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Finite groups having centralizer commutator product property

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Abstract Let α be an automorphism of a finite group G and assume that $G = \big\{ [g, \alpha] : g \in G \big\} \cdot C_G(\alpha)$. We prove that the order of the subgroup $[G, \alpha]$ is bounded above by $n^{\log_2(n+1)}$ where n is the index of $C_G(\alpha)$ in G.

Keywords Automorphism · Commutator · Fixed point subgroup · Centralizer

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1 Introduction

Let A be a finite group that acts on the finite group G. In the case where $(|G|, |A|) = 1$, there are several very useful relations between the groups G and A, some of which are as follows: (i) $G = [G, A] \cdot C_G(A)$, (ii) $[G, A, A] = [G, A]$ and (iii) $C_{G/N}(A) = C_G(A)N/N$ for any A-invariant normal subgroup N of G. Almost all of the research papers studying this kind of action concerned with the situations where the fixed point subgroup $C_G(A)$ has a restricted structure. However, Parker and Quick [1] considered a dual situation by assuming that the index of $C_G(A)$ is bounded. As this assumption clearly gives no restriction to $C_G(A)$, they focused their attention on the group $[G, A]$ and proved that $|[G, A]| \leqslant n^{\log_2(n+1)}$ if $|G : C_G(A)| \leqslant n$.

We consider here a special noncoprime action in view of [1]:

Let α be an automorphism of the finite group G such that for every $x \in G$, $x = [g, \alpha] \cdot z$ for some $g \in G$ and $z \in C_G(\alpha)$.

In the literature a finite group G admitting such an automorphism α is called an *α-CCP group* where the acronym CCP stands for "*centralizer commutator product*". Lemma 2.1 below shows that nice relations indicated above which are valid in the case of a coprime action also survive in the setting of α-CCP groups. The study of α-CCP groups was started

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by Stein [2] who proved that the subgroup $[G, \alpha]$ is solvable. The goal of the present paper is to give an upper bound for the order of $[G, \alpha]$ in terms of the index of $C_G(\alpha)$ in G. Namely, we prove the following:

Theorem A *Let G be an α-CCP group such that $|G : C_G(\alpha)| \leqslant n$. Then $|[G, \alpha]| \leqslant n^{\log_2(n+1)}$.*

An internal reformulation of Theorem A can be stated as

Theorem B *Let H be a finite group containing an element x such that $H = \big\{[h, x] : h \in H\big\} \cdot C_H(x)$. If $|H : C_H(x)| \leqslant n$ then $|[H, x]| \leqslant n^{log_2(n+1)}$.*

Theorem A is the α-CCP analogue of [1, TheoremA]. The key lemma in our proof is Lemma 3.1 which we obtain as the α-CCP analogue of [1, Lemma 2.1]. The rest of the paper contains the proof of Theorem A and some technical results pertaining to the proof of Theorem A; all of which are proven in a similar fashion as in the proofs of [1, Proposition 2.2], [1, Corollary 2.3] and [1, TheoremA] with obvious changes, namely using Lemma 3.1 instead of [1, Lemma 2.1]. For the sake of completeness we present a proof here for each of them.

In Sect. 2 we state and prove some preliminary facts about α-CCP groups. Section 3 is concerned with our key lemma, namely Lemma 3.1, and its consequences. We prove our main result Theorem A and its equivalent Theorem B in Sect. 4.

All groups are assumed to be finite. The notation and terminology are standard.

2 Preliminaries on α-CCP groups

Lemma 2.1 *The following hold for any α-CCP group G.*

(i) *$G = [G, \alpha] \cdot C_G(\alpha)$ and $[G, \alpha, \alpha] = [G, \alpha]$. Furthermore $G = [G, \alpha] \times C_G(\alpha)$ whenever G is abelian.*
(ii) *Every α-invariant subgroup S of G is also an α-CCP group and we have $\big\{[x, \alpha] : x \in S\big\} = \big\{[g, \alpha] : g \in G\big\} \cap S$.*
(iii) *G/N is an α-CCP group for any α-invariant normal subgroup N of G.*
(iv) *If $[g, \alpha]^f \in C_G(\alpha)$ for some f and g in G, then $g \in C_G(\alpha)$.*
(v) *$C_{G/N}(\alpha) = C_G(\alpha)N/N$ for any α-invariant normal subgroup N of G.*
(vi) *$\big\{\, [g, \alpha] : g \in G \,\big\}$ is a transversal to $C_G(\alpha)$. Furthermore α^G is a transversal to $C_H(\alpha^a)$ for any $a \in G$ in the semidirect product $H = G\langle\alpha\rangle$.*

Proof This lemma gives almost the same information as in [2, Proposition 2.2] on an α-CCP group G. We need only to show that $G = [G, \alpha] \times C_G(\alpha)$ when G is abelian: Notice that $[G, \alpha] = \big\{\, [g, \alpha] : g \in G \,\big\}$ when G is abelian and also observe that for any $[g, \alpha] \in C_G(\alpha)$, we have $[g, \alpha] = 1$ by Lemma 2.1(iv). □

The following lemma is crucial in proving our key lemma Lemma 3.1.

Lemma 2.2 *Let G be an α-CCP group and set $H = G\langle\alpha\rangle$. Then*

(i) *the map $f_{\alpha^a} : \alpha^G \longrightarrow \alpha^G$ defined by $f_{\alpha^a}(\alpha^g) = (\alpha^a)^{\alpha^g}$ is a bijection for any $a \in G$,*
(ii) *for any $X \leqslant H$ with $X \cap \alpha^G \neq \phi$ and for any $\alpha^a \in X$ we have*

$$(\alpha^a)^X = (\alpha^a)^{X \cap \alpha^G} = X \cap \alpha^G.$$

Proof α^G is a transversal to $C_H(\alpha^a)$ by Lemma 2.1(vi). If g and h are elements of G such that $(\alpha^a)^{\alpha^g} = (\alpha^a)^{\alpha^h}$, then $\alpha^g(\alpha^h)^{-1} \in C_H(\alpha^a)$ and so $\alpha^g = \alpha^h$. This proves (i) since α^G is finite.

It is straightforward to verify that $(\alpha^a)^{X\cap\alpha^G} \subseteq (\alpha^a)^X \subseteq X \cap \alpha^G$. If $\alpha^y \in X \cap \alpha^G$, then $\alpha^y = (\alpha^a)^{\alpha^h}$ for some $h \in G$ by part (i). This yields $\alpha^y \in (\alpha^a)^{\alpha^G}$. Notice that $f_{\alpha^a}(X\cap\alpha^G) \subseteq X\cap\alpha^G$ as $\alpha^a \in X$, and so $f_{\alpha^a}(X\cap\alpha^G) = X\cap\alpha^G$ since f_{α^a} is a bijection. Then $\alpha^h \in X$ and hence $X\cap\alpha^G \subseteq (\alpha^a)^{X\cap\alpha^G}$ which establishes the claim (ii). □

3 Some technical lemmas pertaining to the proof of Theorem A

The following results are modifications of Lemma 2.1, Proposition 2.2 and Corollary 2.3 in [1] for α-CCP groups.

Lemma 3.1 *Let G be an α-CCP group and let $\mathcal{O} = \alpha^G$. If $I \subseteq \mathcal{O}$ and Θ is an orbit of $\langle I\rangle$ on $\mathcal{O}$, then $\langle I\rangle \leqslant \langle\Theta\rangle$. Furthermore if some member of Θ is not contained in $\langle I\rangle$, then $\langle I\rangle < \langle\Theta\rangle$.*

Proof To ease the notation set $K = \langle I\rangle$ and let $\Theta = (\alpha^x)^K$. It should be noted that $K\langle\Theta\rangle$ is a subgroup of G because K normalizes $\langle\Theta\rangle$. Set now $L = K\langle\Theta\rangle$. Since $L\cap\alpha^G \neq \phi$ and $\alpha^x \in L$, we have

$$(\alpha^x)^L = (\alpha^x)^{L\cap\alpha^G} = L\cap\alpha^G$$

by Lemma 2.2(ii). Then, for any generator α^y of K, we have

$$\alpha^y \in L\cap\alpha^G = (\alpha^x)^{K\langle\Theta\rangle} \subseteq \langle(\alpha^x)^K\rangle = \langle\Theta\rangle.$$

This completes the proof. □

Lemma 3.2 *When G is an α-CCP group the group $\langle\alpha^G\rangle$ can be generated by $\log_2\left(\frac{2(n+p-1)}{p}\right)$ conjugates of α where p is the smallest positive divisor of the order of α and $|G : C_G(\alpha)| \leqslant n$.*

Proof We let $\mathcal{O} = \alpha^G$ and consider the action of $\langle\alpha\rangle$ on $\mathcal{O}$ by conjugation. Suppose first that $\langle\alpha\rangle$ has a fixed point α^x which is different from α. Then $[\alpha, x] \in C_G(\alpha)$ and hence $[\alpha, x] = 1$ by Lemma 2.1(iv). This contradiction shows that α is the only fixed point of $\langle\alpha\rangle$ in its action on $\mathcal{O}$.

Define $K_0 = 1$, $K_1 = \langle\alpha\rangle$ and for $j > 1$, $K_j = \langle K_{j-1}, \alpha_j\rangle$ where at each stage $\alpha_j \in \mathcal{O}$ is chosen to maximize the order of K_j. Since G is finite, there exists k such that $K_k = \langle\alpha^G\rangle$ and $K_{k-1} \neq \langle\alpha^G\rangle$. Now $\langle\alpha^G\rangle = \langle\alpha_1, \alpha_2, \ldots, \alpha_k\rangle$ where $\alpha_1 = \alpha$. Fix $j \in \{1, \ldots, k\}$ and let $I = \{\alpha_1, \alpha_2, \ldots, \alpha_j\}$. Now $K_j = \langle I\rangle$. Choose an orbit Θ of K_j with representative B where $B \nleq K_j$. Then $K_j < \langle\Theta\rangle$ by Lemma 3.1. If Θ were also an orbit of K_{j-1}, then we would have

$$K_j < \langle B^{K_j}\rangle = \langle B^{K_{j-1}}\rangle \leqslant \langle B, K_{j-1}\rangle$$

contradicting the choice of α_j. Therefore Θ is a union of at least two orbits of K_{j-1} on $\mathcal{O}$. Notice also that $B \nleq K_i$ for each $i = 1, \ldots, j$. Thus Θ is a union of at least 2^{j-1} orbits of $\langle\alpha\rangle$ on $\mathcal{O}$, each of which has length at least p. Since $\alpha_j \leqslant K_j$ for $i \leqslant j$ we see that the set $\Omega = \{\alpha_1\} \cup \{{\alpha_2}^{K_1}\} \cup \ldots \cup \{{\alpha_i}^{K_{i-1}}\}$ is contained in K_j. Therefore $\Omega \cap {\alpha_{i+1}}^{K_i}$ is empty as

$\alpha_{i+1} \not\leq K_i$. Then $\mathcal{O} \supseteq \{\alpha_1\} \cup \{\alpha_2{}^{K_1}\} \cup \ldots \cup \{\alpha_k{}^{K_{k-1}}\}$ and the right hand side is a disjoint union. So

$$n \geqslant |\mathcal{O}| \geqslant 1 + p(1 + 2 + \cdots + 2^{k-2}) = 1 + p(2^{k-1} - 1).$$

Consequently we have $k - 1 \leqslant \log_2\left(\frac{n+p-1}{p}\right)$ as claimed. □

Lemma 3.3 *Let G be an α-CCP group. Suppose that G is a p-group for some prime p with $|G : C_G(\alpha)| \leqslant p^m$. Then $|[G, \alpha]| \leqslant p^{\frac{m^2+m}{2}}$.*

Proof Firstly we handle the case where G is of class at most two by induction on the order of G. By Lemma 2.1(i) we have $[G, \alpha] = [G, \alpha, \alpha]$ and $G/G' = [G/G', \alpha] \times C_{G/G'}(\alpha)$. Then $G = [G, \alpha]$ by induction and hence $C_{G/G'}(\alpha) = 1$, that is $C_G(\alpha) \leqslant G'$. Thus $|G : G'| \leqslant p^m$. In this case the proof is in a similar fashion as in the proof of [1, Proposition 3.1]. For the sake of completeness we present it here. Let the abelian group $\bar{G} = G/G'$ be the direct product of nontrivial cyclic subgroups $\langle \bar{x_i} \rangle$ for $i = 1, \ldots, d$ where $|\bar{x_i}| = p^{m_i}$. We have $G = \langle x_1, \ldots, x_d \rangle$ since $G' \leqslant \Phi(G)$. It is straightforward now to verify that $G' = \langle [x_j, x_i] : 1 \leqslant i < j \leqslant d \rangle$ since $G' \leqslant Z(G)$. Set $H_i = \langle x_{i+1}, \ldots, x_d, G' \rangle$. Then $G' = \prod_{i=1}^{d-1} [H_i, x_i]$ for $i = 1, \ldots, d-1$. We have $|[H_i, x_i]| \leqslant |H_i/G'| = p^{m_{i+1}+\cdots+m_d}$ due to the fact that $h \longmapsto [h, x_i]$ defines a homomorphism from H_i/G' onto $[H_i, x_i]$. Thus $|G| \leqslant \prod_{i=1}^{d} p^{m_i} \prod_{i=1}^{d-1} |[H_i, x_i]| \leqslant p^M$ where $M = \sum_{i=1}^{d} i m_i$. It can be proven by induction on d that $M \leqslant (m^2 + m)/2$. This completes the proof when G is of class at most two.

Suppose now that G has class c with $c \geqslant 3$. Again assume $|G|$ minimal, therefore $G = [G, \alpha]$. The proof in this case is in a very similar fashion as in the proof of [1, Theorem B]. Note that $\gamma_{c-1}(G)$ is abelian. We also observe that $[\gamma_{c-1}(G), \alpha] \neq 1$, because otherwise $[\gamma_{c-1}(G), \alpha, G] = 1 = [G, \gamma_{c-1}(G), \alpha]$ and hence $\gamma_{c-1}(G) \leqslant Z(G)$ by the Three Subgroup Lemma. Let now N be of minimal order among all normal α-invariant subgroups of G contained in $\gamma_{c-1}(G)$ and are not centralized by α. Let $|G/N : C_{G/N}(\alpha)| = p^r$. As $C_{G/N}(\alpha) = C_G(\alpha)N/N$ by Lemma 2.1(v) we have $|G : C_G(\alpha)N| = p^r$. Note that G/N and N are both α-CCP groups by Lemma 2.1(i). It follows by induction that $|[G/N, \alpha]| \leqslant p^{\frac{r^2+r}{2}}$. As $[G/N, \alpha] = [G, \alpha]N/N = G/N$ we have $|G/N| \leqslant p^{\frac{r^2+r}{2}}$. Let now $|N : C_N(\alpha)| = p^s$. Since N is abelian we have $N = [N, \alpha] \times C_N(\alpha)$ and so $|[N, \alpha]| = p^s$. It remains to bound $|N/[N, \alpha]|$ suitably. As N is contained in $\gamma_{c-1}(G)$ we have $[N, G] \leqslant \gamma_c(G) \leqslant Z(G)$. Hence for $g \in G$ the map $x \mapsto [x, g]$ for $x \in N$, is a homomorphism with kernel $C_N(g)$, in particular $[N, G]$ lies in the kernel and $|N : C_N(g)| = |[N, g]|$. Set now $H = [N, \alpha][N, G]$. Observe that $1 \neq [N, \alpha] = [N, \alpha, \alpha] \leqslant [H, \alpha]$ by Lemma 2.1(i). It follows by minimality of N that $H = N$. Thus

$$|[N, g]| = |N : C_N(g)| \leqslant |N : [N, G]| = |[N, \alpha][N, G] : [N, G]| \leqslant |[N, \alpha]|.$$

We also observe that $[N, G'] = 1$ by the three subgroup Lemma as $[N, G, G] = 1 = [G, N, G]$. This gives that $NC_G(\alpha) \leqslant G' \leqslant C_G(N)$. As $N \leqslant \gamma_{c-1}(G) \leqslant G'$ we get $NC_G(\alpha) \leqslant G' \leqslant C_G(N)$. Therefore $|G : C_G(N)| \leqslant p^r$. Let Y be a minimal generating set for G modulo $C_G(N)$. Then $|Y| \leqslant r$. Since $[N, G] \leqslant Z(G)$ we also see that $[N, G] = \prod_{y \in Y} [N, y]$. Thus $|[N, G]| \leqslant |[N, \alpha]|^{|Y|} \leqslant p^{sr}$. So $|N| = |[N, G][N, \alpha]| \leqslant p^{s(r+1)}$ whence $|G| = |G/N| \cdot |N| \leqslant p^{(r^2+r)+s(r+1)}$. This establishes the claim as

$$\begin{aligned} 1/2\big((r^2 + r) + s(r + 1)\big) &\leqslant 1/2(r^2 + r) + 1/2(s^2 + s) + sr \\ &\leqslant 1/2\left((r + s)^2 + r + s\right) \leqslant 1/2(m^2 + m). \end{aligned}$$

□

4 Proof of the main results

In this section we present a proof of Theorem A and deduce Theorem B.

Proof of Theorem A Let G be a minimal counterexample to the theorem. Then $G = [G, \alpha]$ by induction as $[G, \alpha] = [G, \alpha, \alpha]$ by Lemma 2.1(i). As a consequence $C_G(\alpha) \leqslant G'$, and G is nonabelian. The main result of [2] gives that the group G is solvable and hence $F(G) \neq 1$. If $[F(G), \alpha] = 1$, then $G \leqslant C_G(F(G)) = Z(F(G))$ by the Three Subgroup Lemma, which is a contradiction as G is nonabelian. Thus $[F(G), \alpha] \neq 1$ and hence there is a prime p dividing $|F(G)|$ such that $\left[O_p(G), \alpha\right] \neq 1$. Notice that if $\left[Z_2(O_p(G)), \alpha\right] = 1$, then $Z_2\left(O_p(G)\right) \leqslant Z(G)$ by the Three Subgroup Lemma as $[G, \alpha] = G$. This forces $O_p(G) = Z_2(O_p(G)) = Z(O_p(G))$ which contradicts the fact that $\left[O_p(G), \alpha\right] \neq 1$. Let Q be minimal element of the set $\{S : S$ is a normal α-invariant subgroup of G which is contained in $Z_2\left(O_p(G)\right)$ such that $[S, \alpha] \neq 1\}$. Clearly $[Q', \alpha] = 1$ by the minimality of Q and so $Q' \leqslant Z(G)$ by the Three Subgroup Lemma. Set now $Q_0 = \langle [Q, \alpha]^G \rangle$. Note that both Q and $|G/Q|$ are α-CCP groups. So we have $[Q, \alpha] = [Q, \alpha, \alpha]$ by Lemma 2.1(i). Thus $1 \neq \left[[Q, \alpha]\right] \leqslant [Q_0, \alpha]$ and hence $Q = Q_0$ by the minimality of Q. Now $|QC_G(\alpha) : C_G(\alpha)| = |Q : C_Q(\alpha)| = p^m$ for some m. Let $|G : QC_G(\alpha)| = r$. Then $r \leqslant \frac{n}{p^m}$. We observe by Lemma 3.3 that $|[Q, \alpha]| \leqslant p^{\frac{m^2+m}{2}}$. Set $R = C_{[Q,\alpha]}(\alpha)$. Then $R \leqslant [Q, \alpha]' \leqslant Q'$ and hence $R \leqslant Z(G)$. Now

$$|[Q, \alpha]/R| = |[Q, \alpha]C_Q(\alpha) : C_Q(\alpha)| = |Q : C_Q(\alpha)| = p^m.$$

So $|R| = \frac{|[Q,\alpha]|}{p^m} \leqslant p^{\frac{m^2-m}{2}}$. It remains to bound $|G/R|$ suitably.

Set $\bar{G} = G/R$. The group $\bar{Q}$ is the product of at most $log_2(r+1)$ of the conjugates of $[\bar{Q}, \alpha]$ in $\bar{G}$: To see this let $H = G \rtimes \langle\alpha\rangle$. Note that $Q \rtimes H$ and $C_G(\alpha)\langle\alpha\rangle Q \leqslant N_H(Q\langle\alpha\rangle)$. Set $\tilde{H} = H/Q$. Now $|\tilde{H} : N_{\tilde{H}}(\langle\tilde{\alpha}\rangle)| \leqslant |H : Q\langle\alpha\rangle C_G(\alpha)| = r$. By Lemma 3.2 $\langle\langle\tilde{\alpha}\rangle^{\tilde{H}}\rangle$ can be generated by at most $k = log_2(r+1)$ conjugates of $\langle\tilde{\alpha}\rangle$. That is $\langle(\langle\tilde{\alpha}\rangle)^{\tilde{H}}\rangle = \langle\tilde{\alpha_1}, \ldots, \tilde{\alpha_k}\rangle$ where each α_i is a conjugate of α and $\alpha_1 = \alpha$. Note that $H = [G, \alpha]\langle\alpha\rangle = \langle\alpha^H\rangle C_G(\alpha) = MQC_G(\alpha)$ where $M = \langle\alpha_1, \ldots, \alpha_k\rangle C_G(\alpha)$. Therefore

$$\langle [Q, \alpha]^G \rangle = \langle [Q, \alpha]^M \rangle = [Q, \alpha][Q, \alpha, M] \leqslant [Q, M] = \prod_{i=1}^{k}[Q, \alpha_i].$$

We are now ready to complete the proof of Theorem A. By the above paragraph we have $|\bar{Q}| = |\langle[\bar{Q}, \alpha]^{\bar{G}}\rangle| \leqslant |[\bar{Q}, \alpha]|^k = p^{mk}$ and so $|Q| \leqslant p^{mk+(\frac{m^2-m}{2})}$. Notice that $|G/Q| \leqslant r^k$ by induction. Thus

$$|G| = |G/Q||Q| \leqslant r^k p^{mk+\frac{m^2-m}{2}} = r^k (p^m)^{k+\frac{m-1}{2}} \leqslant r^k (p^m)^{log_2(n+1)} \leqslant n^{log_2(n+1)}.$$

This contradiction completes the proof of Theorem A. □

Remark 4.1 As indicated in the introduction one can reformulate Theorem A as Theorem B. Their equivalence can be easily seen as follows:

Suppose that Theorem A is true. Set $H = G \rtimes \langle\alpha\rangle$ and $x = \alpha$ in H. Then $[G, \alpha] = [H, x]$ and $\left\{[g, \alpha] : g \in G\right\} = \left\{[h, x] : h \in H\right\}$ and $|G : C_G(\alpha)| = |H : C_H(x)| = n$. Therefore $|[G, \alpha]| = |[H, x] \leqslant n^{log_2(n+1)}$ by Theorem A. Conversely suppose that Theorem B is true and let H be a finite group containing an element x such that $H = \left\{[h, x] : h \in H\right\}C_H(x)$ holds. Set $G = H$ and let α denote the inner automorphism of G induced by x. Then by applying Theorem B we have $|[H, x]| \leqslant n^{log_2(n+1)}$ as desired.

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