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Polynomial decay rate estimate for bilinear parabolic systems under weak observability condition

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Abstract In this paper, we shall study the stability for distributed bilinear systems on a Hilbert state space that can be decomposed in two subspaces: unstable finite-dimensional and stable infinite-dimensional with respect to the evolution generator. Then, we shall show under a weaker observability assumption that stabilizing such a system with a feedback control of the form $p_r(t) = -\|y(t)\|^{-r}\langle y(t), By(t)\rangle$ for $r < 2$, reduces stabilizing only its projection on the finite-dimension subspace which make the whole system stable. To this end, we shall propose a new family of continuous feedback controls that guarantee the uniform stabilizability with an explicit optimal decay rate estimate of the stabilized state. Two illustrating examples and simulations are provided.

Keywords Bilinear parabolic systems · Stabilization · Polynomial decay estimate

Mathematics Subject Classification 93D15

1 Introduction

Bilinear systems have been considered since the early 1960s as a gateway between linear and nonlinear systems that are defined to be linear in both state and control when considered independently, with the nonlinearity (or bilinearity) arising from coupled terms involving products of system state and control (see [8,19]). By formulating the model appropriately, the bilinear term could also be represented by products of system output and control input, i.e. the output is defined as a system state. Therefore many researchers have focused their studies on this class of systems and their applications ever since. By the beginning of 1970s,

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indeed, several comprehensive review articles and monographs (see [3,8]) had already been published. The thrust for the study of bilinear systems comes from the intrinsic limits of linear models when dealing with practical applications in the field of ecology (e.g. populations models), biology as well as in nuclear fission (see [8]), transmission and power systems (see [6,9]). Furthermore, there are several characteristics that render bilinear systems appealing also from the theoretical point of view. In this paper, we are concerned with the question of feedback stabilization of bilinear systems that can be described in the following form

$$\frac{dy(t)}{dt} = Ay(t) + p(t)By(t), \quad y(0) = y_0, \tag{1}$$

on a real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and corresponding norm $\|\cdot\|$, where A generates a semigroup of contractions $S(t)$ on H and $B \in \mathcal{L}(H)$. While the real valued function $p(\cdot) \in L^2(0,+\infty;\mathbb{R})$ is a control. The obvious choice of the stabilizing feedback control is

$$p_0(t) = -\langle y(t), By(t)\rangle, \tag{2}$$

(see [2]). In the case where B is sequentially continuous from H_w (H endowed with the weak topology) to H, the quadratic feedback $p_0(t)$ weakly stabilizes the system (1), provided that the following approximate observability assumption

$$\langle BS(t)y, S(t)y\rangle = 0, \quad \forall t \geq 0 \Longrightarrow y = 0, \tag{3}$$

holds (see [2]). Under the exact observability assumption

$$\int_0^T |\langle BS(t)y, S(t)y\rangle| dt \geq \delta\|y\|^2, \quad \forall y \in H, \ (T, \delta > 0), \tag{4}$$

a strong stabilization result has been obtained using the control $p_0(t)$ (see [11]). However, in this way the convergence of the resulting closed loop state is not better than $\|y(t)\| = \mathcal{O}(\frac{1}{\sqrt{t}})$.

The control $p_2(t) = -\frac{\langle y(t), By(t)\rangle}{\|y(t)\|^2}$ has been considered for infinite-dimensional bilinear system in [13], where exponential stabilization results have been established provided that the observation assumption (4) holds. In [15] the rational decay rates are established i.e.,

$$\|y(t)\| = \mathcal{O}(t^{\frac{-1}{2-r}}), \quad r \in (-\infty, 2), \tag{5}$$

using the following continuous feedback control

$$p_r(t) = \begin{cases} -\frac{<y(t), By(t)>}{\|y(t)\|^r}, & y(t) \neq 0 \\ 0, & y(t) = 0. \end{cases} \tag{6}$$

In the sequel of this section, we shall present an appropriate decomposition of the state space H and the system (1) via the spectral properties of the operator A, and we shall apply this approach to study the stabilization problem of the system (1). This problem has been considered in the finite-dimensional case [i.e., $H = \mathbb{R}^n$ and $A, B \in \mathbb{R}^{n^2}$ (see [4])], who showed that the condition

$$\text{span}\{Az, ad^0(A,B)z, ad^1(A,B)z, \ldots, ad^k(A,B)z, \ldots\} = I\!R^n, \quad \forall z \in I\!R^n - \{0\}, \tag{7}$$

where $ad^k(A,B)$ is defined recursively as $ad^0(A,B) = B$, $ad^1(A,B) = AB - BA$ and $ad^{k+1}(A,B) = ad^1(A, ad^k(A,B))$, $\forall k \in I\!N$ is sufficient for exponential stabilization of the system (1) controlled by the feedback $p_2(t)$. In [7,14], it has been shown that if the spectrum $\sigma(A)$ of A can be decomposed into $\sigma_u(A) = \{\lambda : \mathcal{R}e(\lambda) \geq -\gamma\}$ and $\sigma_s(A) = \{\lambda : \mathcal{R}e(\lambda) < -\gamma\}$, (for some $\gamma > 0$), then the state space H can be decomposed according to

$$H = H_u \oplus H_s, \tag{8}$$

where $H_u = P_u H = \text{vect}\{\varphi_j,\ 1 \leq j \leq N\}$, $H_s = P_s H = \text{vect}\{\varphi_j,\ j > N\}$, P_u is given by

$$P_u = \frac{1}{2\pi i}\int_C (\lambda I - A)^{-1} d\lambda, \tag{9}$$

where C is a curve surrounding $\sigma_u(A)$, $P_s = I - P_u$ and for all $j \geq 1$, φ_j is the eigenvector associated to the eigenvalue λ_j. The projection operators P_u and P_s commute with A, and we have $A = A_u + A_s$ with $A_u = P_u A P_u$ and $A_s = P_s A P_s$. Also, for all $y(t) \in H$, we set $y_u = P_u y$ and $y_s = P_s y$. If A_s satisfies the spectrum growth assumption:

$$\lim_{t\to+\infty} \frac{\ln(\|S_s(t)\|)}{t} = \sup \text{Re}(\sigma(A_s)), \tag{10}$$

which is equivalent to:

$$\|S_s(t)\| \leq M_1 \exp(-\gamma t), \quad \forall t \geq 0 \ (\text{for some } M_1, \gamma > 0), \tag{11}$$

where $S_s(t)$ denotes the semigroup generated by A_s in H_s. In the sequel, we suppose that the operator B satisfies

$$B = B_u \oplus B_s, \tag{12}$$

where $B_u = P_u B P_u$ and $B_s = P_s B P_s$, (see [14]).

Let us consider that the system (1) can be decomposed in the following two subsystems:

$$\frac{dy_u(t)}{dt} = A_u y_u(t) + p(t) B_u y_u(t), \quad y_u(0) = y_{0u} \in H_u, \tag{13}$$

$$\frac{dy_s(t)}{dt} = A_s y_s(t) + p(t) B_s y_s(t), \quad y_s(0) = y_{0s} \in H_s, \tag{14}$$

in the state spaces H_u and H_s respectively. For linear systems and based on the above decomposition (8), it has been shown that the whole system can be divided into two uncoupled subsystems, one of which is exponentially stable without applying controls, while the other one is unstable (see [18]). So the instability problem in some sense reduces to a finite-dimensional one. In this paper, we shall study the uniform stabilizability of the system (1) and we shall establish the explicit decay estimate (5) under a weaker observability assumption based on the decomposition method described above of the system (1). The rest of the paper is as follows: the second and the third section are devoted to the main results. In the last section we give some applications and simulations.

Remark 1.1 1. For finite-dimensional systems, the conditions (3) and (4) are equivalent (see [4,17]). However, in infinite-dimensional case, and if B is compact, then the condition (4) is impossible. Indeed, if (φ_j) is an orthonormal basis of H, then applying (4) for $y = \varphi_j$ and using the fact that $\varphi_j \rightharpoonup 0$, as $j \to +\infty$, we obtain the contradiction $\delta = 0$.

2. Since $S(t)$ is a semigroup of contractions, so $S_u(t) = I_u$ (the identity operator). Indeed, the assumption that A generates a contraction semigroup implies that $\sup(\lambda_j) \leq 0$. On the other hand, remarking that for all $0 < \gamma_1 < \gamma$ the set $\{\lambda_j | \lambda_j > -\gamma_1\}$ is also finite, we can take $0 < \gamma < \inf_{\lambda_j < 0}(-\lambda_j)$ so that $\{\lambda_j | \lambda_j > -\gamma\} = \{0\}$, and hence $S_u(t) = I_u$. Then, in this case the state space $H_u = \ker(A_u)$.

2 Stabilization results

Let us now recall the following definition concerning the asymptotic behavior of the system (1) (see e.g. [2]).

Definition 2.1 The system (1) is weakly (resp. strongly) stabilizable if there exists a feedback control $p(t) = f(y(t))$, $t \geq 0$, $f : H \longrightarrow K := I\!R, C\!\!\!\!I$ such that the corresponding mild solution $y(t)$ of the system (1) satisfies the properties:

1. for each initial state y_0 of the system (1) there exists a unique mild solution defined for all $t \in I\!R^+$ of the system (1),
2. $\{0\}$ is an equilibrium state of the system (1),
3. $y(t) \to 0$, weakly (resp. strongly), as $t \to +\infty$, for all $y_0 \in H$.

The following definition concerns the uniform and the uniform polynomial stabilization of the system (1).

Definition 2.2 1. We say a dynamical system $(t, y_0) \mapsto y(t, y_0)$ on a normed state space H is uniformly stable (with respect to bounded sets) if for any bounded set $B \subset H$, we have

$$\lim_{t\to+\infty} \sup\{\|y(t, y_0)\| : y_0 \in B\} = 0. \tag{15}$$

2. A dynamical system $(t, y_0) \mapsto y(t, y_0)$ is uniformly polynomially stable, if for $R > 0$ there exists a constant $C_R > 0$ such that every solution of (1) with $\|y_0\| \leq R$ satisfies

$$\|y(t)\| \leq C_R t^{-\frac{1}{2-r}}, \quad \text{as } t \to +\infty. \tag{16}$$

Remark 2.1 1. The uniform polynomial stability implies the uniform stability and the last result is stronger than the strong stability as is in the Definition 2.1.
2. The positive constant C_R depends on R and is independent of t.
3. The uniform stability is consistent with the concept of uniform attraction in dynamical systems (see [5]).

Before we state our main results, the following lemmas will be needed.

Lemma 2.1 [1] *Let (s_k) be a sequence of positive real numbers satisfying*

$$s_{k+1} + C s_{k+1}^{2+\alpha} \leq s_k, \quad \forall k \geq 0, \tag{17}$$

where $C > 0$ and $\alpha > -1$ are constants. Then there exists a positive constant M_2 (depending on α and C) such that

$$s_k \leq \frac{M_2}{(k+1)^{\frac{1}{\alpha+1}}}\ k \geq 0. \tag{18}$$

Let us now recall the following existing result.

Lemma 2.2 [15] *Let A generate a semigroup of contractions $S(t)$ on H and let B be linear and bounded operator from H into itself. Then the system (1), controlled by (6) possesses a unique mild solution $y(t)$ on $\mathbb{R}^+$ for each $y \in H$ which satisfies*

$$\int_0^T |\langle S(t)y, BS(t)y\rangle| dt = \mathcal{O}\left(\left(\|y(t)\|^r \int_t^{t+T} \frac{\langle y(s), By(s)\rangle^2}{\|y(s)\|^r} ds\right)^{1/2}\right), \quad \text{as } t \to +\infty, \tag{19}$$

for almost all $T > 0$.

The next result concerns the uniform stabilization of the system (1).

Theorem 2.1 1. *Let A generate a linear C_0-semigroup $S(t)$ of contractions on H,*
2. *A allow the decomposition (8) of H with $\dim H_u < +\infty$ such that (11) holds, and*
3. *$B \in \mathcal{L}(H)$ be compact and satisfies (3).*

Then the system (1) is uniformly stabilizable using the feedback control law

$$p_r(t) = -\rho \frac{\langle y(t), By(t)\rangle}{\|y(t)\|^r} \mathbf{1}_\Lambda, \quad r < 2, \tag{20}$$

where the parameter $\rho > 0$ is chosen sufficiently small and $\Lambda = \{t \geq 0 \colon y(t) \neq 0\}$.

Proof The system (1) controlled by (20) possesses a unique mild solution $y(t)$ defined on a maximal interval $[0, t_{\max}[$ and given by the variation of constants formula:

$$y(t) = S(t)y_0 + \int_0^t S(t-\tau) f(y(\tau)) d\tau, \tag{21}$$

where $f(y) = -\rho \frac{\langle y, By\rangle}{\|y\|^r} By$, for all $y \neq 0$, $f(0) = 0$ corresponds to (20). Since $S(t)$ is a semigroup of contractions, we have:

$$\frac{d\|y(t)\|^2}{dt} \leq -2\langle f(y(t)), y(t)\rangle, \quad \forall y_0 \in \mathcal{D}(A). \tag{22}$$

It follows that

$$\|y(t)\| \leq \|y_0\|. \tag{23}$$

From (21) and using the fact that $S(t)$ is a semigroup of contractions and Gronwall inequality, we deduce that the map $y_0 \longrightarrow y(t)$ is continuous from H to H. We deduce that (23) holds for all $y_0 \in H$ by density argument and hence $t_{\max} = +\infty$ (see [16]). By virtue of (22) we have

$$2\int_0^t \frac{|\langle y(\tau), By(\tau)\rangle|^2}{\|y(\tau)\|^r} d\tau \leq \|y_0\|^2, \quad \forall t \geq 0,\ y_0 \in \mathcal{D}(A). \tag{24}$$

This last inequality holds, by density, for all $y_0 \in H$. It follows that the integral

$$\int_0^t \frac{|\langle y(\tau), By(\tau)\rangle|^2}{\|y(\tau)\|^r} d\tau$$

converges. Consequently, we deduce from the Cauchy criterion that

$$\int_t^{t+T} \frac{|\langle y(\tau), By(\tau)\rangle|^2}{\|y(\tau)\|^r} d\tau \to 0, \quad \text{as } t \to +\infty. \tag{25}$$

Now let us show that $y(t) \rightharpoonup 0$, as $t \to +\infty$, (where $\rightharpoonup$ refers to weak convergence). Let $t_n \to +\infty$ such that $y(t_n)$ weakly converge in H, and let $z \in H$ such that $y(t_n) \rightharpoonup z$, as $t \to +\infty$. (The existence of such (t_n) and z are ensured by (23) and by the fact that space H is reflexive). Since $S(t)$ is bounded and B is a compact operator, then for all $t \geq 0$, we have $S(t)y(t_n) \rightharpoonup S(t)z$ and $BS(t)y(t_n) \to BS(t)z$, as $n \to +\infty$. Then

$$\lim_{n\to+\infty} \langle BS(t)y(t_n), S(t)y(t_n)\rangle = \langle BS(t)z, S(t)z\rangle.$$

Hence, by the dominated convergence theorem

$$\lim_{n\to+\infty} \int_0^T |\langle BS(t)y(t_n), S(t)y(t_n)\rangle| dt = \int_0^T |\langle BS(t)z, S(t)z\rangle| dt.$$

Then using (19) and (25), we get $\int_0^T |\langle BS(t)z, S(t)z\rangle| dt = 0$, so $\langle BS(t)z, S(t)z\rangle = 0$, $\forall t \geq 0$, which gives, by virtue of (3) that $z = 0$. Hence $y_u(t) \rightharpoonup 0$, as $t \to +\infty$, and using the fact that dim $H_u < +\infty$, we deduce that $y_u(t) \to 0$ as $t \to +\infty$. The last result together with (23) (i.e., $y_u(t)$ approaches to 0 depends only on the norm $\|y_0\|$) show that $y_u(t)$ converges uniformly to zeros as t tend to $+\infty$. For the component $y_s(t)$ of $y(t)$, we have

$$y_s(t) = S_s(t)y_{0s} - \rho \int_0^t S_s(t-\tau) \frac{\langle y(\tau), By(\tau)\rangle}{\|y(\tau)\|^r} B_s y_s(\tau) d\tau. \tag{26}$$

It follows from (11) that

$$\|y_s(t)\| \leq M_1 e^{-\gamma t}\|y_{0s}\| + \rho M_1 \|B\|^2 \|y_0\|^{2-r} \int_0^t e^{-\gamma(t-\tau)} \|y_s(\tau)\| d\tau. \tag{27}$$

From Gronwall inequality, we obtain:

$$\|y_s(t)\| \leq M_1 e^{(\rho M_1 \|B\|^2 \|y_0\|^{2-r} - \gamma)t} \|y_{0s}\|, \quad \forall t \geq 0.$$

Then, for all $R > 0$, there exists a constant $C_R > 0$ such that $\|y_0\| \leq R$, we have

$$\|y_s(t)\| \leq M_1 C_R e^{(\rho M_1 C_R \|B\|^2 - \gamma)t}, \quad \forall t \geq 0.$$

Taking $\rho < \frac{\gamma}{M_1 C_R \|B\|^2}$, we deduce that $y_s(t) \to 0$ uniformly as $t \to +\infty$.

Hence $y(t) = y_u(t) + y_s(t) \to 0$ uniformly as $t \to +\infty$. □

Remark 2.2 If $y_0 = 0$ then $y(t) = 0$ for almost all $t \geq 0$.

3 Polynomial decay rate estimate of the stabilized state

In this section, based on the decomposition method described above, we are concerned with the question of the uniform polynomial stabilizability of the system (1) with the decay estimate (16), using a continuous feedback control, which depends only on the unstable part, without compactness assumption of the operator B and under a weaker condition than (3).

Before we state our main result in this section, the following lemma will be needed.

Lemma 3.1 *Let H be a Hilbert space such that* $\dim(H) < +\infty$, *then the two inequality* (3) *and* (4) *are equivalent.*

Proof Firstly, it is evident that the inequality (4) implies (3).

Conversely, we shall show that (3) implies (4), for this purpose two situations are provided:
Case 1 $z = 0$, it is trivial that the estimate (4) holds.
Case 2 $z \in H - \{0\}$, so

$$\int_0^T |\langle BS(t)z, S(t)z\rangle| dt > 0, \quad \forall z \in H - \{0\}. \tag{28}$$

Indeed, we assume that there exists $z \in H - \{0\}$, such that

$$\int_0^T |\langle BS(t)z, S(t)z\rangle| dt = 0,$$

which implies that $\langle BS(t)z, S(t)z\rangle = 0$, since the map $t \longrightarrow S(t)z$ is continuous on $[0, +\infty[$. From (3) we get $z = 0$, which contradict with the fact that $z \in H - \{0\}$ so (28) is satisfied. Furthermore, note that the map $z \longrightarrow \int_0^T |\langle BS(t)z, S(t)z\rangle| dt$ depends continuously on z.

Using now the fact that the set $S = \{z \in H|\ \|z\| = 1\}$ is compact in H $(\dim(H) < +\infty)$. Then, from (28) one concludes that there exists a positive constant $\delta > 0$, such that

$$\inf_{z\in S} \int_0^T |\langle BS(t)z, S(t)z\rangle| dt = \delta. \tag{29}$$

Hence, we deduce that (4) holds for all $z \in S$. Now, for all $z \in H - \{0\}$, one can take $y = \frac{z}{\|z\|} \in S$, then from (29) we get

$$\inf_{z\in H-\{0\}} \int_0^T \left| \left\langle BS(t)\frac{z}{\|z\|}, S(t)\frac{z}{\|z\|} \right\rangle \right| dt = \delta,$$

which immediately gives

$$\inf_{z\in H-\{0\}} \int_0^T |\langle BS(t)z, S(t)z\rangle| dt = \delta\|z\|^2.$$

This completes the proof of Lemma 3.1. □

The main result in this section is as follows.

Theorem 3.1 1. *Let A generate a linear C_0-semigroup $S(t)$ such that $S_u(t)$ is of isometries and* (11) *holds,*
2. *A allow the decomposition* (8) *of H with* $\dim H_u < +\infty$,
3. *$B \in \mathcal{L}(H)$ such that for all $y_u \in H_u$, we have*

$$\langle B_u S_u(t) y_u, S_u(t) y_u\rangle = 0, \quad \forall t \geq 0 \Rightarrow y_u = 0. \tag{30}$$

Then, for all $R > 0$ there exists a constant $C_R > 0$ such that every solution of (1) *with $\|y_0\| \leq R$, the feedback control law*

$$p_{r,u}(t) = -\rho \frac{\langle y_u(t), B_u y_u(t)\rangle}{\|y_u(t)\|^r} \mathbf{1}_{\Lambda_u}, \quad r < 2, \tag{31}$$

satisfies

$$\|y(t)\| \leq C_R t^{-\frac{1}{2-r}}, \quad as\ t \to +\infty, \tag{32}$$

where the parameter $\rho > 0$ is chosen sufficiently small and $\Lambda_u = \{t \geq 0 : y_u(t) \neq 0\}$.

Proof Let us consider the system:

$$\frac{dy_u(t)}{dt} = A_u y_u(t) - \rho \frac{\langle y_u(t), B_u y_u(t)\rangle}{\|y_u(t)\|^r} B_u y_u(t), \quad y_u(0) = y_{0u}. \tag{33}$$

Multiplying the system (33) by $y_u(t)$ and integrating over Ω and using the fact that $S_u(t)$ is a semigroup of isometries, (so that $\langle A_u\phi, \phi\rangle = 0,\ \forall \phi \in \mathcal{D}(A_u)$), we obtain

$$\frac{d\|y_u(t)\|^2}{dt} = -2\rho \frac{|\langle y_u(t), B_u y_u(t)\rangle|^2}{\|y_u(t)\|^r} \leq 0, \quad \forall y_{0u} \in \mathcal{D}(A_u), \tag{34}$$

which proves that the real function $t \longrightarrow \|y_u(t)\|$ is decreasing on $\mathbb{R}^+$, and we have

$$\|y_u(t)\| \leq \|y_u(0)\|, \quad \forall t \geq 0. \tag{35}$$

Hence, the system (33) admits a unique mild solution defined for almost all $t \geq 0$ (see [15]).

Integrating now the last inequality over the interval $[kT, (k+1)T]$, $k \in \mathbb{N}$ and $T > 0$, we obtain:

$$\|y_u((k+1)T)\|^2 - \|y_u(kT)\|^2 \leq -2\rho \int_{kT}^{(k+1)T} \frac{|\langle y_u(\tau), B_u y_u(\tau)\rangle|^2}{\|y_u(\tau)\|^r} d\tau,$$

using now the estimate (19), we deduce that

$$\begin{aligned} &\|y_u((k+1)T)\|^2 - \|y_u(kT)\|^2 \\ &\leq -M\|y_u(kT)\|^{-r} \left(\int_0^T |\langle S_u(\tau)y_u, B_u S_u(\tau)y_u\rangle| d\tau \right)^2, \quad M > 0 \end{aligned} \tag{36}$$

Using the fact $\dim(H_u) < +\infty$, then from Lemma 3.1, the assumption (30) implies

$$\int_0^T |\langle B_u S_u(t)y_u, S_u(t)y_u\rangle| dt \geq \delta\|y_u\|^2, \quad \forall y_u \in H_u, \ (T, \delta > 0). \tag{37}$$

Then, from (36) and (37) we get

$$\|y_u((k+1)T)\|^2 - \|y_u(kT)\|^2 \leq -M\delta^2\|y_u(kT)\|^{4-r},$$

using the fact that $t \longrightarrow \|y_u(t)\|$ is a decreasing function, we obtain

$$\|y_u((k+1)T)\|^2 - \|y_u(kT)\|^2 \leq -M\delta^2\|y_u((k+1)T)\|^{4-r},$$

which implies that

$$\|y_u((k+1)T)\|^2 + C\|y_u((k+1)T)\|^{4-r} \leq \|y_u(kT)\|^2, \quad C = M\delta^2.$$

Letting $s_k = \|y_u(kT)\|^2$, the last inequality can be written as

$$s_{k+1} + Cs_{k+1}^{2-\frac{r}{2}} \leq s_k, \quad \forall k \geq 0.$$

Applying Lemma 2.1, we deduce that $s_k \leq \frac{M_2}{(k+1)^{\frac{2}{2-r}}}$. For $k = [\frac{t}{T}]$ ($[\frac{t}{T}]$ designed the integer part of $\frac{t}{T}$), we obtain: $s_k \leq \frac{M_3}{t^{\frac{2}{2-r}}}$, ($M_3(\|y_{0u}\|) = M_3 > 0$) which gives $\|y_u(t)\|^2 \leq \frac{M_3}{t^{\frac{2}{2-r}}}$. Hence, for all $R_1 > 0$, there exists a constant $C_{R_1} > 0$ such that $\|y_{0u}\| \leq R_1$ implies

$$\|y_u(t)\| \leq C_{R_1} t^{-\frac{1}{2-r}}, \quad \text{as } t \to +\infty. \tag{38}$$

For the component $y_s(t)$, we shall show that $y_s(t)$ is defined for all $t \geq 0$ and uniformly exponentially converges to 0, as $t \to +\infty$. The system (1) excited by the control (31) admits a unique mild solution defined for almost all t in a maximal interval $[0, t_{\max}[$ by

$$y(t) = S(t)y_0 + \int_0^t S(t-\tau)p_{r,u}(\tau)By(\tau)d\tau.$$

Thus

$$y_s(t) = S_s(t)y_{0s} + \int_0^t S_s(t-\tau)p_{r,u}(\tau)B_s y_s(\tau)d\tau, \quad \forall t \in [0, t_{\max}[. \tag{39}$$

It follows from (11), (35) and (39) that

$$\|y_s(t)\| \leq M_1 e^{-\gamma t}\|y_{0s}\| + \rho M_1\|B\|^2\|y_{0u}\|^{2-r} \int_0^t e^{-\gamma(t-\tau)}\|y_s(\tau)\|d\tau$$

for almost all $t \in [0, t_{\max}[$.

Then the scalar function $z(t) = \|y_s(t)\| \, e^{\gamma t}$ satisfies

$$z(t) \le M_1 \|y_{0s}\| + \rho M_1 \|B\|^2 \|y_{0u}\|^{2-r} \int_0^t z(\tau) d\tau.$$

Gronwall inequality then yields

$$z(t) \le M_1 \|y_{0s}\| e^{\rho M_1 \|B\|^2 \|y_{0u}\|^{2-r} t}.$$

In other words, for all $R_2 > 0$, there exists a positive constant $C_{R_2} > 0$ such that $\|y_{0s}\| \le R_2$, we have

$$\|y_s(t)\| \le M_1 C_{R_3} e^{(\rho M_1 C_{R_3} \|B\|^2 - \gamma) t}, \tag{40}$$

where $C_{R_3} = \max\{C_{R_1}, C_{R_2}\}$. Taking $\rho < \frac{\gamma}{M_1 C_{R_3} \|B\|^2}$, it follows from (40) that $y_s(t)$ is bounded on $[0, t_{\max}[$, so $t_{\max} = +\infty$, and the estimate (40) holds for all $t \ge 0$. From the inequality (40) together with the choice of ρ sufficiently small, we deduce that there exists a constant $C_{R_4} > 0$ for which we have

$$\|y_s(t)\| \le C_{R_4} t^{-\frac{1}{2-r}}, \quad \text{as } t \to +\infty, \ \ (C_{R_4} > 0). \tag{41}$$

Then, we conclude from (38) and (41) that the solution of the system (1) satisfies

$$\|y(t)\| \le C_{R_5} t^{-\frac{1}{2-r}}, \quad \text{as } t \to +\infty, \tag{42}$$

such that $C_{R_5} = C_{R_1} + C_{R_4}$. This completes the proof of Theorem 3.1. □

Remark 3.1 1. Since $\|y_u(t)\|$ decreases, then we have

$$\exists t_0 \ge 0; \ y_u(t_0) = 0 \Leftrightarrow y_u(t) = 0, \quad \forall t \ge t_0.$$

In this case, we have

$$p_{r,u}(t) = 0, \quad \forall t \ge t_0 \Longrightarrow y(t) = S_s(t - t_0) y_s(t_0), \quad \forall t \ge t_0.$$

Hence, using (11) the system (1) is exponentially stable.

2. In Theorems 2.1 and 3.1, the parameter $\rho > 0$ is chosen sufficiently small which ensures that the exponential decay on the subspace corresponding to $\sigma_s(A)$ overcomes the presence of the nonlinear control.
3. The feedback (31) depends only on the unstable part $y_u(t)$ and is uniformly bounded with respect to initial states.
4. We note that (30) is weaker than both (3) and (4).
5. In the case $\dim H_u = +\infty$ and B is nonlinear and locally Lipschitz such that $B(0) = 0$. Then using the techniques as in [15], we can obtain the result of Theorem 3.1 if (4) is changed to (37).
6. In the case $r = 2$, and under the weak observability condition (30), the exponential stability of the system (1) has been established (see [14]).
7. The feedback (20) and (31) are continuous, whereas $p_2(t)$ is not.
8. If $r = 0$, we retrieve the result of [12].
9. The decay estimate (42) is the best one.
10. Taking $\varphi(t) = \|y_u(t)\|^2$ in (34), one can deduce that $-\frac{1}{2\rho} \varphi^{-2+\frac{r}{2}}(t) \varphi'(t) \le \|B\|^2$. Integrating now the last inequality over $[0, t]$, we obtain

$$\varphi(t) \ge \frac{1}{\left(2\rho(1 - \frac{r}{2}) \|B\|^2 t + \frac{1}{\varphi^{1-\frac{r}{2}}(0)} \right)^{\frac{2}{2-r}}}, \quad \forall t \ge 0,$$

so $\|y_u(t)\| \geq \dfrac{1}{\left(2\rho(1-\frac{r}{2})\|B\|^2 t+\frac{1}{\|y_u(0)\|^{2-r}}\right)^{\frac{1}{2-r}}}$, $r<2$. Consequently, from (38) we deduce

$$\frac{M_1}{t^{\frac{1}{2-r}}} \leq \|y_u(t)\| \leq \frac{M_2}{t^{\frac{1}{2-r}}}, \quad \text{as } t \to +\infty, \; (M_2 \geq M_1 > 0). \tag{43}$$

4 Applications

In this part, we shall give two illustrating examples of the established results.

Example 4.1 Let us consider the following fourth-order PDE equation:

$$\begin{cases} \frac{\partial y(x,t)}{\partial t} = -\frac{\partial^4 y(x,t)}{\partial x^4} - \frac{\partial^2 y(x,t)}{\partial x^2} + p(t)By(x,t) & \text{in } (-\pi,\pi)\times(0,\infty) \\ y(x,0) = y_0(x), & x \in (-\pi,\pi), \end{cases} \tag{44}$$

where the state space is $H = L^2(-\pi,\pi)$, and the dynamic is defined by $A = -\frac{\partial^4}{\partial x^4} - \frac{\partial^2}{\partial x^2}$, with $\mathcal{D}(A) = \{y \in H^4(-\pi,\pi) | \; \frac{\partial^n y}{\partial x^n}(-\pi) = \frac{\partial^n y}{\partial x^n}(\pi) = 0, \;\; n = 0,\ldots,3\}$. The operator A is a infinitesimal generator of a C_0-semigroup of contractions in $L^2(-\pi,\pi)$, with eigenvalues given by $\lambda_j = -j^4 + j^2$, $(j \in \mathbb{N}^*)$, associated to the eigenfunctions $(\varphi_j)_{j\geq 1}$ expressed by $\varphi_j(x) = \frac{1}{\sqrt{\pi}}\sin(jx)$, $\forall j \geq 1$, (see [10]). In this case $S_u(t) = I_{H_u}$ (the identity operator), namely $S_u(t)$ is a semigroup of isometries. Moreover, taking the operator of control by $By = \sum_{j=1}^{+\infty} \frac{1}{j^2}\langle y,\varphi_j\rangle\varphi_j$. The family $(\varphi_j)_{j\geq 1}$ is an orthonormal basis of $L^2(-\pi,\pi)$, and the solution $y(t)$ of the system (44) can be written as:

$$y(x,t) = \sum_{j=1}^{+\infty} a_j(t)\varphi_j(x) = \sum_{j=1}^{+\infty}\langle y(t),\varphi_j\rangle_{L^2(-\pi,\pi)}\varphi_j(x).$$

The fact that $\lambda_1 = 0$ and $\lambda_j < 0$, $\forall j \geq 2$, implies that the subspace $H_u = \mathbb{R}\varphi_1$ i.e., the solution $y_1(t) \in H_u$ is written as:

$$y_1(x,t) = y_u(x,t) = a_1(t)\varphi_1(x),$$

where the function a_1 satisfies

$$a_1'(t) = -|a_1(t)|^{2-r}a_1(t), \quad \forall t \geq 0. \tag{45}$$

It is easy to see that the operator B is compact, and A generates the semigroup $S(t)$ such that $S(t)y = \sum_{j=1}^{+\infty} e^{\lambda_j t}\langle y,\varphi_j\rangle\varphi_j$, and we have $\langle BS(t)y, S(t)y\rangle = \sum_{j=1}^{+\infty}\frac{e^{\lambda_j t}}{j^2}|\langle y,\varphi_j\rangle|^2$, $\forall t \geq 0$, so, it is clear that (3) holds. Consequently, the assumptions of the Theorem 2.1 are satisfied and the control (20) in this case is explicitly expressed by

$$p_r(t) = -\rho\frac{\sum_{j=1}^{+\infty}\frac{1}{j^2}\left(\int_{-\pi}^{\pi} y(\tau,t)\sin(j\tau)d\tau\right)^2}{\left\{\sum_{j=1}^{+\infty}\left(\int_{-\pi}^{\pi} y(\tau,t)\sin(j\tau)d\tau\right)^2\right\}^{\frac{r}{2}}}, \tag{46}$$

which ensures the uniform stability of the system (44). Moreover, for the application of the Theorem 3.1, we have $\langle B_u S_u(t)y_u, S_u(t)y_u\rangle = |\langle y_u,\varphi_1\rangle|^2 = |a_1(t)|^2$, and therefore the hypothesis (30) is verified. Furthermore, from (45) we get:

$$|a_1(t)| = \frac{1}{\left[(2-r)\rho t + \frac{1}{|a_1(0)|^{2-r}}\right]^{\frac{1}{2-r}}}, \quad \forall t \geq 0,$$

which gives:

$$\|y_1(t)\|_{L^2(-\pi,\pi)} = \frac{1}{\left[(2-r)\rho t + \frac{1}{|a_1(0)|^{2-r}}\right]^{\frac{1}{2-r}}}, \quad \forall t \geq 0,$$

for any $a_1(0) \neq 0$. Then, using the following control

$$p_{r,u}(t) = -\frac{\rho}{(2-r)\rho t + \frac{1}{|a_1(0)|^{2-r}}}, \quad \forall t \geq 0, \tag{47}$$

we obtain

$$\|y_1(t)\|_{L^2(-\pi,\pi)} \leq C_{R_1} t^{-\frac{1}{2-r}}, \quad \text{as } t \to +\infty,$$

where $C_{R_1} > 0$. Using now the fact that $\lambda_j < 0$ for any $j \geq 2$ and $p_{r,u}(t) < 0, \ \forall t \geq 0$, we deduce that

$$|a_j(t)| \leq e^{(-j^4+j^2)t}|a_j(0)|, \quad \forall t \geq 0.$$

Hence,

$$\|y_s(t)\|_{L^2(-\pi,\pi)} \leq e^{-t}\|y_s(0)\|_{L^2(-\pi,\pi)}, \quad \forall t \geq 0,$$

which gives

$$\|y_s(t)\|_{L^2(-\pi,\pi)} \leq C_{R_2} t^{-\frac{1}{2-r}} \quad \text{as } t \to +\infty,$$

where $C_{R_2} > 0$. Consequently, the system (44) is uniformly polynomially stabilizable with the decay estimate

$$\|y(t)\|_{L^2(-\pi,\pi)} \leq C_R t^{-\frac{1}{2-r}}, \quad \text{as } t \to +\infty,$$

where $C_R = C_{R_1} + C_{R_2}$.

Remark 4.1 The two negative controls (46) and (47) satisfying the inequality $|p_r(t)| = \mathcal{O}\left(\frac{1}{t}\right)$, as $t \to +\infty$, and $|p_{r,u}(t)| = \mathcal{O}\left(\frac{1}{t}\right)$, as $t \to +\infty$.

The second example in this section is the following.

Example 4.2 Let us consider the following 1-d bilinear heat equation:

$$\begin{cases} \frac{\partial y(x,t)}{\partial t} = \frac{\partial^2 y(x,t)}{\partial x^2} + p(t)By(x,t), & x \in (0,1), \ t > 0, \\ \frac{\partial y(0,t)}{\partial x} = \frac{\partial y(1,t)}{\partial x} = 0, & \forall t > 0, \end{cases} \tag{48}$$

where $y(t)$ is the temperature profile at time t. We suppose that the system is controlled via the flow of a liquid $p(t)$ in an adequate metallic pipeline. Here we take the state space $H = L^2(0,1)$ and the operator A is defined by $Ay = \frac{\partial^2 y}{\partial x^2}$, with

$$\mathcal{D}(A) = \left\{ y \in H^2(0,1) | \ \frac{\partial y(0,t)}{\partial x} = \frac{\partial y(1,t)}{\partial x} = 0 \right\}.$$

The domain of A gives the homogeneous Neumann boundary condition imposed at the ends of the bar which require specifying how the heat flows out of the bar and mean that both ends are insulated. The spectrum of A is given by the simple eigenvalues $\lambda_j = -\pi^2(j-1)^2,\ j \in \mathbb{N}^*$ and eigenfunctions $\varphi_1(x) = 1$ and $\varphi_j(x) = \sqrt{2}\cos((j-1)\pi x)$ for all $j \geq 2$. Then the subspace H_u is the one-dimensional space spanned by the eigenfunction φ_1, and we have $S_u(t)y_u = \langle y_u, \varphi_1\rangle\varphi_1$ so $S_u(t) = I_{H_u}$ (the identity) and hence $S_u(t)$ is a semigroup of isometries. For the operator of control B, we consider:

$By = \sum_{j=1}^{+\infty} \alpha_j < y, \varphi_j > \varphi_j,\ \alpha_j \geq 0,\ \forall j \geq 1$ and $\sum_{j=1}^{+\infty} \alpha_j^2 < \infty$ (see [13]). From the relation:

$$\langle B_u S_u(t)y_u, S_u(t)y_u\rangle = \alpha_1|\langle y_u, \varphi_1\rangle|^2,$$

we can see that (30) holds if $\alpha_1 > 0$. To examine the estimate (32), remarking for the scalar functions $y_j(t) = \langle y(t), \varphi_j\rangle,\ \forall j \geq 1$, we have

$$|y_1(t)| = |y_u(t)| = \frac{1}{\left[(2-r)\rho\alpha_1^2 t + \frac{1}{|y_{0u}|^{2-r}}\right]^{\frac{1}{2-r}}}, \quad \forall t \geq 0,$$

for all y_0 such that $y_{0u} \neq 0$, where $r \in (-\infty, 2)$. This implies

$$|y_u(t)| \leq C_{R_1} t^{\frac{-1}{2-r}}, \quad \text{as } t \to +\infty, \tag{49}$$

where $C_{R_1} > 0$. Then the control in this case is defined by

$$p_{r,u}(t) = -\frac{\rho\alpha_1}{(2-r)\rho\alpha_1^2 t + \frac{1}{|y_{0u}|^{2-r}}}. \tag{50}$$

For $j \geq 2$, the functions $y_j(t)$ are characterised by $y_j(0) = \langle y(0), \varphi_j\rangle,\ \forall j \geq 2$ and satisfy

$$\frac{\partial}{\partial t} y_j(t) = \left(\lambda_j - \frac{\rho\alpha_1\alpha_j}{(2-r)\rho\alpha_1^2 t + \frac{1}{|y_{0u}|^{2-r}}}\right) y_j(t).$$

which implies that

$$|y_j(t)| \leq e^{-\pi^2(j-1)^2 t}|y_j(0)|, \quad j \geq 2.$$

Then

$$\|y_s(t)\| \leq e^{-\pi^2 t}\|y_s(0)\|, \quad \forall t \geq 0,$$

which implies

$$\|y_s(t)\| \leq C_{R_2} t^{-\frac{1}{2-r}}, \quad \text{as } t \to +\infty, \tag{51}$$

where $C_{R_2} > 0$. The inequality (49) together with (51) give

$$\|y(t)\| \leq C_{R_3} t^{-\frac{1}{2-r}}, \quad \text{as } t \to +\infty, \tag{52}$$

where $C_{R_3} = C_{R_1} + C_{R_2}$.

Remark 4.2 With the usual homogeneous Dirichlet boundary condition, the eigenvalues of the operator Δ are all $\lambda_j < 0$, for any $j \geq 1$. In this, case the system (48) is exponentially stable (taking $p(t) = 0$).

4.1 Simulations

In this part, taking in the system (48), the operator $B = I$, i.e., $\alpha_j = 1$, $(j \geq 1)$ and $y_0(x) = 6.75x$. Then we obtain the results shown in the Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10. The Figs. 1, 2, 3, 4 and 8 are plotted in log scale, while the Figs. 5, 6, 7 and 9, 10 are plotted in the linear scale (takes the negative values).

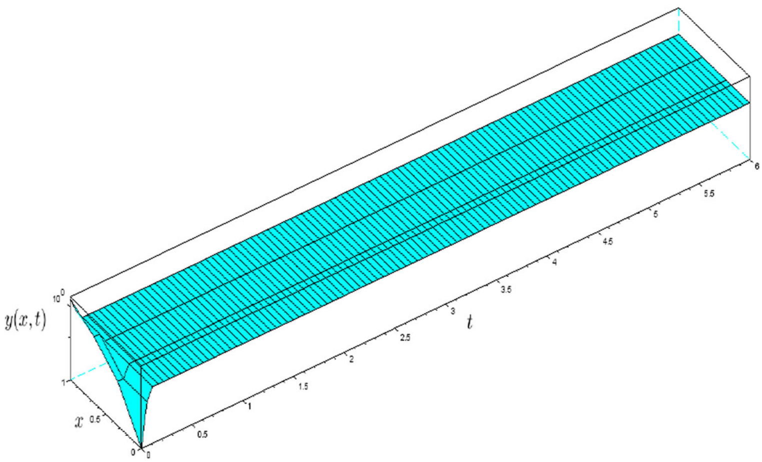

Fig. 1 The evolution of the free state

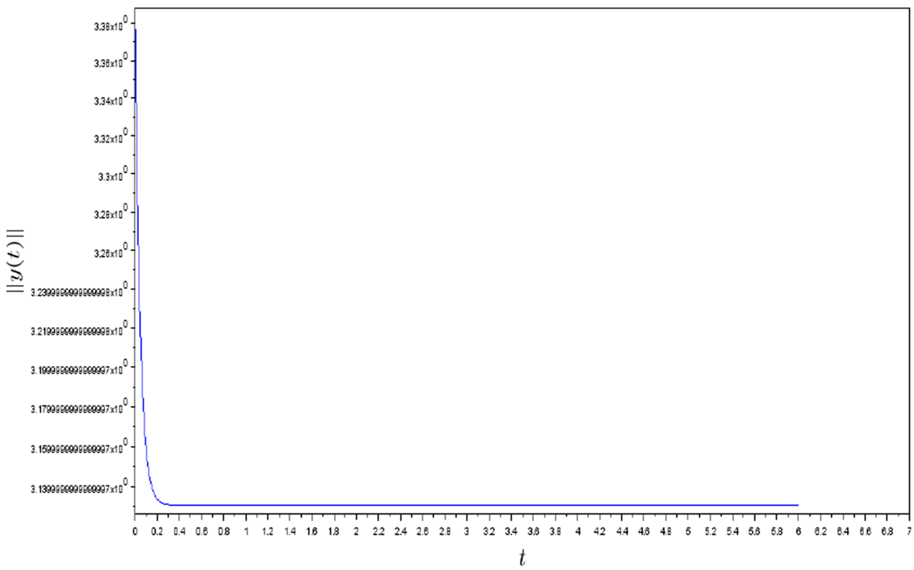

Fig. 2 The norm of the free state

The Figs. 1 and 2 represent the evolution of state and the norm of the free state ($p(t) = 0$). By injecting in the system (48) the control $p_{1.8,u}(t)$, ($r = 1.8$), we obtain the Figs. 3, 4, 5 and 6, which show the evolution of the stabilized state, the norm and its logarithm. The Fig. 6 shows the evolution of logarithm of the norm divided by $\log(t)$. Figure 7 represents the evolution of the stabilizing control.

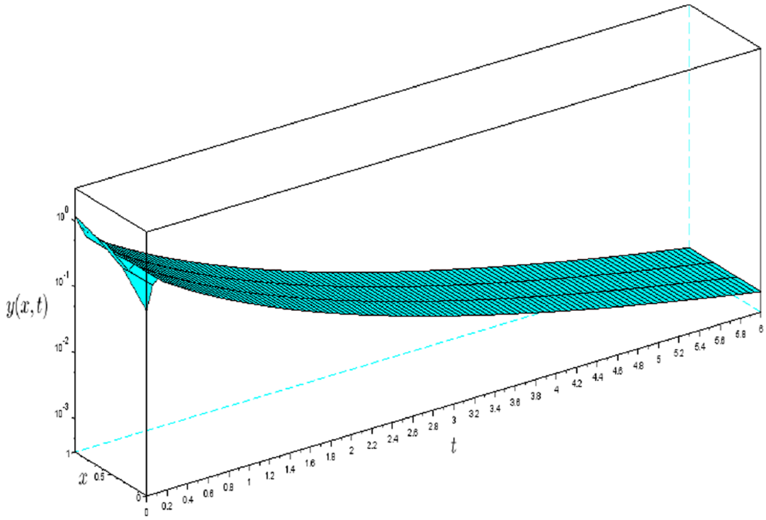

Fig. 3 The evolution of the stabilized state

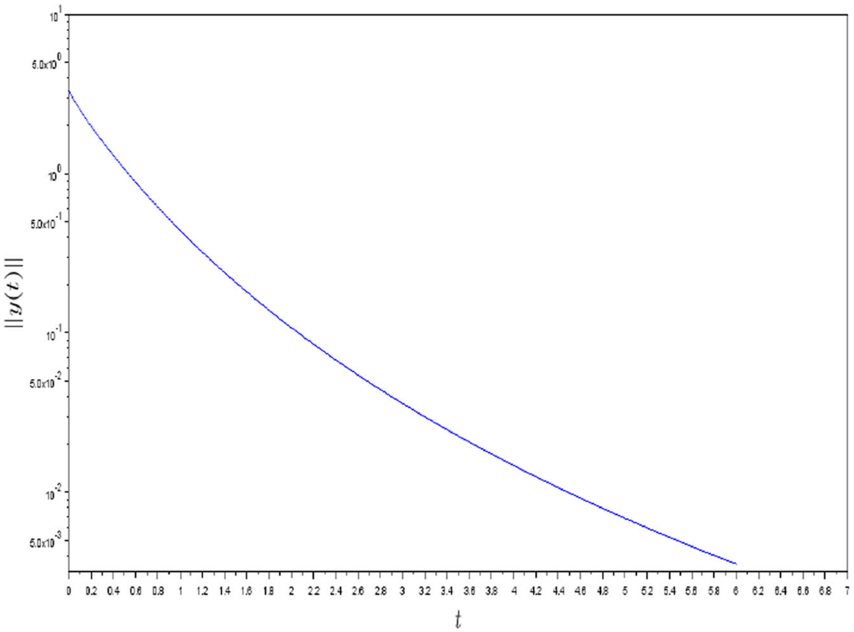

Fig. 4 The norm of the stabilized state

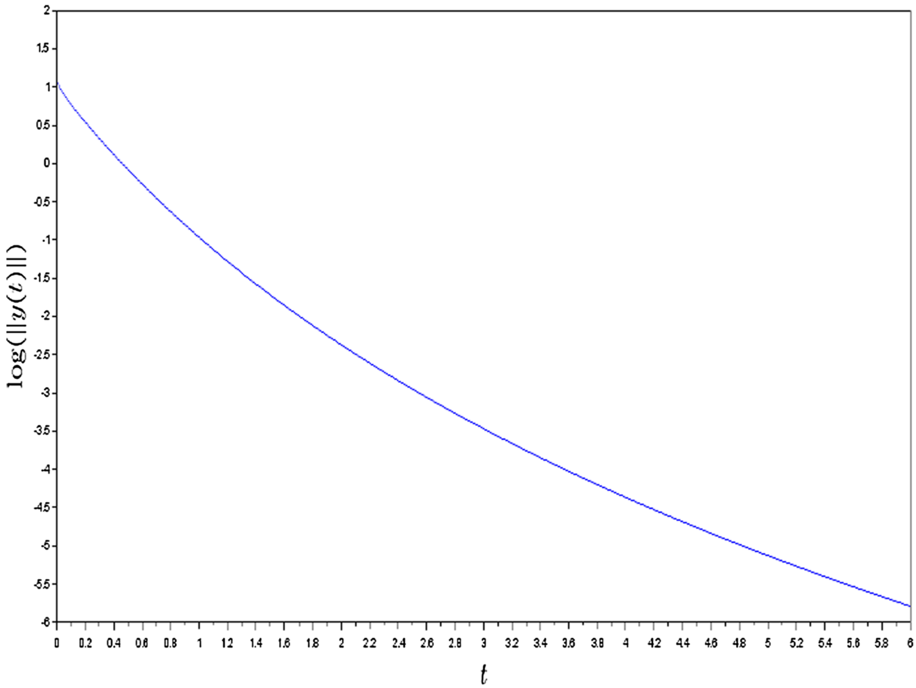

Fig. 5 The logarithm of the norm of the stabilized state

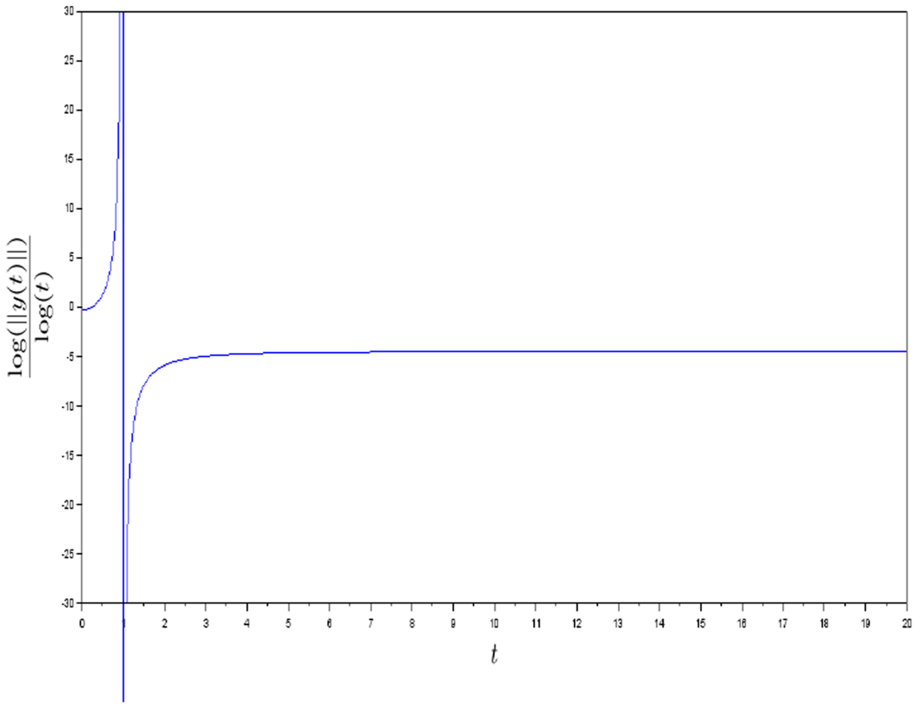

Fig. 6 The evolution of $\frac{\log(\|y(\cdot)\|)}{\log(\cdot)}$ of the stabilized state in the interval [0, 20]

The algebraic rate predicted by the Theorem 3.1 is -5 for $r = 1.8$, which is presumably approach the estimated rate exponent as t increases, as is shown in the Fig. 6.

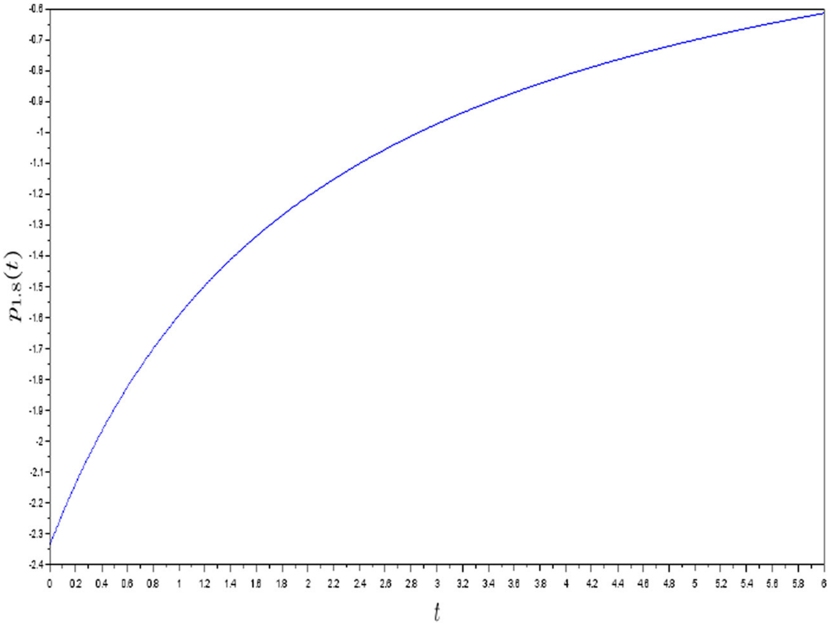

Fig. 7 The evolution of the stabilizing control

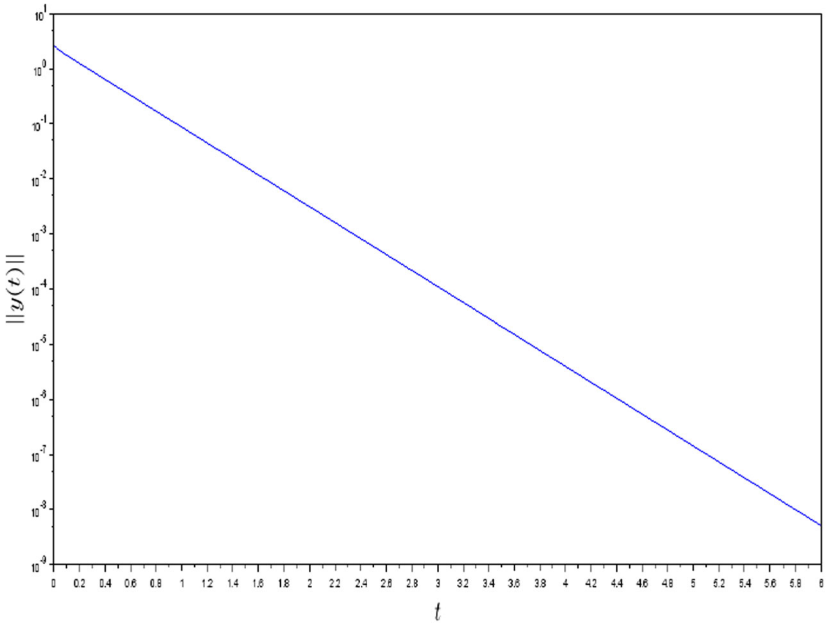

Fig. 8 The norm of the stabilized state

In the special case $r = 2$, the exponential decay of the stabilized state has been established (see [14]). The control in this case is defined by $p_{r,u}(t) = -\rho \mathbf{1}_{\{t \geq 0: y_u(t) \neq 0\}}$, and we have the Figs. 8, 9 and 10.

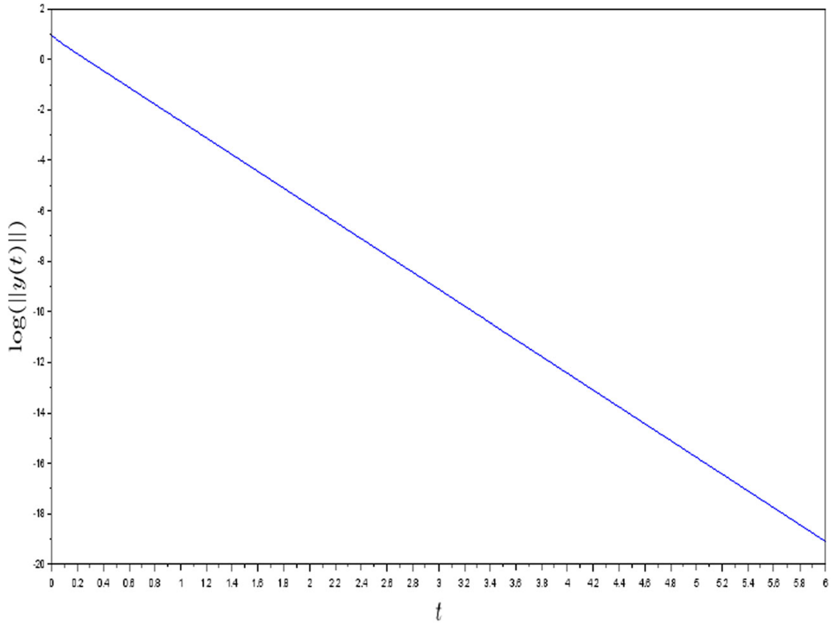

Fig. 9 The logarithm of the norm of the stabilized state

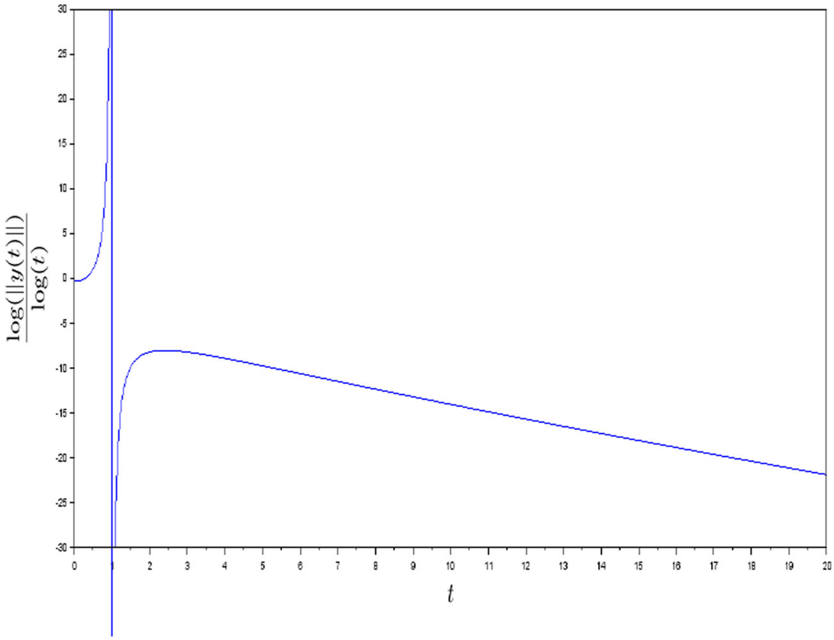

Fig. 10 The evolution of $\frac{\log(\|y(\cdot)\|)}{\log(\cdot)}$ of the stabilized state in the interval [0, 20]

Remark 4.3 In the simulations subsection, the norm $\|\cdot\|$ means $\|\cdot\|_{L^2(0,1)}$.

4.2 Conclusion

In this work, we have considered the problem of the uniform stabilization with polynomial decay rate estimate of the stabilized state for bilinear parabolic systems, that can be decomposed in the stable and unstable parts (14) and (13) under a weaker condition (30). We also have considered the problem of using a stabilizing feedback control depending only on the unstable part (13) that can make the whole system (1) stable. Also, the simulations illustrate perfectly the established theoretical results. Various topics of interest remain open, in case of extending this work to regional stabilization problem.

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