

Mixed equilibrium problems on Hadamard manifolds

S. Jana¹ · C. Nahak¹

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Abstract In this paper we study the existence of solutions of mixed equilibrium problems on Hadamard manifolds. We also introduce the implicit and explicit algorithms to solve these problems. Under reasonable assumptions, we show that the sequence generated by both implicit and explicit algorithms converges to a solution of mixed equilibrium problems, whenever it exists. Moreover our results generalize some corresponding results, existing in the literature.

Keywords Hadamard manifolds · Equilibrium problems · Variational inequalities · Monotonicity · KKM mappings

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1 Introduction

The theory of equilibrium problems has many important applications in many fields of mathematics such as optimization problems, variational inequality problems, fixed point problems, Nash equilibria problems, complementarity problems etc. It has been studied extensively in finite and infinite dimensional linear spaces, see for example [2–4, 7] and the references therein.

Recently many researchers [5, 8, 16, 17] extended the concepts and techniques of the theory of equilibrium problems from Euclidean spaces to nonlinear spaces like Hadamard manifolds. One motivation is to study equilibrium problems in Hadamard manifolds is that some

✉ C. Nahak
cnahak@maths.iitkgp.ernet.in

S. Jana
shreyasi.iitkgp@gmail.com

¹ Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur 721302, India

equilibrium problems on Euclidean spaces can not be solved by the classical technique but they can be solved on Hadamard manifolds [5].

Therefore, the extension of the concepts and techniques of the theory of equilibrium problems from Euclidean spaces to Riemannian manifolds is natural. Colao et al. [5] proved the existence of solutions of equilibrium problems on Hadamard manifolds. Nemeth [10] introduced geodesic monotone vector fields, Wang et al. [22] studied monotone and accretive vector fields on Riemannian manifolds. Nemeth [11] generalized some basic existence and uniqueness theorems of the classical theory of variational inequalities from Euclidean spaces to Hadamard manifolds. Zhou and Huang [23] introduced the notion of the (KKM) mapping and proved a generalized (KKM) theorem on the Hadamard manifold. Li and Huang [8], studied the generalized vector quasi-equilibrium problems.

Tang et al. [20] introduced the proximal point algorithm for pseudomonotone variational inequalities on Hadamard manifolds. Implicit methods for solving equilibrium problems on Hadamard manifolds were proposed by Noor et al. [17]. Also Noor et al. [16] introduced explicit iterative methods for solving equilibrium problems on Hadamard manifolds.

Motivated by the research work mentioned above, we prove the existence of solutions of mixed equilibrium problems on Hadamard manifolds. We also introduce the implicit and explicit methods for solving mixed equilibrium problems on Hadamard manifolds and observe that the results of [14–17] are particular cases of our work. Our results may stimulate further research in this area.

2 Preliminaries

In this section, we recall some fundamental definitions, basic properties and notations which will be needed in this paper. These materials can be found in any textbook on Riemannian geometry, for example [19,21].

Let M be an n -dimensional connected manifold. We denote by $T_x M$ the n -dimensional tangent space of M at x and by $TM = \cup_{x \in M} T_x M$, the tangent bundle of M . When M is endowed with a Riemannian metric $\langle \cdot, \cdot \rangle$ on the tangent space $T_x M$ with corresponding norm denoted by $\|\cdot\|$, then M is a Riemannian manifold. The length of a piecewise smooth curve $\gamma : [a, b] \rightarrow M$ joining x to y such that $\gamma(a) = x$ and $\gamma(b) = y$, is defined by

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\|_{\gamma(t)} dt.$$

Then for any $x, y \in M$ the Riemannian distance $d(x, y)$ which induces the original topology on M is defined by minimizing this length over the set of all curves joining x to y .

On every Riemannian manifold there exists exactly one covariant derivation called Levi-Civita connection denoted by $\nabla_X Y$ for any vector fields X, Y on M . Let γ be a smooth curve in M . A vector field X is said to be parallel along γ if $\nabla_{\gamma'} X = 0$. If γ' itself is parallel along γ , we say that γ is a geodesic. A geodesic joining x to y in M is said to be a minimal geodesic if its length equals $d(x, y)$.

A Riemannian manifold is complete if for any $x \in M$ all geodesics emanating from x are defined for all $t \in \mathbb{R}$. By the Hopf–Rinow theorem, we know that if M is complete then any pair of points in M can be joined by a minimal geodesic. Moreover, (M, d) is a complete metric space and bounded closed subsets are compact.

Assuming that M is complete the exponential mapping $\exp_x : T_x M \rightarrow M$ is defined by $\exp_x v = \gamma_v(1)$, where γ_v is the geodesic defined by its position x and velocity v at x .

Recall that a Hadamard manifold is a simply connected complete Riemannian manifold with nonpositive sectional curvature.

Let us recall that a geodesic triangle $\Delta(x_1x_2x_3)$ of a Hadamard manifold is the set consisting of three distinct points x_1, x_2, x_3 called the vertices and three minimizing geodesic segments γ_{i+1} joining x_{i+1} to x_{i+2} called the sides, where $i = 1, 2, 3(mod\ 3)$.

Theorem 1 [19] *Let M be a Hadamard manifold, $\Delta(x_1x_2x_3)$ a geodesic triangle and $\gamma_{i+1} : [0, l_{i+1}] \rightarrow M$ geodesic segments joining x_{i+1} to x_{i+2} and set $l_{i+1} = l(\gamma_{i+1})$, $\theta_{i+1} = \angle(\gamma'_{i+1}(0), -\gamma'_i(l_i))$, for $i = 1, 2, 3(mod\ 3)$. Then*

$$\begin{aligned} \theta_1 + \theta_2 + \theta_3 &\leq \pi, \\ l_{i+1}^2 + l_{i+2}^2 - 2l_{i+1}l_{i+2} \cos \theta_{i+2} &\leq l_i^2, \\ d^2(x_{i+1}, x_{i+2}) + d^2(x_{i+2}, x_i) - 2\langle \exp_{x_{i+2}}^{-1} x_{i+1}, \exp_{x_{i+2}}^{-1} x_i \rangle &\leq d^2(x_i, x_{i+1}). \end{aligned} \tag{1}$$

Lemma 1 ([9]) *Let $x_0 \in M$ and $\{x_n\} \in M$ such that $x_n \rightarrow x_0$. Then the following assertions hold.*

(i) *For any $y \in M$*

$$\exp_{x_n}^{-1} y \rightarrow \exp_{x_0}^{-1} y \quad \text{and} \quad \exp_y^{-1} x_n \rightarrow \exp_y^{-1} x_0.$$

(ii) *If $\{v_n\}$ is a sequence such that $v_n \in T_{x_n}M$ and $v_n \rightarrow v_0$, then $v_0 \in T_{x_0}M$.*

(iii) *Given the sequence $\{u_n\}$ and $\{v_n\}$ with $u_n, v_n \in T_{x_n}M$, if $u_n \rightarrow u_0$ and $v_n \rightarrow v_0$ with $u_0, v_0 \in T_{x_0}M$, then $\langle u_n, v_n \rangle \rightarrow \langle u_0, v_0 \rangle$.*

Definition 1 ([18]) *A subset K of M is said to be geodesic convex if and only if for any two points $x, y \in K$, the geodesic joining x to y is contained in K . That is if $\gamma : [0, 1] \rightarrow M$ is a geodesic with $x = \gamma(0)$ and $y = \gamma(1)$, then $\gamma(t) \in K$, for $0 \leq t \leq 1$.*

Definition 2 ([18]) *A real-valued function $f : M \rightarrow \mathbb{R}$ defined on a geodesic convex set K is said to be geodesic convex if and only if for $0 \leq t \leq 1$,*

$$f(\gamma(t)) \leq (1 - t)f(\gamma(0)) + tf(\gamma(1)).$$

Definition 3 ([24]) *Let $K \subset M$ be a nonempty closed geodesic convex set and $G : K \rightarrow 2^K$ be a set-valued mapping. We say that G is a (KKM) mapping if for any $\{x_1, \dots, x_m\} \subset K$, we have*

$$co(\{x_1, \dots, x_m\}) \subset \bigcup_{i=1}^m G(x_i).$$

Lemma 2 ([5]) *Let K be a nonempty closed geodesic convex set and $G : K \rightarrow 2^K$ be a set-valued mapping such that for each $x \in K$, $G(x)$ is closed. Suppose that*

- (i) *there exists $x_0 \in K$ such that $G(x_0)$ is compact.*
- (ii) *$\forall x_1, \dots, x_m \in K, co(\{x_1, \dots, x_m\}) \subset \bigcup_{i=1}^m G(x_i)$.*

Then $\bigcap_{x \in K} G(x) \neq \emptyset$.

Throughout the rest part of the paper we take M to be a finite dimensional Hadamard manifold and K denote a nonempty closed geodesic convex subset of M , unless explicitly stated otherwise.

3 Main results

Let $F : K \times K \rightarrow \mathbb{R}$ be a bifunction satisfying the property $F(x, x) = 0$ for all $x \in K$. Then the equilibrium problem introduced by Colao et al. [5] is to find a point $\bar{x} \in K$, such that

$$F(\bar{x}, y) \geq 0 \quad \text{for all } y \in K. \quad (2)$$

We introduce the mixed equilibrium problems on the Hadamard manifold M . Let $\psi : K \rightarrow \mathbb{R}$ be a mapping and $F : K \times K \rightarrow \mathbb{R}$ be a bifunction satisfying the property $F(x, x) = 0$ for all $x \in K$. Then the problem is to find $\bar{x} \in K$ such that

$$F(\bar{x}, y) + \psi(y) - \psi(\bar{x}) \geq 0, \quad \forall y \in K, \quad (3)$$

is called a mixed equilibrium problem on K . We denote by SOL(MEP), the solution set of the mixed equilibrium problem (3).

Some particular cases of mixed equilibrium problems are as follows.

- (i) Equilibrium problem: If $\psi \equiv 0$, then the mixed equilibrium problem (3) reduces to the equilibrium problem (2).
- (ii) Variational inequality problem: Let $V : K \rightarrow TM$ be a vector field, that is, $V_x \in T_x M$ for each $x \in K$ and \exp^{-1} denote the inverse of the exponential map. Then the problem introduced by Nemeth ([11]), is to find $x \in K$ such that

$$\langle V_x, \exp_x^{-1} y \rangle \geq 0, \quad \forall y \in K, \quad (4)$$

is called a variational inequality problem on K . If we define

$$F(x, y) = \langle V_x, \exp_x^{-1} y \rangle,$$

and $\psi \equiv 0$, then the mixed equilibrium problem (3) and the variational inequality problem (4) are equivalent.

- (iii) Mixed variational inequality problem: let $\psi : K \rightarrow \mathbb{R}$ be a mapping. Then the mixed variational inequality problem ([5]) is to find $x \in K$ such that

$$\langle V_x, \exp_x^{-1} y \rangle + \psi(y) - \psi(x) \geq 0, \quad \forall y \in K. \quad (5)$$

If we take

$$F(x, y) = \langle V_x, \exp_x^{-1} y \rangle,$$

then the mixed equilibrium problem (3) and the mixed variational inequality problem (5) are equivalent.

- (iv) Optimization problem: Let $f : K \rightarrow \mathbb{R}$ be a function and consider the minimization problem

$$(P) \text{ find } x \in K \text{ such that } f(x) = \min_{y \in K} f(y).$$

If we set $F(x, y) = f(y) - f(x)$, for all $x, y \in K$. Then the problems (P) and (2) are equivalent.

3.1 Existence of solutions of mixed equilibrium problems

Colao et al. [5] studied existence of solutions of equilibrium problems under monotonicity assumptions on Hadamard manifolds.

Definition 4 ([5]) We call a bifunction F to be monotone on K if for any $x, y \in K$, we have

$$F(x, y) + F(y, x) \leq 0. \tag{6}$$

In this section we study the existence of solutions of mixed equilibrium problems under pseudomonotonicity assumptions.

Definition 5 A bifunction F is said to be pseudomonotone with respect to the function ψ if

$$F(x, y) + \psi(y) - \psi(x) \geq 0 \Rightarrow F(y, x) + \psi(x) - \psi(y) \leq 0. \tag{7}$$

We show by an example that pseudomonotonicity is a generalization of monotonicity.

Example 1 Let $H^1 = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 - x_2^2 = -1, x_2 > 0\}$ be the hyperbolic 1-space which forms a Hadamard manifold ([1]) endowed with the metric defined by

$$\langle x, y \rangle = x_1y_1 - x_2y_2, \quad \forall x = (x_1, x_2), \quad y = (y_1, y_2) \in \mathbb{R}^2.$$

Let K be a subset of H^1 defined by $K = \{x = (x_1, x_2) \in H^1 : -1 \leq x_1 \leq 1\}$.

Now we define the bifunction $F : K \times K \rightarrow \mathbb{R}$ by

$$F(x, y) = x_2(x_1 - y_1). \tag{8}$$

To show that F is pseudomonotone on K but not monotone.

We take $\psi \equiv 0$.

$F(x, y) \geq 0$ on K when $x_1 \geq y_1$ (as $x_2 > 0$),

then $F(y, x) = y_2(y_1 - x_1) \leq 0$ (as $y_2 > 0$).

Therefore, F is pseudomonotone.

Particularly if we take $x = (1, \sqrt{2}) \in K$ and $y = (0, 1) \in K$,

then $F(x, y) + F(y, x) = \sqrt{2} - 1 > 0$.

That is, F is not monotone.

Definition 6 A function $F : K \rightarrow \mathbb{R}$ is said to be hemicontinuous if for every geodesic $\gamma : [0, 1] \rightarrow K$, whenever $t \rightarrow 0$, $F(\gamma(t)) \rightarrow F(\gamma(0))$.

Next we give the following lemma which will be needed in the sequel. Throughout the rest of the paper we denote F as a bifunction with $F(x, x) = 0$, unless otherwise stated.

Lemma 3 Let $F : K \times K \rightarrow \mathbb{R}$ be hemicontinuous in the first argument and for fixed $x \in K$ the mapping $z \mapsto F(x, z)$ be geodesic convex. Also assume that the map $\psi : K \rightarrow \mathbb{R}$ is geodesic convex and the bifunction F is pseudomonotone with respect to ψ . Then $\bar{x} \in K$ is a solution of the mixed equilibrium problem (3), if and only if

$$F(y, \bar{x}) + \psi(\bar{x}) - \psi(y) \leq 0 \quad \text{for all } y \in K. \tag{9}$$

Proof Let $\bar{x} \in K$ is a solution of the equilibrium problem (3), then

$$F(\bar{x}, y) + \psi(y) - \psi(\bar{x}) \geq 0, \quad \text{for all } y \in K. \tag{10}$$

Since F is pseudomonotone with respect to the function ψ , we have

$$F(y, \bar{x}) + \psi(\bar{x}) - \psi(y) \leq 0.$$

Conversely, let $\bar{x} \in K$ be a solution of (9). Let $\gamma(t)$ be a geodesic joining \bar{x} and y such that $\gamma(0) = \bar{x}$.

As K is geodesic convex, we have

$$F(\gamma(t), \bar{x}) + \psi(\bar{x}) - \psi(\gamma(t)) \leq 0 \quad \text{for } 0 \leq t \leq 1. \quad (11)$$

As ψ is geodesic convex then

$$\psi(\gamma(t)) \leq t\psi(y) + (1-t)\psi(\bar{x}) \Rightarrow \psi(\gamma(t)) - \psi(\bar{x}) \leq t[\psi(y) - \psi(\bar{x})].$$

Also as $z \mapsto F(x, z)$ is geodesic convex,

$$\begin{aligned} 0 &= F(\gamma(t), \gamma(t)) \leq tF(\gamma(t), y) + (1-t)F(\gamma(t), \bar{x}), \\ &\Rightarrow \psi(\gamma(t)) - \psi(\bar{x}) \leq tF(\gamma(t), y) + (1-t)F(\gamma(t), \bar{x}) + \psi(\gamma(t)) - \psi(\bar{x}) \\ &\leq tF(\gamma(t), y) + (1-t)F(\gamma(t), \bar{x}) + t[\psi(y) - \psi(\bar{x})] \\ &\Rightarrow t[F(\gamma(t), y) - F(\gamma(t), \bar{x}) + \psi(y) - \psi(\bar{x})] \geq -[F(\gamma(t), \bar{x}) + \psi(\bar{x}) - \psi(\gamma(t))] \geq 0 \end{aligned}$$

[by (11)].

That is, $F(\gamma(t), y) - F(\gamma(t), \bar{x}) + \psi(y) - \psi(\bar{x}) \geq 0$, as $t \geq 0$.

Since F is hemicontinuous in the first argument taking $t \rightarrow 0$, we have $F(\bar{x}, y) - F(\bar{x}, \bar{x}) + \psi(y) - \psi(\bar{x}) \geq 0 \Rightarrow F(\bar{x}, y) + \psi(y) - \psi(\bar{x}) \geq 0$, for all $y \in K$. This completes the proof. \square

Next we prove the main existence theorem. First we consider the case when the set K is bounded.

Theorem 2 *Let K be a bounded subset of M and $F : K \times K \rightarrow \mathbb{R}$ be hemicontinuous in the first argument. Suppose for fixed $x \in K$, the mappings $z \mapsto F(x, z)$ and $\psi : K \rightarrow \mathbb{R}$ are geodesic convex, lower semicontinuous. Also assume that the bifunction F is pseudomonotone with respect to ψ . Then the mixed equilibrium problem (3) has a solution.*

Proof Consider the two set-valued mappings $G_1 : K \rightarrow 2^K$ and $G_2 : K \rightarrow 2^K$ such that

$$\begin{aligned} G_1(y) &= \{x \in K : F(x, y) + \psi(y) - \psi(x) \geq 0\}, \quad \text{for all } y \in K, \\ G_2(y) &= \{x \in K : F(y, x) + \psi(x) - \psi(y) \leq 0\}, \quad \text{for all } y \in K. \end{aligned}$$

It is easy to see that $\bar{x} \in K$ solves the mixed equilibrium problem (3) if and only if $\bar{x} \in \bigcap_{y \in K} G_1(y)$. Thus it suffices to prove that $\bigcap_{y \in K} G_1(y) \neq \emptyset$. First we show G_1 is a (KKM) mapping. So we have to prove that for any choice of $x_1, \dots, x_m \in K$

$$co(\{x_1, \dots, x_m\}) \subset \bigcup_{i=1}^m G_1(x_i). \quad (12)$$

Suppose on the contrary that there exists a point x_0 in K , such that $x_0 \in co(\{x_1, \dots, x_m\})$ but $x_0 \notin \bigcup_{i=1}^m G_1(x_i)$. That is

$$F(x_0, x_i) + \psi(x_i) - \psi(x_0) < 0, \quad \forall i \in \{1, \dots, m\}. \quad (13)$$

This implies that for any $i \in \{1, \dots, m\}$, $x_i \in \{y \in K : F(x_0, y) + \psi(y) - \psi(x_0) < 0\}$. Now the function $y \mapsto F(x_0, y)$ is geodesic convex, also ψ is geodesic convex. Being the sum of two geodesic convex function, $y \mapsto F(x_0, y) + \psi(y)$ is geodesic convex. Hence the set $\{y \in K : F(x_0, y) + \psi(y) - \psi(x_0) < 0\}$ is a geodesic convex set. Then

$$x_0 \in co(\{x_1, \dots, x_m\}) \subseteq \{y \in K : F(x_0, y) + \psi(y) - \psi(x_0) < 0\}.$$

Therefore $F(x_0, x_0) + \psi(x_0) - \psi(x_0) < 0$. But we have $F(x_0, x_0) = 0$, a contradiction. Hence G_1 is a (KKM) mapping.

From Lemma 3, we have $G_1(y) \subset G_2(y), \forall y \in K$. That is,

$$co(\{x_1, \dots, x_m\}) \subset \bigcup_{i=1}^m G_2(x_i).$$

Hence G_2 is also a (KKM) mapping.

Since $F(y, \cdot)$ and ψ are lower semicontinuous, $G_2(y)$ is closed for all $y \in K$.

Now $G_2(y)$ is a closed subset of a compact set K . So $G_2(y)$ is compact for all $y \in K$.

Hence by Lemma 2, there exists a point $\bar{x} \in K$ such that $\bar{x} \in \bigcap_{y \in K} G_2(y)$.

By Lemma 3, we have $\bigcap_{y \in K} G_1(y) = \bigcap_{y \in K} G_2(y)$. That is $\bar{x} \in \bigcap_{y \in K} G_1(y)$.

So there exists a point $\bar{x} \in K$, such that

$$F(\bar{x}, y) + \psi(y) - \psi(\bar{x}) \geq 0, \quad \forall y \in K.$$

Therefore, $\bar{x} \in K$ solves the equilibrium problem (3). □

Suppose K is an unbounded subset of M . Given a point $\mathbf{0} \in M$, let $\Sigma_R = \{x \in M : d(\mathbf{0}, x) \leq R\}$ be the closed geodesic ball of radius R and center $\mathbf{0}$.

Theorem 3 *Let K be an unbounded subset of M and $F : K \times K \rightarrow \mathbb{R}$ be hemicontinuous in the first argument. Suppose for fixed $x \in K$, the mappings $z \mapsto F(x, z)$ and $\psi : K \rightarrow \mathbb{R}$ are geodesic convex, lower semicontinuous. Also assume that the bifunction F is pseudomonotone with respect to ψ . If there exists a point $x_0 \in K$, such that*

$$F(x, x_0) + \psi(x_0) - \psi(x) < 0, \quad \text{whenever } d(\mathbf{0}, x) \rightarrow +\infty, \quad x \in K, \tag{14}$$

holds, then the mixed equilibrium problem (3) has a solution.

Proof Let $K_R = K \cap \Sigma_R$. If $K_R \neq \emptyset$, then there exists at least one $x_R \in K_R$ such that

$$F(x_R, y) + \psi(y) - \psi(x_R) \geq 0, \quad \forall y \in K_R, \tag{15}$$

by Theorem 2.

We now take a point $x_0 \in K$ satisfying (14) with $d(\mathbf{0}, x_0) < R$, so $x_0 \in K_R$.

Hence by (15), we have

$$F(x_R, x_0) + \psi(x_0) - \psi(x_R) \geq 0. \tag{16}$$

If $d(\mathbf{0}, x_R) = R$ for all R , we may choose R large enough so that $d(\mathbf{0}, x_R) \rightarrow +\infty$.

Hence by (14), $F(x_R, x_0) + \psi(x_0) - \psi(x_R) < 0$ contradicts (16). So there exists an R such that $d(\mathbf{0}, x_R) < R$.

Given $y \in K$, let $\gamma(t)$ be a geodesic joining x_R to y with $\gamma(0) = x_R$. Now since $d(\mathbf{0}, x_R) < R$, we can choose $0 < t < 1$, sufficiently small so that $\gamma(t) \in K_R$.

Hence

$$\begin{aligned} 0 &\leq F(x_R, \gamma(t)) + \psi(\gamma(t)) - \psi(x_R) \\ &\leq tF(x_R, y) + (1-t)F(x_R, x_R) + t[\psi(y) - \psi(x_R)] \\ &= t[F(x_R, y) + \psi(y) - \psi(x_R)], \\ &\text{or, } F(x_R, y) + \psi(y) - \psi(x_R) \geq 0, \quad \text{for } y \in K. \end{aligned}$$

That is x_R is a solution of the mixed equilibrium problem (3). □

3.2 Implicit method for solving mixed equilibrium problem

In this section we introduce the implicit iterative method for solving mixed equilibrium problems on Hadamard manifolds. We now consider the following implicit iterative (proximal point) algorithm [17].

Algorithm 1 *At stage n , given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, as a solution of the following iterative scheme.*

$$F(x_{n+1}, y) + \frac{1}{\rho} \langle \exp_{x_n}^{-1} x_{n+1}, \exp_{x_{n+1}}^{-1} y \rangle + \psi(y) - \psi(x_{n+1}) \geq 0, \quad \forall y \in K. \quad (17)$$

Next we deduce some special cases of Algorithm 1.

(i) When $\psi \equiv 0$, the Algorithm 1 reduces to the following implicit iterative algorithm.

Algorithm 2 *At stage n , given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, such that*

$$F(x_{n+1}, y) + \frac{1}{\rho} \langle \exp_{x_n}^{-1} x_{n+1}, \exp_{x_{n+1}}^{-1} y \rangle \geq 0, \quad \forall y \in K.$$

Algorithm 2 is the implicit algorithm solving for the equilibrium problems introduced by Noor et al. [17].

(ii) If K is a convex set in \mathbb{R}^n , then Algorithm 1 reduces into the following algorithm ([12, 13]).

Algorithm 3 *At stage n , given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, as a solution of the iterative scheme*

$$F(x_{n+1}, y) + \frac{1}{\rho} \langle x_{n+1} - x_n, y - x_{n+1} \rangle + \psi(y) - \psi(x_{n+1}) \geq 0, \quad \forall y \in K.$$

(iii) If we take $F(x, y) = \langle V_x, \exp_x^{-1} y \rangle$, then Algorithm 1 reduces to the following.

Algorithm 4 *At stage n , given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, as a solution of the iterative scheme*

$$\langle \rho V_{x_{n+1}} + \exp_{x_n}^{-1} x_{n+1}, \exp_{x_{n+1}}^{-1} y \rangle + \psi(y) - \psi(x_{n+1}) \geq 0, \quad \forall y \in K,$$

which is an algorithm for solving mixed variational inequalities and is studied by Noor et al. [15].

(iv) When $\psi \equiv 0$, the Algorithm 4 reduces to the following implicit iterative algorithm ([20]) for solving variational inequalities.

Algorithm 5 *At stage n , given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, as a solution of the iterative scheme*

$$\langle \rho V_{x_{n+1}} + \exp_{x_n}^{-1} x_{n+1}, \exp_{x_{n+1}}^{-1} y \rangle \geq 0, \quad \forall y \in K.$$

We now consider the convergence analysis of Algorithm 1. For which we recall the notion of Fejer convergence and the following related results which can be found in [6] and [9].

Definition 7 Let X be a complete metric space and $A \subseteq X$ be a nonempty set. A sequence $\{x_n\} \subset X$ is said to be Fejer convergent to A if

$$d(x_{n+1}, y) \leq d(x_n, y), \quad \forall y \in A \text{ and } n = 0, 1, 2, \dots \quad (18)$$

Lemma 4 *Let X be a complete metric space and let A be a nonempty subset of X . Suppose $\{x_n\} \subset X$ be Fejer convergent to K and any limit point of $\{x_n\}$ belongs to A . Then $\{x_n\}$ converges to a point of A .*

Theorem 4 *Let $F : K \times K \rightarrow \mathbb{R}$ be pseudomonotone with respect to the function ψ and continuous in the first argument and $SOL(MEP) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ generated by (17) is well defined and $\psi : K \rightarrow \mathbb{R}$ is continuous. Then $\{x_n\}$ converges to a solution of the mixed equilibrium problem (3).*

Proof We first prove that $\{x_n\}$ is Fejer convergent to $SOL(MEP)$. Let $x \in K$ be a solution of (3). Then

$$F(x, y) + \psi(y) - \psi(x) \geq 0, \quad \forall y \in K. \tag{19}$$

Taking $y = x_{n+1}$ in (19), we get

$$F(x, x_{n+1}) + \psi(x_{n+1}) - \psi(x) \geq 0. \tag{20}$$

Since F is pseudomonotone with respect to ψ , then

$$F(x_{n+1}, x) + \psi(x) - \psi(x_{n+1}) \leq 0. \tag{21}$$

From (17), taking $y = x$ we have

$$\begin{aligned} F(x_{n+1}, x) + \frac{1}{\rho} \langle \exp_{x_n}^{-1} x_{n+1}, \exp_{x_{n+1}}^{-1} x \rangle + \psi(x) - \psi(x_{n+1}) &\geq 0 \quad \forall y \in K. \\ \Rightarrow \frac{1}{\rho} \langle \exp_{x_n}^{-1} x_{n+1}, \exp_{x_{n+1}}^{-1} x \rangle &\geq -(F(x_{n+1}, x) + \psi(x) - \psi(x_{n+1})) \\ &\geq 0 \text{ (by equation (21)).} \end{aligned} \tag{22}$$

So we finally get as $\rho > 0$,

$$\langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} x \rangle \leq 0. \tag{23}$$

Considering the geodesic triangle $\Delta(x_n, x_{n+1}, x)$, by using (1) we get

$$d^2(x_{n+1}, x) + d^2(x_{n+1}, x_n) - 2 \langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} x \rangle \leq d^2(x_n, x),$$

or

$$d^2(x_{n+1}, x) + d^2(x_{n+1}, x_n) \leq d^2(x_n, x) \quad \text{(by (23))}. \tag{24}$$

This clearly implies that $d^2(x_{n+1}, x) \leq d^2(x_n, x)$, so $\{x_n\}$ is Fejer convergent to $SOL(MEP)$. From (24) it follows that

$$d^2(x_{n+1}, x_n) \leq d^2(x_n, x) - d^2(x_{n+1}, x). \tag{25}$$

Since the sequence $\{d(x_n, x)\}$ is monotone decreasing and bounded below by 0, it is also convergent. Hence by (25) $\lim_{n \rightarrow \infty} d^2(x_{n+1}, x_n) = 0$. That is

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = 0. \tag{26}$$

Next we prove that any limit point of $\{x_n\}$ belongs to $SOL(MEP)$. Let x be a limit point of $\{x_n\}$. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \rightarrow x$. Hence $d(x_{n_k+1}, x_{n_k}) \rightarrow 0$, by the assertion just proved, and so $x_{n_k+1} \rightarrow x$. It follows from (17) with $n = n_k$,

$$F(x_{n_k+1}, y) + \frac{1}{\rho} \langle \exp_{x_{n_k}}^{-1} x_{n_k+1}, \exp_{x_{n_k+1}}^{-1} y \rangle + \psi(y) - \psi(x_{n_k+1}) \geq 0 \quad \forall y \in K. \tag{27}$$

Passing to the limit as $k \rightarrow \infty$ in (27) we get

$$F(x, y) + \psi(y) - \psi(x) \geq 0, \quad \forall y \in K. \quad (28)$$

That is $x \in \text{SOL}(MEP)$. Hence by Lemma 4, $\{x_n\}$ converges to point of $\text{SOL}(MEP)$. This completes the proof. \square

3.3 Explicit method for solving mixed equilibrium problem

In this section we prove the convergence of explicit iterative methods ([14]) for solving mixed equilibrium problems on Hadamard manifolds.

Definition 8 The bifunction F is said to be partially relaxed pseudomonotone with respect to the function ψ if there exists $\alpha > 0$ such that $\forall x, y, z \in K$

$$F(x, y) + \psi(y) - \psi(x) \geq 0 \Rightarrow F(z, x) + \psi(x) - \psi(z) \leq \alpha d^2(y, z). \quad (29)$$

If we take $z = y$, then F reduces to a pseudomonotone function.

We now consider the following explicit iterative scheme for solving mixed equilibrium problems.

Algorithm 6 At stage n , given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, as a solution of the iterative scheme

$$F(x_n, y) + \frac{1}{\rho} \langle \exp_{x_n}^{-1} x_{n+1}, \exp_{x_{n+1}}^{-1} y \rangle + \psi(y) - \psi(x_n) \geq 0 \quad \forall y \in K. \quad (30)$$

Some particular cases of Algorithm 6 are given as:

- (i) When $\psi \equiv 0$, the Algorithm 6 reduces to the following explicit iterative algorithm [16] for equilibrium problems.

Algorithm 7 At stage n , given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, such that

$$F(x_n, y) + \frac{1}{\rho} \langle \exp_{x_n}^{-1} x_{n+1}, \exp_{x_{n+1}}^{-1} y \rangle \geq 0, \quad \forall y \in K.$$

- (ii) If K is a convex set in \mathbb{R}^n , then Algorithm 6 reduces to the following ([12, 13]).

Algorithm 8 At stage n , given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, as a solution of the iterative scheme

$$F(x_n, y) + \frac{1}{\rho} \langle x_{n+1} - x_n, y - x_{n+1} \rangle + \psi(y) - \psi(x_{n+1}) \geq 0, \quad \forall y \in K.$$

- (iii) If we take $F(x, y) = \langle V_x, \exp_x^{-1} y \rangle$, then Algorithm 6 turns into the following algorithm for solving mixed variational inequalities.

Algorithm 9 At stage n , given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, as a solution of the iterative scheme

$$\langle \rho V_{x_n} + \exp_{x_n}^{-1} x_{n+1}, \exp_{x_{n+1}}^{-1} y \rangle + \psi(y) - \psi(x_{n+1}) \geq 0, \quad \forall y \in K.$$

- (iv) When $\psi \equiv 0$, the Algorithm 9 reduces to the following explicit iterative algorithm [14].

Algorithm 10 At stage n , given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, as a solution of the iterative scheme

$$\langle \rho V_{x_n} + \exp_{x_n}^{-1} x_{n+1}, \exp_{x_{n+1}}^{-1} y \rangle \geq 0, \quad \forall y \in K.$$

We now study the convergence analysis of Algorithm 6.

Theorem 5 Let $F : K \times K \rightarrow \mathbb{R}$ be a partially relaxed pseudomonotone bifunction with respect to the function ψ with a constant $\alpha > 0$, and continuous in the first argument. Suppose that the sequence $\{x_n\}$ generated by (30) is well defined, $\psi : K \rightarrow \mathbb{R}$ is continuous and $SOL(MEP) \neq \emptyset$. Then

$$d^2(x_{n+1}, x) \leq d^2(x_n, x) - (1 - 2\rho\alpha)d^2(x_{n+1}, x_n).$$

If in addition $\rho < \frac{1}{2\alpha}$, then $\{x_n\}$ converges to a solution of the mixed equilibrium problem (3).

Proof We first prove that $\{x_n\}$ is Fejer convergent to $SOL(MEP)$. Let $x \in K$ be a solution of (3). Then

$$F(x, y) + \psi(y) - \psi(x) \geq 0, \quad \forall y \in K. \tag{31}$$

Taking $y = x_{n+1}$ in (31), we get

$$F(x, x_{n+1}) + \psi(x_{n+1}) - \psi(x) \geq 0. \tag{32}$$

Since F is partially relaxed pseudomonotone with a constant $\alpha > 0$, then

$$F(x_n, x) + \psi(x) - \psi(x_n) \leq \alpha d^2(x_{n+1}, x_n). \tag{33}$$

From (30), taking $y = x$ we have

$$\begin{aligned} F(x_n, x) + \frac{1}{\rho} \langle \exp_{x_n}^{-1} x_{n+1}, \exp_{x_{n+1}}^{-1} x \rangle + \psi(x) - \psi(x_n) &\geq 0 \quad \forall y \in K. \\ \Rightarrow \frac{1}{\rho} \langle \exp_{x_n}^{-1} x_{n+1}, \exp_{x_{n+1}}^{-1} x \rangle &\geq -(F(x_n, x) + \psi(x) - \psi(x_n)) \\ &\geq -\alpha d^2(x_{n+1}, x_n) \text{ (by (33))}. \end{aligned} \tag{34}$$

So we finally get

$$\langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} x \rangle \leq \rho \alpha d^2(x_{n+1}, x_n). \tag{35}$$

Considering the geodesic triangle $\Delta(x_n, x_{n+1}, x)$ we get

$$\begin{aligned} d^2(x_{n+1}, x) + d^2(x_{n+1}, x_n) - 2 \langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} x \rangle &\leq d^2(x_n, x) \\ \Rightarrow d^2(x_{n+1}, x) + d^2(x_{n+1}, x_n) &\leq d^2(x_n, x) + 2 \langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} x \rangle. \end{aligned}$$

It follows from (35)

$$\begin{aligned} d^2(x_{n+1}, x) + d^2(x_{n+1}, x_n) &\leq d^2(x_n, x) + 2\rho\alpha d^2(x_{n+1}, x_n) \\ \Rightarrow d^2(x_{n+1}, x) &\leq d^2(x_n, x) - (1 - 2\rho\alpha)d^2(x_{n+1}, x_n). \end{aligned} \tag{36}$$

As $\rho < \frac{1}{2\alpha}$, from (36) this clearly follows

$$d^2(x_{n+1}, x) \leq d^2(x_n, x).$$

So $\{x_n\}$ is Fejer convergent to $\text{SOL}(\text{MEP})$. From (36) we get

$$\Rightarrow (1 - 2\rho\alpha)d^2(x_{n+1}, x_n) \leq d^2(x_n, x) - d^2(x_{n+1}, x). \quad (37)$$

Since the sequence $\{d(x_n, x)\}$ is monotone decreasing and bounded below by 0, it is also convergent. Hence by (36) it follows that $\lim_{n \rightarrow \infty} d^2(x_{n+1}, x_n) = 0$. That is

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = 0. \quad (38)$$

Next we prove that any limit point of $\{x_n\}$ belongs to $\text{SOL}(\text{MEP})$. Let x be a limit point of $\{x_n\}$. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \rightarrow x$. Hence $d(x_{n_k+1}, x_{n_k}) \rightarrow 0$, by the assertion just proved, and so $x_{n_k+1} \rightarrow x$. It follows from (25) with $n = n_k$,

$$F(x_{n_k}, y) + \frac{1}{\rho} \langle \exp_{x_{n_k}}^{-1} x_{n_k+1}, \exp_{x_{n_k+1}}^{-1} y \rangle + \psi(y) - \psi(x_{n_k}) \geq 0 \quad \forall y \in K. \quad (39)$$

Passing to the limit as $k \rightarrow \infty$ in (39), we get

$$F(x, y) + \psi(y) - \psi(x) \geq 0, \quad \forall y \in K. \quad (40)$$

That is $x \in \text{SOL}(\text{MEP})$. Hence by Lemma 4, $\{x_n\}$ converges to point of $\text{SOL}(\text{MEP})$. This completes the proof. \square

4 Conclusions

This paper is devoted to the study of existence of solutions of mixed equilibrium problems on Hadamard manifolds. We also prove the convergence of the implicit and explicit iterative methods for solving mixed equilibrium problems under generalized monotonicity assumptions. The results presented in this paper are completely new and some existing results followed as a special case of our results.

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