Rend. Circ. Mat. Palermo (2015) 64:209–220
DOI 10.1007/s12215-015-0191-0

Complete controllability of semi-linear stochastic system with delay

Anurag Shukla · N. Sukavanam · D. N. Pandey

Received: 11 December 2014 / Accepted: 27 January 2015 / Published online: 14 February 2015

Abstract This paper deals with the complete controllability of semilinear stochastic system with delay under the assumption that the corresponding linear system is completely controllable. The control function for this system is suitably constructed by using the controllability operator. With this control function, the sufficient conditions for the complete controllability of the proposed problem in finite dimensional are established. The results are obtained by using Banach fixed point theorem. Finally, one example is provided to illustrate the application of the obtained results.

Keywords Complete controllability · Semilinear systems · Stochastic control system · Control delayed

Mathematics Subject Classification 34A34 · 34K35 · 93B05

1 Introduction

Controllability concepts play a vital role in deterministic control theory. It is well known that controllability of deterministic equation is widely used in many fields of science and technology. But in many practical problems such as fluctuating stock prices or physical system subject to thermal fluctuations, Population dynamics etc, some randomness appear, so the system should be medelled stochastic form.

In setting of deterministic systems: Kalman [1] introduced the concept of controllability for finite dimensional deterministic linear control systems. The basic concepts of control theory in finite dimensional spaces has been introduced in [2]. In [3] Naito established sufficient conditions for approximate controllability of deterministic semilinear control system dominated by the linear part using Schuder's fixed point theorem. Balachandran et al. [4] obtained results for controllability of nonlinear systems in Banach spaces. In [5,6] Wang extended

A. Shukla (✉) · N. Sukavanam · D. N. Pandey
Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee 247667, India
e-mail: anuragshukla259@gmail.com

the results of [3] and established sufficient conditions for delayed deterministic semilinear systems using Schauder's fixed point theorem and concept of fundamental solution. In [7,8] Sukavanam et al. obtained the results for approximate controllability of a delayed semilinear control system with growing nonlinear term using Schauder's fixed point theorem.

In setting of stochastic systems: in [9] Bashirov et al. provides some concepts for controllability of linear stochastic systems. Using these concepts Mahmudov [10] established sufficient conditions for controllability of linear stochastic systems in Hilbert spaces. In [11–15] Klamka obtained some sufficient conditions for controllablity of delay linear systems in finite dimensional using Rank theorem. In [16,17] Mahmudov obtained results for controllability of semilinear stochastic systems using Banach fixed point theorem. Shen et al. [18] extended the results of [12] in infinite dimensional using technique of [16] and obtained sufficient conditions for Relative controllability of stochastic nonlinear systems with delay in control. In [19] Sukavanam et al. obtained some results for stochastic controllability of an abstract first order semilinear control system using Schauder's fixed point theorem. Recently Shukla et al. [20] obtained some sufficient conditions for approximate controllability of retarded semilinear stochastic system with non local conditions using Banach fixed point theorem. However in best of our knowledge, there is no result on complete controllability of semilinear stochastic system with delay as treated in this paper. In this paper results are obtained in L_2 norm and delay is considered in both state and control term simultaneously which in not previously discussed up to now in the literature.

In this paper we adopt the following notations:

(i) $(\Omega, \mathcal{F}, P)$:Let $\mathcal{F}$ be the σ algebra generated by $\Omega \subset R^n$ and $P : \mathcal{F} \to [0, 1]$ be the probability measure on $\mathcal{F}$. Then the triple $(\Omega, \mathcal{F}, P)$ is called a probability space.

(ii) Let ω be the Wiener process and $\{\mathcal{F}_t | t \in [0, T]\}$ is the filtration generated by $\{\omega(s) : 0 \le s \le t\}$.

(iii) $L_2(\Omega, \mathcal{F}_T, R^n)$=the Hilbert space of all $\mathcal{F}_T$-measurable square integrable variables with values in R^n.

(iv) $L_P^{\mathcal{F}}([0, T], R^n)$ is the Banach space of all p-integrable and $\mathcal{F}_t$-measurable processes with values in R^n, for $p \ge 2$.

(v) X_2=the Banach space of all square integrable and $\mathcal{F}_t$-adapted processes $\varphi(t)$ with norm

$$||\varphi||^2 = \left(\int_{-h}^{t} \mathbf{E}||\varphi(t)||^2 \right), \quad where \ \ \mathbf{E} \ \ is \ \ Expected \ \ value.$$

(vi) $\mathbb{L}(X, Y)$ is the space of all linear bounded operators from a Banach space X in to a Banach space Y.

(vii) Let set of admissible controls is $U_{ad} = L_2^{\mathcal{F}}([0, T], R^m)$. □

Complete controllability of linear stochastic system with control delay

$$dx(t) = [Ax(t) + B_0 u(t) + B_1 u(t - h)]dt + \sigma d\omega(t) \tag{1.1}$$

given the initial condition as a random function

$$x(0) = x_0 \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n) \quad and \quad u(t) = 0 \quad for \quad t \in [-h, 0)$$

has been studied by various authors under Supremum norm. (see Klamka [12] and the references there in).

The problem of controllability of semi-linear stochastic system

$$dx(t) = [Ax(t) + Bu(t) + f(t, x(t))]dt + \sigma(t, x(t))d\omega(t)$$
$$x(0) = x_0 \in R^n$$

has been studied by various authors under Supremum norm. (see Mahmudov [16] and Sukavanam [8])

In this paper we examine the complete controllability of the following semi-linear stochastic system with delay in L_2 norm:

$$dx(t) = (Ax(t) + B_0u(t) + B_1u(t-h) + f(t, x_t))dt + \sigma(t, x_t)d\omega(t) \quad \text{for t } \in (0, T] \tag{1.2}$$

$$x(t) = \psi(t), \quad for\ t \in [-h, 0], \quad x(0) = \psi(0) = x_0(let) \tag{1.3}$$

$$u(t) = 0, \quad for\ \in [-h, 0] \tag{1.4}$$

where the state $x(t) \in L_2(\Omega, \digamma_t, R^n) = X$ and the control $u(t) \in L_2^{\digamma}([0, T], R^m) = U$, A is $n \times n$ constant matrix, B_0 and B_1 are an $n \times m$ constant matrices .$x_t \in L^2([-h, 0], R^n)$-valued stochastic processes and defined as $x_t(s) = \{x(t+s)| - h \leq s \leq 0|\}$ and $\psi = \{\psi(s)| - h \leq s \leq 0\} \in L^2([-h, 0], R^n)$-valued stochastic processes and $h \geq 0$ is the upper bound for the time delay.Moreover, the functions $f(.,.), \sigma(.,.)$ are defined as $\sigma : [0, T] \times L^2([-h, 0], R^n) \rightarrow R^{n \times n}$, $f : [0, T] \times L^2([-h, 0], R^n) \rightarrow R^n$ are nonlinear functions and ω is a n-dimensional Wiener process.

2 Preliminaries

It is well known [8,16] that for given initial conditions (1.3), (1.4), any admissible control $u \in U_{ad}$, for $t \in [-h, T]$ and suitable nonlinear functions $f(t, x_t)$ and $\sigma(t, x_t)$ there exists unique solution $x(t; x_0, u) \in L_2(\Omega, \digamma_t, R^n)$ of the semilinear stochastic differential state equation (1.2) which can be represented in the following integral form

$$x(t; x_0, u) = \begin{cases} exp(At)x_0 + \int_0^t exp(A(t-s))(B_0u(s) + B_1u(s-h) + f(s, x_s))ds \\ \quad + \int_0^t exp(A(t-s))\sigma(s, x_s)d\omega(s) \quad for \quad t > 0 \\ \psi(t) \quad for \quad t \in [-h, 0] \end{cases} \tag{2.1}$$

Since $u(t) = 0$ for $t \in [-h, 0]$ therefore solution for $t \in [0, h]$ has the following form

$$x(t; x_0, u) = exp(At)x_0 + \int_0^t \exp(A(t-s))(B_0u(s) + f(s, x_s)ds + \int_0^t exp(A(t-s))\sigma(s, x_s)d\omega(s) \tag{2.2}$$

for $t > h$ we have

$$\begin{aligned} x(t; x_0, u) = exp(At)x_0 &+ \int_0^t exp(A(t-s))(B_0u(s) + f(s, x_s))ds \\ &+ \int_0^{t-h} exp(A(t-s-h))B_1u(s)ds \\ &+ \int_0^t exp(A(t-s))\sigma(s, x_s)d\omega(s) \end{aligned}$$

or equivalent

$$x(t;x_0,u) = exp(At)x_0 + \int_0^{t-h}(exp(A(t-s))B_0 + exp(A(t-s-h))B_1)u(s)ds$$
$$+\int_{t-h}^{t} exp(A(t-s))B_0u(s)ds$$
$$+\int_0^t exp(A(t-s))f(s,x_s)ds + \int_0^t exp(A(t-s))\sigma(s,x_s)d\omega(s) \quad (2.3)$$

for $T > h$, let us introduce the following operators and sets.

$L_T \in \mathbb{L}((L_2^{\mathcal{F}}([0,T]),R^m),L_2(\Omega,\mathcal{F}_T,R^n))$, defined by

$$L_Tu = \int_0^{T-h}(exp(A(T-s))B_0 + exp(A(T-s-h))B_1)u(s)ds$$
$$+\int_{T-h}^{T} exp(A(T-s))B_0u(s)ds$$

Then it can be seen that the adjoint operator $L_T^* \in L_2(\Omega,\mathcal{F}_T,R^n) \to L_2^{\mathcal{F}}([0,T]),R^m)$ is given by

$$L_T^*z = \begin{cases} (B_0^*exp(A^*(T-t)) + B_1^*exp(A^*(T-t-h)))E\{z|\mathcal{F}_t\} & \text{for t } \in [0,T-h] \\ B_0^*exp(A^*(T-t))E\{z|\mathcal{F}_t\} & \text{for t } \in (T-h,T] \end{cases}$$

The set of all states reachable in time T from initial state $x(0) = x_0 \in L_2(\Omega,\mathcal{F}_T,R^n)$, using admissible controls is defined as

$$R_T(U_{ad}) = \{x(T;x_0,u) \in L_2(\Omega,\mathcal{F}_T,R^n) : u \in U_{ad}\}$$
$$where \quad x(T;x_0,u) = exp(At)x_0 + L_Tu + \int_0^T exp(A(T-s))(f(s,x_s)ds + \sigma(s,x_s)d\omega(s))$$

Now we introduce the linear controllability operator $\Pi_0^T \in \mathbb{L}(L_2(\Omega,\mathcal{F}_T,R^n),L_2(\Omega,\mathcal{F}_T,R^n))$ as follows:

$$\Pi_0^T\{.\} = L_T(L_T)^*\{.\}$$
$$= \int_0^{T-h}(exp(A(T-t))B_0B_0^*exp(A^*(T-t)) + exp(A(T-t-h))$$
$$\times B_1B_1^*exp(A^*(T-t-h)))E\{.|\mathcal{F}_t\}dt$$
$$+\int_{T-h}^{T}(exp(A(T-t))B_0B_0^*exp(A(T-t)))E\{.|\mathcal{F}_t\}dt$$

The corresponding controllability $n \times n$ matrix for deterministic model is:

$$\Gamma_s^T = L_T(s)L_T^*(s)$$
$$= \int_s^{T-h}(exp(A(T-t))B_0B_0^*exp(A^*(T-t)) + exp(A(T-t-h))$$
$$\times B_1B_1^*exp(A^*(T-t-h)))dt$$
$$+\int_{T-h}^{T} exp(A(T-t))B_0B_0^*exp(A^*(T-t))dt$$

Definition 2.1 A control system is said to be completely controllable in the interval $I = [0, T]$ if for every initial state x_0 and desired final state x_1, there exists a control $u(t)$ such that the solution $x(t)$ of the system corresponding to this control u satisfies $x(T) = x_1$.

Remark 2.1 For dynamical system (1.2) it is possible to define many different concepts of controllability. Using this admissible controls in [21,22] Klamka obtained complete controllability with constrained admissible controls of nonlinear systems. It is generally assumed that the control values are in a convex and closed cone with vertex at zero, or in a cone with nonempty interior. Klamka obtained sufficient conditions for constrained exact local controllability using the generalized open mapping theorem. Let $U_0 \subset U$ be a closed convex cone with nonempty interior. The set of admissible controls for the system (1.2) is given by $U_{ad} = L_\infty([0, T], U_0)$ (for more detail see [21,22]).

In this paper some sufficient conditions for complete controllability with unconstrained admissible controls of system (1.2) is obtained. Unconstrained admissible control for the system (1.2) in this paper is defined in notation (vii).

3 Main results

Lemma 1 *Assume that the operator Π_0^T is invertible. Then for arbitrary $x_T \in L_2(\Omega, F_T, R^n)$, $f(.,.) \in L_2^F([0, T], R^n)$, $\sigma(.,.) \in (L_2^F([0, T], R^{n\times n})$, the control*

$$u(t)=\begin{cases} B_0^* exp(A^*(T-t)) \\ \times E\left\{(\Pi_0^T)^{-1}\left(x_T - exp(AT)x_0 - \int_0^{T-h} exp(A(T-s))(f(s,x_s)ds + \sigma(s,x_s)d\omega(s))\right)|F_t\right\} & for\ t\in[0,h] \\ (B_0^* exp(A^*(T-t)) + B_1^* exp(A^*(T-h-t))) \\ \times E\left\{(\Pi_0^T)^{-1}\left(x_T - exp(AT)x_0 - \int_{T-h}^{T} exp(A(T-s))(f(s,x_s)ds + \sigma(s,x_s)d\omega(s))\right)|F_t\right\} & for\ t\in(h,T] \end{cases}$$

transfers the system (2.1) from $x_0 \in R^n$ to x_T at time T and

$$\begin{aligned} x(t) &= \exp(At)x_0 + \Pi_0^t[exp(A^*(T-t))(\Pi_0^T)^{-1} \times (x_T - exp(AT)x_0 \\ &\quad - \int_0^T exp(A(T-r))f(r,x_r)dr - \int_0^T exp(A(T-r))\sigma(r,x_r)d\omega(r))] \\ &\quad + \int_0^t exp(A(t-s))f(s,x_s)ds + \int_0^t exp(A(t-s))\sigma(s,x_s)d\omega(s) \end{aligned} \tag{3.1}$$

provided the solution of (3.1) exists.

Proof By substituting $u(t)$ in (2.2) and (2.3), we can easily obtain the following

For $t \in [0, h]$

$$\begin{aligned} x(t; x_0, u) &= exp(At)x_0 + \int_0^t exp(A(t-s))B_0B_0^* exp(A^*(t-s)) \\ &\quad \times E\left\{(\Pi_0^T)^{-1}\left(x_T - exp(AT)x_0 - \int_0^{T-h} exp(A(T-s))(f(s,x_s)ds\right.\right. \\ &\quad \left.\left. + \sigma(s,x_s)d\omega(s))\right)|F_s\right\}ds + \int_0^t exp(A(t-s))f(s,x_s)ds \\ &\quad + \int_0^t exp(A(t-s))\sigma(s,x_s)d\omega(s) \end{aligned}$$

For $t \in (h, T]$

$$
\begin{aligned}
x(t; x_0, u) = exp(At)x_0 + \int_0^{t-h} & (exp(A(t-s))B_0 B_0^* exp(A^*(t-s)) \\
& + exp(A(T-h-s))B_1 B_1^* exp(A^*(T-h-s))) \\
& \times E\Big\{(\Pi_0^T)^{-1}\Big(x_T - exp(AT)x_0 - \int_{T-h}^{T} exp(A(T-s))(f(s, x_s)ds \\
& + \sigma(s, x_s)d\omega(s)))\,|F_s\Big\}ds + \int_{t-h}^{t} exp(A(t-s))B_0 B_0^* exp(A^*(T-s)) \\
& \times E\Big\{(\Pi_0^T)^{-1}\Big(x_T - exp(AT)x_0 - \int_0^{T-h} exp(A(T-s))(f(s, x_s)ds \\
& + \sigma(s)d\omega(s, x_s)))\,|F_s\Big\}ds + \int_0^t exp(A(t-s))f(s, x_s)ds \\
& + \int_0^t exp(A(t-s))\sigma(s, x_s)d\omega(s)
\end{aligned}
$$

Thus, taking into account of the form of the operator Π_0^T [12,19] we have

$$
\begin{aligned}
x(t; x_0, u) = exp(At)x_0 + \Pi_0^t & \Big[exp(A^*(T-t))(\Pi_0^T)^{-1}\Big(x_T - exp(AT)x_0 \\
& - \int_0^T exp(A(T-s))(f(s, x_s)ds + \sigma(s, x_s)d\omega(s))\Big)\Big] \\
& + \int_0^t exp(A(t-s))f(s, x_s)ds + \int_0^t exp(A(t-s))\sigma(s, x_s)d\omega(s)
\end{aligned}
$$

Substitute $t = T$ in above equation we get

$$
\begin{aligned}
x(T; x_0, u) = exp(AT)x_0 + \Pi_0^T & \Big[(\Pi_0^T)^{-1} \\
& \times \Big(x_T - exp(AT)x_0 - \int_0^T exp(A(T-s))(f(s, x_s)ds + \sigma(s, x_s)d\omega(s)\Big)\Big] \\
& + \int_0^T exp(A(T-s))f(s, x_s)ds + \int_0^T exp(A(T-s))\sigma(s, x_s)d\omega(s) \\
x(T; x_0, u) = x_T &
\end{aligned}
$$

Remark 3.1 In Theorem 3.1 sufficient condition are given for the existence and uniqueness of solution of (3.1).

Lemma 2 (See [23]) *For every $z \in L_2(\Omega, F_T, R^n)$, there exists a process $\varphi(.) \in L_2([0, T], R^{n\times n})$ such that*

$$
z = \boldsymbol{E}z + \int_0^T \varphi(s)d\omega(s)
$$

$$
\Pi_0^T z = \Gamma_0^T \boldsymbol{E}z + \int_0^T \Gamma_s^T \varphi(s)d\omega(s)
$$

Moreover

$$
\begin{aligned}
E||\Pi_0^T z||^2 &\le ME||E\{z|F_T\}||^2 \\
&\le ME||z||^2, \quad z \in L_2(\Omega, F_T, R^n)
\end{aligned}
$$

Note that if the assumption (A3) holds, then for some $\gamma > 0$

$$E\langle \Pi_0^T z, z\rangle \geq \gamma E||z||^2, \quad for \quad all \quad z \in L_2(\Omega, F_T, R^n)$$

consequently

$$E||(\Pi_0^T)^{-1}||^2 \leq \frac{1}{\gamma} = l_4$$

Now let us assume the following conditions

(A1) (f, σ) satisfies the Lipschitz condition with respect to x i.e., $||f(t, x_t) - f(t, y_t)||^2 \leq L_1||x_t - y_t||^2$, $||\sigma(t, x_t) - \sigma(t, y_t)||^2 \leq L_2||x_t - y_t||^2$ for all $x_t, y_t \in L^2([-h, 0], R^n)$, $0 < t \leq T$
(A2) (f, σ) is continuous on $[0, T] \times R^n$ and satisfies $||f(t, x_t)||^2 \leq L_3(||x_t||^2 + 1)$, $||\sigma(t, x_t)||^2 \leq L_4(||x_t||^2 + 1)$
(A3) The linear system (1.1) is completely controllable.

Let **S** be an operator defined as

$$\mathbf{S}(x)(t) = \begin{cases} \psi(t) \quad for \quad t \in [-h, 0] \\ exp(At)x_0 + \Pi_0^t\Big[exp(A^*(T-t))\Big((\Pi_0^T)^{-1} \times \Big(x_T - exp(AT)x_0 \\ \quad - \int_0^T exp(A(T-r))f(r, x_r)dr - \int_0^T exp(A(T-r))\sigma(r, x_r)d\omega(r)\Big)\Big)\Big] \\ \quad + \int_0^t exp(A(t-s))f(s, x_s)ds + \int_0^t exp(A(t-s))\sigma(s, x_s)d\omega(s) \quad for \quad t \in [0, T] \end{cases}$$

From Lemma 1, the control $u(t)$ transfer the system (3.1) from the initial state x_0 to the final state x_T provided that the operator **S** has a fixed point. So, if the operator **S** has a fixed point then the system (1.2) has a unique mild solution and completely controllable.

Now for convenience, let us introduce the notation

$$l_1 = max||exp(At)||^2 : t \in [0, T], \quad l_2 = max(||B_0||^2, ||B_1||^2)$$
$$l_3 = E||x_T||^2, \quad M = max||\Gamma_s^T||^2 : s \in [0, T]$$

Theorem 3.1 *Assume that the conditions (A1), (A2) and (A3) hold. In addition if the inequality*

$$\Big(4l_1L(Ml_1l_4 + 1)(T + 1)T^2\Big)^{\frac{1}{2}} < 1 \tag{3.2}$$

holds, then the system (1.2) *is completely controllable.*

Proof As mentioned above, to prove the complete controllability it is enough to show that **S** has a fixed point in X_2. To do this, we use the contraction mapping principle. To apply the contraction mapping principle, first we show that **S** maps X_2 into itself.

Now by Lemma 1 we have

$$
\begin{aligned}
&E\left\|(\mathbf{S}x)(t)\right\|^2 \\
&= E\Big\|\psi(t) + exp(At)x_0 + \Pi_0^t\Big[exp(A^*(T-t)) \times (\Pi_0^T)^{-1}(x_T - exp(AT)x_0 \\
&\quad -\int_0^T exp(A(T-r))f(r,x_r)dr - \int_0^T exp(A(T-r))\sigma(r,x_r)d\omega(r))\Big] \\
&\quad +\int_0^t exp(A(t-s))f(s,x_s)ds + \int_0^t exp(A(t-s))\sigma(s,x_s)d\omega(s)\Big\|^2 \\
&\le 5||\psi(t)||^2 + 5||exp(At)x_0||^2 + 5E\Big\|\Pi_0^t[exp(A^*(T-t)) \times (\Pi_0^T)^{-1}(x_T - exp(AT)x_0 \\
&\quad -\int_0^T exp(A(T-r))f(r,x_r)dr - \int_0^T exp(A(T-r))\sigma(r,x_r)d\omega(r))]\Big\|^2 \\
&\quad +5t\int_0^t ||exp(A(t-r))||^2 E||f(r,x_r||^2 dr + 5\int_0^t ||exp(A(t-r))||^2 E||\sigma(r,x_r)||^2 dr \\
&\le 5||\psi(t)||^2 + 5l_1||x_0||^2_{L^2[-h,0]} + 20Ml_1l_4\Big(l_3 + l_1||x_0||^2 + Tl_1\int_0^T E||f(r,x_r)||^2 dr \\
&\quad +l_1\int_0^T E||\sigma(r,x_r)||^2 dr\Big) + 5l_1\int_0^t \left(TE||f(r,x_r)||^2 + E||\sigma(r,x_r)||^2\right) dr \\
&\le B_1 + B_2\left(\int_0^T \left(TE||f(r,x_r)||^2 + E||\sigma(r,x_r)||^2\right) dr\right)
\end{aligned}
$$

where $B_1 > 0$ and $B_2 > 0$ are suitable constants. It follows from the above and the condition (A2) that there exists $C_1 > 0$ such that

$$
\begin{aligned}
E\left\|(Sx)(t)\right\|^2 &\le C_1\left(1 + \int_0^T E||x_r||^2 dr\right) \\
&= C_1\left(1 + \left(\int_0^T E\int_{-h}^0 ||x(r+s)||^2 ds dr\right)\right) \\
&= C_1\left(1 + \left(\int_0^T E\int_{r-h}^r ||x(v)||^2 dv dr\right)\right) \\
&\le C_1\left(1 + \left(\int_0^T E\int_{-h}^T ||x(v)||^2 dv dr\right)\right) \\
&\le C_1\left(1 + T\left(\int_{-h}^T E||x(v)||^2 dv\right)\right)
\end{aligned}
$$

taking L_2 norm both side

$$
\begin{aligned}
\left(\int_{-h}^T E||(Sx)(t)||^2 dt\right)^{1/2} &\le \left(\int_{-h}^T C_1\left(1 + T\left(\int_{-h}^T E||x(v)||^2 dv\right)\right) dt\right)^{1/2} \\
&\le \sqrt{C_1}\sqrt{T+h}\left(1 + T\left(\int_{-h}^T E||x(t)||^2 dt\right)\right)^{1/2}
\end{aligned}
$$

for all $t \in [-h, T]$. Therefore $\mathbf{S}$ maps X_2 into itself. Secondly, we show that $\mathbf{S}$ is a contraction mapping. Indeed

$$
\begin{aligned}
&E\left|\left|(\mathbf{S}x)(t)-(\mathbf{S}y)(t)\right|\right|^2\\
&=E\Bigg|\Bigg|\Pi_0^t[exp(A^*(T-t))(\Pi_0^T)^{-1}\times\left(\int_0^T exp(A(T-s))(f(s,y_s)-f(s,x_s))\right)ds\\
&\quad+\int_0^T exp(A(T-s))(\sigma(s,y_s)-\sigma(s,x_s)))d\omega(s))]\\
&\quad+\int_0^t exp(A(t-s))(f(s,x_s)-f(s,y_s))ds\\
&\quad+\int_0^t exp(A(t-s))(\sigma(s,x_s)-\sigma(s,y_s))d\omega(s)\Bigg|\Bigg|^2\\
&\le 4Ml_1^2l_4(T\int_0^T E||f(s,x_s)-f(s,y_s)||^2ds+\int_0^T E||\sigma(s,x_s)-\sigma(s,y_s)||^2ds)\\
&\quad+4l_1(T\int_0^t E||f(s,x_s)-f(s,y_s)||^2ds+\int_0^t E||\sigma(s,x_s)-\sigma(s,y_s)||^2ds)\\
&=4Ml_1^2l_4L(T+1)\int_0^T E||x_s-y_s)||^2ds+4l_1L(T+1)\int_0^t E||x_s-y_s||^2ds\\
&\le 4l_1L(Ml_1l_4+1)(T+1)\int_0^T E||x_s-y_s||^2ds\\
&=4l_1L(Ml_1l_4+1)(T+1)\left(\int_0^T E\int_{-h}^0||x(t+s)-y(t+s)||^2dtds\right)\\
&=4l_1L(Ml_1l_4+1)(T+1)\left(\int_0^T E\int_{s-h}^s||x(v)-y(v)||^2dvds\right)\\
&\le 4l_1L(Ml_1l_4+1)(T+1)\left(\int_0^T E\int_{-h}^T||x(v)-y(v)||^2dvds\right)\\
&\le 4l_1L(Ml_1l_4+1)(T+1)T\left(\int_{-h}^T E||x(v)-y(v)||^2dv\right)
\end{aligned}
$$

taking L_2 norm both side we get

$$
\begin{aligned}
&\left(\int_{-h}^T E\left|\left|(\mathbf{S}x)(t)-(\mathbf{S}y)(t)\right|\right|^2dt\right)^{1/2}=\left(\int_0^T E\left|\left|(\mathbf{S}x)(t)-(\mathbf{S}y)(t)\right|\right|^2dt\right)^{1/2}\\
&\quad\le\left(\int_0^T 4l_1L(Ml_1l_4+1)(T+1)T\left(\int_{-h}^T E||x(v)-y(v)||^2dv\right)dt\right)^{1/2}\\
&\quad\le\left(4l_1L(Ml_1l_4+1)(T+1)T^2\right)^{\frac{1}{2}}\left(\int_{-h}^T E||x(t)-y(t)||^2dt\right)^{1/2}
\end{aligned}
$$

Therefore $\mathbf{S}$ is a contraction mapping if the inequality (3.2) holds. Then the mapping $\mathbf{S}$ has a unique fixed point $x(.)$ in X_2 which is the solution of the Eq. (1.2). Thus the system (1.2) is completely controllable. The theorem is proved.

Remark 3.2 In this paper the sufficient conditions for complete controllability are obtained in Theorem (3.1) for the system (1.2)–(1.4) using L_2 norm. In [21] Mahmudov et al. have considered a particular case of system (1.2)–(1.4) by taking $B_1=0$, $f(t,x_t)=f(t,x(t))$, $\sigma(t,x_t)=\sigma(t,x(t))$ and obtained the results using L_∞ (supremum) norm.

Remark 3.3 In [12] Balachandran et al. have considered a particular case of system (1.2)–(1.4) for deterministic system by taking $B_1 = 0,\ f(t, x_t) = f(t, x(t)),\ \sigma(t, x_t) = 0$ and obtained the results using L_∞ (supremum) norm.

Remark 3.4 Inequality (3.2) is fulfilled if L is sufficiently small.

4 Example

Example 1 Consider a two-dimensional semi-linear stochastic system with delay in state and control terms

$$dx(t) = [A_0 x(t) + B_0 u(t) + B_1 u(t-h) + f(t, x_t)]dt + \sigma(t, x_t) d\omega(t) \quad \text{for } t \in [0, T] \tag{4.1}$$

with initial condition (1.3)

where $\omega(t)$ is a one dimensional Wiener process and

$$A_0 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix},\quad B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad B_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$f(t, x_t) = \frac{1}{a}\begin{bmatrix} \sin x_t \\ x_t \end{bmatrix},\quad \sigma(t, x_t) = \frac{1}{b}\begin{bmatrix} x_t & 0 \\ 0 & \cos x_t \end{bmatrix}$$

If we take Euclidean norm then

$$||f(t, x_t) - f(t, y_t)||^2 \leq \frac{2}{a^2}||x_t - y_t||^2 \quad and$$

$$||\sigma(t, x_t) - \sigma(t, y_t)||^2 \leq \frac{2}{b^2}||x_t - y_t||^2 \quad so,$$

$$||f(t, x_t) - f(t, y_t)||^2 + ||\sigma(t, x_t) - \sigma(t, y_t)||^2 \leq L||x_t - y_t||^2 \tag{4.2}$$

$$where \quad L = \left(\frac{2}{a^2} + \frac{2}{b^2}\right) \tag{4.3}$$

$$||A_0|| = 2,\ ||B_0|| = \sqrt{2},\ ||B_1|| = \sqrt{2}$$

We can see that conditions of Theorem 3.1 with the help definition of M and Lemma 2 for sufficiently small L [using Eq. (4.3)] for any time T are satisfied. So system (4.1) is completely controllable.

Example 2 In particular let us consider the system (1.2) without any delay means $B_1 = 0$, $f(t, x_t) = f(t, x(t))$ and $\sigma(t, x_t) = \sigma(t, x(t))$

$$dx(t) = [Ax(t) + Bu(t) + f(t, x(t))]dt + \sigma(t, x(t))d\omega(t)$$
$$x(0) = x_0 \in R^n \tag{4.4}$$

$$A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ f(t, x(t)) = \begin{bmatrix} \sin x(t) \\ x(t) \end{bmatrix},\ \sigma(t, x(t)) = \begin{bmatrix} x(t) & 0 \\ 0 & \cos x(t) \end{bmatrix}$$

Here $f(t, x(t))$ and $\sigma(t, x(t))$ are satisfying conditions (A1) and (A2). For $x = (x_1, x_2)$ with the initial value x_0 and final point $x_T \in R^2$. For this system the controllability matrix is

$$\Gamma_s^t = \frac{1}{2}\begin{bmatrix} 1 - \exp^{-2(t-s)} & 0 \\ 0 & 1 - \exp^{-2(t-s)} \end{bmatrix}$$

If we take Euclidean norm then

$$||A|| = 2, \ ||B|| = \sqrt{2}, \ ||\Gamma_s^t|| = \frac{1 - \exp^{-2(t-s)}}{\sqrt{2}} > 0 \quad \forall \ 0 \leq s < t;$$

In [16] obtained sufficient condition for controllability of system (4.4) in L_∞ (supremum) norm as below

$$4l_1 L(Ml_1l_4 + 1)(T + 1)T < 1$$

In this paper sufficient condition for controllability of system (4.4) is obtained in L_2 norm as below

$$\left(4l_1 L(Ml_1l_4 + 1)(T + 1)T^2\right)^{1/2} < 1$$

Suppose $C = (4l_1 L(Ml_1l_4 + 1))$ then sufficient condition for controllability for [16] will be $C(T + 1)T < 1$. Substitute $T \ (time) = 5 \ unit$ then $C \in (0, 1/30)$. For all these values of C system will be controllable. Now in this paper Left hand side for sufficient condition is $\sqrt{C(T+1)} \ T = C_1 \ (let)$ substitute values of C and T we get $C_1 \in (0, \sqrt{5})$. It means C_1 can be greater than 1 for some cases. For these cases sufficient condition of Theorem (3.1) is not satisfied but the system (4.4) is completely controllable form [16] in L_∞ norm.

But if we proceed in similar manner as above and suppose the sufficient conditions of this paper are satisfied for some values of $T < 1$ then sufficient conditions of [16] paper may not be satisfied.

References

1. Kalman, R.E.: Controllability of linear systems. Contrib. Differ. Equ. **1**, 190–213 (1963)
2. Barnett, S.: Introduction to Mathematical Control Theory. Clarendon press, Oxford (1975)
3. Naito, K.: Controllability of semilinear control systems dominated by the linear part. SIAM J. Control Optim. **25**(3), 715–722 (1987)
4. Balachandran, K., Dauer, J.P.: Controllability of nonlinear systems in Banach spaces: a survey. J. Optim. Theory Appl. **115**(1), 7–28 (2002)
5. Wang, L.W.: Approximate controllability of integrodifferential equations with multiple delays. J. Optim. Theory Appl. **143**, 185–206 (2009)
6. Wang, L.W.: Approximate controllability of delayed semilinear control of control system. J. Appl. Math. Stoch. Anal. **1**, 67–76 (2005)
7. Sukavanam, N., Tafesse, S.: Approximate controllability of a delayed semilinear control system with growing nonlinear term. Nonlinear Anal. **74**(18), 6868–6875 (2011)
8. Sukavanam, N.: Approximate controllability of semilinear control systems with growing nonlinearity. In: Mathematical Theory of Control Proceedings of International Conference, pp. 353–357. Marcel Dekker, New York (1993)
9. Bashirov, A.E., Kerimov, K.R.: On controllability conception for stochastic systems. SIAM J. Control Optim. **35**(2), 384–398 (1997)
10. Mahmudov, N.I.: Controllability of linear stochastic systems in Hilbert spaces. J. Math. Anal. Appl. **259**(1), 64–82 (2001)
11. Klamka, J., Socha, L.: Some remarks about stochastic controllability. IEEE Trans. Autom. Control **AC–22**(5), 880–881 (1977)
12. Klamka, J.: Stochastic controllability of linear systems with delay in control. Bull. Pol. Acad. Sci. Tech. Sci. **55**(1), 23–29 (2007)
13. Klamka, J.: Stochastic controllability of linear systems with state delays. Int. J. Appl. Math. Comput. Sci. **17**(1), 5–13 (2007)
14. Klamka, J.: Stochastic controllability of linear systems with variable delay in control. Bull. Pol. Acad. Sci. Tech. Sci. **56**(3), 279–284 (2008)

15. Klamka, J.: Stochastic controllability and minimum energy control of systems with multiple delays in control. Appl. Math. Comput. **206**(2), 704–715 (2008)
16. Mahmudov, N.I., Zorlu, S.: Controllability of nonlinear stochastic systems. J. Control **76**, 95–104 (2003)
17. Mahmudov, N.I., Semi, N.: Approximate controllability of semilinear control systems in Hilbert spaces. TWMS J. Appl. Eng. Math. **2**(1), 67–74 (2012)
18. Shen, L., Sun, J.: Relative controllability of stochastic nonlinear systems with delay in control. Nonlinear Anal. Real World Appl. **13**(6), 2880–2887 (2012)
19. Sukavanam, N., Kumar, M.: *S*-Controllability of an abstract first order semilinear control system. Numer. Funct. Anal. Optim. **31**(7–9), 1023–1034 (2010)
20. Shukla, A., Arora, U.,Sukavanam, N.: Approximate controllability of retarded semilinear stochastic system with non local conditions. J. Appl. Math. Comput. (2014). doi:10.1007/s12190-014-0851-9
21. Klamka, J.: Constrained controllability of semilinear systems with delayed controls. Bull. Pol. Acad. Sci. Tech. Sci. **56**(4), 333–337 (2008)
22. Klamka, J.: Constrained controllability of nonlinear systems. J. Math. Anal. Appl. **201**(2), 365–374 (1996)
23. Bernt, O.: Stochastic Differential Equations an Introduction with Applications, 6th edn. Springer