sigma(T)=\{0\}=\sigma_w(T)$ and $E(T)=\emptyset$. But $T+N$ does not satisfy property (k), because $\sigma(T+N)=\sigma_w(T+N)=\{0\}$ and $\{0\}=E(T+N)$.
- The stability of property (k) showed in Proposition 2.2 cannot be extended to commuting quasi-nilpotent operators, as we can see in the next example:

Example 2.3 Let T be the operator defined on $\ell^2(\mathbb{N})$ by $T(x_1,x_2,\ldots)=(0,\frac{x_1}{2},\frac{x_2}{3},\ldots)$. Put $R=-T$, clearly R is quasi-nilpotent, compact and commutes with T. As it is already mentioned, T satisfies property (k). But $T+R=0$ does not satisfy this property. Indeed, $\sigma(T+R)=\{0\}=\sigma_w(T+R)$, $E(T+R)=\{0\}$. Note also that $\Pi^0(T+R)=\emptyset$, $\Pi^0(T)=\emptyset$.

Theorem 2.4 *Let $R \in \mathcal{R}(X)$ and let $T \in \mathcal{B}(X)$ be a commuting operator with T. If T satisfies property (k), then the following statements are equivalent:*

(i) *$T+R$ satisfies property (k);*
(ii) *$\Pi^0(T+R)=E(T+R)$;*
(iii) *$E(T+R)\cap\sigma(T)\subset\Pi^0(T)$.*

Proof (i) $\Longleftrightarrow$ (ii) If $T+R$ satisfies property (k) then from [9, Theorem 2.5], $\Pi^0(T+R)=E(T+R)$. Conversely, since T satisfies property (k), then it satisfies Browder's theorem, and from Lemma 2.1, $T+R$ satisfies Browder's theorem too. So $\sigma(T+R)\setminus\sigma_w(T+R)=\Pi^0(T+R)$. Thus $T+R$ satisfies property (k).

(ii) $\Longleftrightarrow$ (iii) Suppose that $\Pi^0(T+R)=E(T+R)$ and let $\lambda_0\in E(T+R)\cap\sigma(T)$ be arbitrary. Then $\lambda_0\in\Pi^0(T+R)\cap\sigma(T)$ and hence $\lambda_0\in\sigma(T)\setminus\sigma_b(T)=\Pi^0(T)$. Consequently, $E(T+R)\cap\sigma(T)\subset\Pi^0(T)$. Conversely, suppose that $E(T+R)\cap\sigma(T)\subset\Pi^0(T)$ and let $\mu_0\in E(T+R)$ be arbitrary. We distinguish two cases: the first is $\mu_0\in\sigma(T)$. Then $\mu_0\in E(T+R)\cap\sigma(T)\subset\Pi^0(T)$. It follows that $\mu_0\notin\sigma_b(T+R)$ and since $\mu_0\in\sigma(T+R)$, then $\mu_0\in\Pi^0(T+R)$. The second case is $\mu_0\notin\sigma(T)$. This implies that $\mu_0\notin\sigma_b(T+R)$ and then $\mu_0\in\Pi^0(T+R)$. In the two cases we have $\Pi^0(T+R)\supset E(T+R)$ and as the opposite inclusion is always true, then $\Pi^0(T+R)=E(T+R)$. Remark that the statements (ii) and (iii) are always equivalent without the assumption that T satisfies property (k). □

As an application of Theorem 2.4 to commuting isoloid operators, we give the following corollary. Recall that an operator $T\in\mathcal{B}(X)$ is said to be *isoloid* (resp., *polaroid*) if $\operatorname{iso}\sigma(T)=E(T)$ (resp., $\operatorname{iso}\sigma(T)=\Pi(T)$).

Corollary 2.5 *Let $S\in\mathcal{F}^0(X)$ and let $T\in\mathcal{B}(X)$ be an isoloid operator commuting with S. If T satisfies property (k) then $T+S$ satisfies property (k).*

Proof Let $\lambda_0 \in E(T+S) \cap \sigma(T)$ be arbitrary. Then $\lambda_0 \notin \operatorname{acc}\sigma(T+S) = \operatorname{acc}\sigma(T)$, see [16, Theorem 2.2]. As $\lambda_0 \in \sigma(T)$, then $\lambda_0 \in \operatorname{iso}\sigma(T) = E(T)$. The property (k) for T implies that $E(T) = \Pi^0(T)$ and hence $E(T+S) \cap \sigma(T) \subset \Pi^0(T)$. But this is equivalent from Theorem 2.4, to say that $T+S$ satisfies property (k). □

The next theorem extends [3, Theorem 2.3] to commuting Riesz perturbations which are not necessary nilpotent or compact. According to Rakočěvić [13], we recall that an operator $T \in \mathcal{B}(X)$ is said to satisfy a-Weyl's theorem if $\sigma_a(T) \backslash \sigma_{sf_+^-}(T) = E_a^0(T)$.

Theorem 2.6 *Let $R \in \mathcal{R}(X)$. If $T \in \mathcal{B}(X)$ satisfies a-Weyl's theorem and commutes with R, then the following statements are equivalent:*

(i) *$T+R$ satisfies a-Weyl's theorem;*
(ii) $\Pi_a^0(T+R) = E_a^0(T+R)$;
(iii) $E_a^0(T+R) \cap \sigma_a(T) \subset E_a^0(T)$.

Proof (i) $\Longleftrightarrow$ (ii) Suppose that $T+R$ satisfies a-Weyl's theorem and let $\mu_0 \in E_a^0(T+R)$ be arbitrary. Then $\mu_0 \in E_a^0(T+R) \Longleftrightarrow \mu_0 \in \operatorname{iso}\sigma_a(T+R) \cap \sigma_{sf_+^-}(T+R)^C \Longleftrightarrow \mu_0 \in \Pi_a^0(T+R)$, where $\sigma_{sf_+^-}(T+R)^C$ is the complement of the semi-Weyl spectrum of $T+R$. Thus $\Pi_a^0(T+R) = E_a^0(T+R)$. For the converse, since T satisfies a-Weyl's theorem, then it satisfies a-Browder's theorem and therefore $T+R$ satisfies a-Browder's theorem too, see Lemma 2.1. So $\sigma_a(T+R) \backslash \sigma_{sf_+^-}(T+R) = \Pi_a^0(T+R) = E_a^0(T+R)$.

(ii) $\Longleftrightarrow$ (iii) Suppose that $\Pi_a^0(T+R) = E_a^0(T+R)$ and let $\lambda_0 \in E_a^0(T+R) \cap \sigma_a(T)$ be arbitrary. Then $\lambda_0 \in \Pi_a^0(T+R) \cap \sigma_a(T)$. Hence $\lambda_0 \in \sigma_a(T) \backslash \sigma_{ub}(T) = \Pi_a^0(T)$. Consequently, $E_a^0(T+R) \cap \sigma_a(T) \subset E_a^0(T)$. Conversely, suppose that $E_a^0(T+R) \cap \sigma_a(T) \subset E_a^0(T)$ and let $\mu_0 \in E_a^0(T+R)$ be arbitrary. We distinguish two cases: the first is $\mu_0 \in \sigma_a(T)$. Then $\mu_0 \in E_a^0(T+R) \cap \sigma_a(T) \subset E_a^0(T)$. It follows that $\mu_0 \notin \sigma_{sf_+^-}(T+R)$ and since $\mu_0 \in \operatorname{iso}\sigma_a(T+R)$, then $\mu_0 \in \Pi_a^0(T+R)$. The second case is $\mu_0 \notin \sigma_a(T)$. This implies that $\mu_0 \notin \sigma_{ub}(T+R)$ and so $\mu_0 \in \Pi_a^0(T+R)$. In the two cases we have $\Pi_a^0(T+R) \supset E_a^0(T+R)$ and as the opposite inclusion is always true, then $\Pi_a^0(T+R) = E_a^0(T+R)$. □

The next theorem extends [3, Theorem 2.8] to commuting Riesz perturbations which are not necessary nilpotent or compact.

Theorem 2.7 *Let $R \in \mathcal{R}(X)$. If $T \in \mathcal{B}(X)$ satisfies Weyl's theorem and commutes with R, then the following statements are equivalent:*

(i) *$T+R$ satisfies Weyl's theorem;*
(ii) $\Pi^0(T+R) = E^0(T+R)$;
(iii) $E^0(T+R) \cap \sigma(T) \subset E^0(T)$.

Proof (i) $\Longleftrightarrow$ (ii) Suppose that $T+R$ satisfies Weyl's theorem and let $\mu_0 \in E^0(T+R)$ be arbitrary. Then $\mu_0 \in E^0(T+R) \Longleftrightarrow \mu_0 \in \operatorname{iso}\sigma(T+R) \cap \sigma_w(T+R)^C \Longleftrightarrow \mu_0 \in \Pi^0(T+R)$, where $\sigma_w(T+R)^C$ is the complement of the Weyl spectrum of $T+R$. Thus $\Pi^0(T+R) = E^0(T+R)$. For the converse, since T satisfies Weyl's theorem, then it satisfies Browder's theorem and therefore $T+R$ satisfies Browder's theorem too. So $\sigma(T+R) \backslash \sigma_w(T+R) = \Pi^0(T+R) = E^0(T+R)$.

(ii) $\Longleftrightarrow$ (iii) Suppose that $\Pi^0(T+R) = E^0(T+R)$ and let $\lambda_0 \in E^0(T+R) \cap \sigma(T)$ be arbitrary. Then $\lambda_0 \in \Pi^0(T+R) \cap \sigma(T)$ and hence $\lambda_0 \in \sigma(T) \backslash \sigma_b(T) = \Pi^0(T)$.

Consequently, $E^0(T+R) \cap \sigma(T) \subset E^0(T)$. Conversely, suppose that $E^0(T+R) \cap \sigma(T) \subset E^0(T)$ and let $\mu_0 \in E^0(T+R)$ be arbitrary. We distinguish two cases: the first is $\mu_0 \in \sigma(T)$. Then $\mu_0 \in E^0(T+R) \cap \sigma(T) \subset E^0(T)$. It follows that $\mu_0 \in \sigma_w(T+R)^C \cap \text{iso}\,\sigma(T+R) \iff \mu_0 \in \Pi^0(T+R)$. The second case is $\mu_0 \notin \sigma(T)$. This implies that $\mu_0 \notin \sigma_b(T+R)$. Thus $\mu_0 \in \Pi^0(T+R)$. In the two cases we have $\Pi^0(T+R) \supset E^0(T+R)$ and as the opposite inclusion is always true, then $\Pi^0(T+R) = E^0(T+R)$. □

Corollary 2.8 *Let $S \in \mathcal{F}^0(X)$ and let $T \in \mathcal{B}(X)$ be a bounded operator commuting with S. The following assertions hold:*

(i) *If T satisfies a-Weyl's theorem and $\text{iso}\,\sigma_a(T) = E_a^0(T)$, then $T+S$ satisfies a-Weyl's theorem.*
(ii) *If T satisfies Weyl's theorem and $\text{iso}\,\sigma(T) = E^0(T)$, then $T+S$ satisfies Weyl's theorem.*

Proof (i) Let $\lambda_0 \in E_a^0(T+S) \cap \sigma_a(T)$ be arbitrary. Then $\lambda_0 \notin \text{acc}\,\sigma_a(T+S) = \text{acc}\,\sigma_a(T)$, see [16, Theorem 2.2]. As $\lambda_0 \in \sigma_a(T)$ then $\lambda_0 \in \text{iso}\,\sigma_a(T) = E_a^0(T)$. So $E_a^0(T+S) \cap \sigma_a(T) \subset E_a^0(T)$. But this is equivalent from Theorem 2.6, to say that $T+S$ satisfies a-Weyl's theorem.
(ii) Goes similarly with the proof of the first assertion, as an application of Theorem 2.7. □

In the next, we explore conditions on $S \in \mathcal{B}(X)$ and $T \in \mathcal{B}(Y)$ so that $S \oplus T$ satisfies property (k). The motivation for this work has come from Duggal and Kubrusly [7]. We begin with an example which shows that even if two operators S and T satisfy property (k), yet there direct sum may fail to satisfy property (k).

Example 2.9 Let R and L be the operators defined on $\ell^2(\mathbb{N})$ by $R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$ and $L(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$. Then property (k) holds for R and L, since $\sigma(R) = \sigma_w(R) = D(0,1)$, $E(R) = \emptyset$, $\sigma(L) = \sigma_w(L) = D(0,1)$ and $E(L) = \emptyset$. But it does not hold for $R \oplus L$. In fact $\sigma(R \oplus L) = D(0,1)$, and as $n(R \oplus L) = d(R \oplus L) = 1$ then $0 \notin \sigma_w(R \oplus L)$. So $\sigma_w(R \oplus L) \subsetneq \sigma(R \oplus L)$. We also remark that $E(R \oplus L) = \emptyset$. Thus $\sigma(R \oplus L) \backslash \sigma_w(R \oplus L) \neq E(R \oplus L)$. Note that S and T are isoloid and $\sigma_w(R \oplus L) \subsetneq \sigma_w(R) \cup \sigma_w(L) = D(0,1)$.

Nonetheless, and under the assumption that S and T are isoloid, we give in the following result a characterization of the stability of property (k) under direct sum.

Theorem 2.10 *Let $S \in \mathcal{B}(X)$ and let $T \in \mathcal{B}(Y)$, X and Y are Banach spaces. If S and T satisfy property (k) and are isoloid, then the following assertions are equivalent:*

(i) *$S \oplus T$ satisfies property (k);*
(ii) *$\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$.*

Proof (i) $\Longrightarrow$ (ii) The property (k) for $S \oplus T$ implies with no other restriction on either S or T that $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$. Indeed, as $S \oplus T$ satisfies property (k) then it satisfies Browder's theorem and so $\sigma_w(S \oplus T) = \sigma_b(S \oplus T)$. Since $\sigma_b(S \oplus T) = \sigma_b(S) \cup \sigma_b(T)$, then $\sigma_w(S \oplus T) = \sigma_b(S) \cup \sigma_b(T)$, and as $\sigma_w(S) \cup \sigma_w(T) \subset \sigma_b(S) \cup \sigma_b(T)$, we then have $\sigma_w(S) \cup \sigma_w(T) \subset \sigma_w(S \oplus T)$. Hence $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$.
(ii) $\Longrightarrow$ (i) Suppose that $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$. As S and T are isoloid then

$$E(S \oplus T) = [E(S) \cap \rho(T)] \cup [E(T) \cap \rho(S)] \cup [E(S) \cap E(T)],$$

where $\rho(.) = \mathbb{C}\backslash\sigma(.)$. On the other hand, since S and T satisfy property (k), i.e. $\sigma(S)\backslash\sigma_w(S) = E(S)$ and $\sigma(T)\backslash\sigma_w(T) = E(T)$, we then have

$$\begin{aligned}
&[\sigma(S) \cup \sigma(T)] \setminus [\sigma_w(S) \cup \sigma_w(T)] \\
&= [(\sigma(S)\backslash\sigma_w(S)) \cap \rho(T)] \cup [(\sigma(T)\backslash\sigma_w(T)) \cap \rho(S)] \\
&\quad \cup [(\sigma(S)\backslash\sigma_w(S)) \cap (\sigma(T)\backslash\sigma_w(T))] \\
&= [E(S) \cap \rho(T)] \cup [E(T) \cap \rho(S)] \cup [E(S) \cap E(T)] .
\end{aligned}$$

Hence $E(S \oplus T) = [\sigma(S) \cup \sigma(T)]\backslash[\sigma_w(S) \cup \sigma_w(T)] = \sigma(S \oplus T)\backslash\sigma_w(S \oplus T)$, and this shows that property (k) is satisfied by $S \oplus T$. □

Remark 2.11 1. Theorem 2.10 extends [9, Theorem 3.2] which proves that if $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ are isoloid operators acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ satisfying property (k) with the supplementary condition $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$, then $S \oplus T$ satisfies property (k).

2. The assumption "S and T are isoloid" is essential in Theorem 2.10. For this define on $\mathbb{C}^n \oplus \ell^2(\mathbb{N})$ the operator $U = 0 \oplus S$ where S is defined by $S(x_1, x_2, \ldots) = (0, \frac{x_1}{2}, \frac{x_2}{3}, \ldots)$. It is clear that the null operator satisfies property (k). Also the operator S satisfies property (k), since $\sigma(S) = \sigma_w(S) = \{0\}$ and $E(S) = \emptyset$. But U does not satisfy this property, since $\sigma(U) = \sigma_w(U) = \{0\}$ and $E(U) = \{0\}$. Here $\sigma_w(0 \oplus S) = \sigma_w(0) \cup \sigma_w(S) = \{0\}$, the null operator is isoloid, but S is not isoloid.

Before we state our next corollary as an application of Theorem 2.10 to the class of (H)-operators, we recall the definition of this class and definitions of some classes of operators which are contained in the class (H).

According to the monograph of Aiena [1], the *quasinilpotent* part $H_0(T)$ of $T \in \mathcal{B}(X)$ is defined as the set $H_0(T) = \{x \in X : \lim_{n\to\infty} \|T^n(x)\|^{\frac{1}{n}} = 0\}$. Note that generally, $H_0(T)$ is not closed and from [1, Theorem 2.31] we have if $H_0(T - \lambda I)$ is closed then T has SVEP at λ. We also recall that T is said to belong to the class (H) if for all $\lambda \in \mathbb{C}$ there exists $p := p(\lambda) \in \mathbb{N}$ such that $H_0(T - \lambda I) = \mathcal{N}((T - \lambda I)^p)$, see Aiena [1] for more details about this class of (H)-operators. Of course, every operator T which belongs the class (H) has SVEP, since $H_0(T - \lambda I)$ is closed. Observe also that $a(T - \lambda I) \leq p$, for every $\lambda \in \mathbb{C}$. The class of operators having the property (H) is rather large. Obviously, it contains every operator having the property (H_1). Recall that an operator $T \in \mathcal{B}(X)$ is said to have the property (H_1) if $H_0(T - \lambda I) = \mathcal{N}(T - \lambda I)$ for all $\lambda \in \mathbb{C}$. Although the property (H_1) seems to be rather strong, the class of operators having the property (H_1) is considerably large. In the sequel we give some important classes of operators which satisfy property (H_1). Every *totally paranormal* operator has property (H_1), and in particular every hyponormal operator has property (H_1). Also every *transaloid* operator or *log-hyponormal* has the property (H_1). Some other operators satisfy property (H); for example *M-hyponormal* operators, *p-hyponormal* operators, *algebraically p-hyponormal* operators, *algebraically M-hyponormal* operators, *subscalar* operators and *generalized scalar* operators. For more details about these definitions and comments which we cited above, we refer the reader to Aiena [1], Curto and Han [5] and, Laursen and Neumann [10].

Corollary 2.12 *Let $S \in \mathcal{B}(X)$ and $T \in \mathcal{B}(Y)$ be isoloid operators and have a shared stable sign index. If S and T satisfy property (k), then $S \oplus T$ satisfies property (k). In particular, if S and T are (H)-operators satisfying property (k) then $S \oplus T$ satisfies property (k).*

Proof Assume that S and T are isoloid and satisfy property (k). Since S and T have a shared stable sign index, then it is easily seen that $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$. But this is equivalent by Theorem 2.10, to say that property (k) holds for $S \oplus T$. In particular if S and T are (H)-operators, then they are polaroid and so isoloid. But every (H)-operator has SVEP. Hence $\text{ind}(T - \lambda I)$ and $\text{ind}(S - \mu I)$ are less or equal than zero, for each $\lambda \in \rho_{sbf}(T)$ and $\mu \in \rho_{sbf}(S)$. Hence $S \oplus T$ satisfies property (k). □

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