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Quartic functional equations in Lipschitz spaces

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Abstract In this paper we approximate the quartic functional equations in Lipschitz spaces.

Keywords Quartic functional equation · Lipschitz space · Stability

Mathematics Subject Classification Primary 39B82; Secondary 39B52

1 Introduction

Let G be an Abelian group and E a vector space. Let $S(E)$ be a family of subsets of E. We say that $S(E)$ is linearly invariant if it is closed under the addition and scalar multiplication defined as usual sense and translation invariant, i.e, $x + A \in S(E)$, for every $A \in S(E)$ and every $x \in E$ (see [1]). It is easy to verify that $S(E)$ contains all singleton subsets of E. In particular, $CB(E)$ the family of all closed balls with center at zero is a linearly invariant family in a normed vector space E. By $B(G \times G, S(E))$ we denote the family of all functions $f : G \times G \longrightarrow E$ such that $\mathtt{Im} f \subset A$ for some $A \in S(E)$, where $G \times G$ is the Cartesian product of G with itself. Obviously, this family is a vector space and contains all constant functions.

We say that $B(G \times G, S(E))$ admits a left invariant mean (briefly LIM), if the family $S(E)$ is linearly invariant and there exists a linear operator $M : B(G \times G, S(E)) \longrightarrow E$ such that

(i) if $\mathtt{Im} f \subset A$ for some $A \in S(E)$, then $M[f] \in A$,
(ii) if $f \in B(G \times G, S(E))$ and $(a, b) \in G \times G$, then $M[f^{a,b}] = M[f]$,

where $f^{a,b}(x, y) = f(x + a, y + b)$. Following [2,3] and for 2-variable functions let $\mathbf{d} : (G \times G) \times (G \times G) \longrightarrow S(E)$ be a set-valued function such that

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$$\begin{aligned}\mathbf{d}\left((x+a, y+b),(w+a, z+b)\right) &= \mathbf{d}\left((a+x, b+y),(a+w, b+z)\right)\\ &= \mathbf{d}((x, y),(w, z))\end{aligned}$$

for all $(a, b), (x, y), (w, z) \in G \times G$. A function $f : G \times G \longrightarrow E$ is said to be **d**-Lipschitz if $f(x, y) - f(w, z) \in \mathbf{d}((x, y), (w, z))$ for all $(x, y), (w, z) \in G \times G$.

Let $(G \times G, d)$ be a metric group and E a normed space. A function $m_f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a module of continuity of $f : G \times G \longrightarrow E$ if $d((x, y), (w, z)) \leq \delta$ implies $||f(x, y) - f(w, z)|| \leq m_f(\delta)$ for every $\delta > 0$ and every $(x, y), (w, z) \in G \times G$. A function $f : G \times G \longrightarrow E$ is called Lipschitz function if it satisfies the condition

$$||f(x, y) - f(w, z)|| \leq Ld((x, y), (w, z)) \tag{1.1}$$

for every $(x, y), (w, z) \in G \times G$. The smallest constant L with this property is denoted by $\mathtt{lip}(f)$.

We define $Lip(G \times G, E)$ to be the Lipschitz space consisting of all bounded Lipschitz functions with the norm

$$||f||_{Lip} := ||f||_{\sup} + \mathtt{lip}(f).$$

The study of stability problems for functional equations is related to a question of Ulam [4] concerning the stability of group homomorphisms, which was affirmatively answered for Banach spaces by Hyers [5] (see for example [6,7] and references therein).

In Lipschitz spaces the stability type problems for some functional equations were studied by Czerwik and Dlutek [8] and Tabor [3,9]. Czerwik and Dlutek [8] established the stability of the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

and the author of the present paper [10] proved the stability of the cubic functional equation in Lipschitz spaces. The stability problem for the following quartic functional equation

$$f(2x + y) + f(2x - y) = 4(f(x + y) + f(x - y)) + 24f(x) - 6f(y)$$

first was considered by Rassias [11] for mappings from a real normed space into a Banach space. Najati [12] proved the generalized Hyers–Ulam stability for the above quartic functional equation for functions from a linear space into a Banach space. Bae [13] obtained the general solution and the stability of the following 2-variable quadratic functional equation

$$f(x + z, y + w) + f(x - z, y - w) = 2f(x, y) + 2f(z, w)$$

in complete normed spaces. In this paper, we verify the stability of the quartic functional equation in the Lipschitz norms.

2 Approximation with d-Lipschitz approach

For a given function $f : G \times G \longrightarrow E$ we define its quadratic difference as follows

$$Qf(x, y, z, w) := 2f(x, y) + 2f(z, w) - f(x + z, y + w) - f(x - z, y - w)$$

for all $(x, y), (z, w) \in G \times G$. By $\Delta(G)$ we denote the diagonal set on G, i.e.,

$$\Delta(G) := \{(x, x) \in G \times G : x \in G\}.$$

Theorem 2.1 *Let G be an Abelian group, and let E be a vector space. Assume that the family $B(G\times G, S(E))$ admits LIM. If $f : G\times G \longrightarrow E$ is a function and $Qf(t,s,\cdot,\cdot) : G\times G \longrightarrow E$ is $\mathbf{d}$-Lipschitz for every $(t,s)\in G\times G$, then there exists a quartic function Q such that $f_{|\Delta(G)} - Q$ is $\frac{1}{2}\mathbf{d}$-Lipschitz.*

Proof For every $(a,b)\in G\times G$ we define $F_{a,b} : G\times G \longrightarrow E$ by

$$F_{a,b}(x,y) := \frac{1}{2}f(x+a,y+b)+\frac{1}{2}f(x-a,y-b)-f(x,y).$$

We prove that $F_{a,b}\in B(G\times G,S(E))$. We have for $(x,y),(a,b)\in G\times G$,

$$\begin{aligned}F_{a,b}(x,y) &= \frac{1}{2}f(x+a,y+b)+\frac{1}{2}f(x-a,y-b)-f(x,y)-f(a,b)\\ &\quad -\frac{1}{2}f(x,y)-\frac{1}{2}f(x,y)+f(x,y)+f(0,0)\\ &\quad +f(a,b)-f(0,0)\\ &= \frac{1}{2}Qf(x,y,0,0)-\frac{1}{2}Qf(x,y,a,b)+f(a,b)-f(0,0).\end{aligned}$$

Set $A := \frac{1}{2}\mathbf{d}((0,0),(a,b)) + f(a,b)+f(0,0)$. It is clear that $A\in S(E)$. In view of our assumptions it follows that $\mathtt{Im}F_{a,b}\subset A$ and so we obtain the result. The fact that the family $B(G\times G,S(E))$ admits LIM ensures there exists a linear operator $M : B(G\times G,S(E))\longrightarrow E$ such that

(i) $M[F_{a,b}]\in A$ for some $A\in S(E)$,
(ii) if for $(z,w)\in G\times G$, $F_{a,b}^{z,w} : G\times G\longrightarrow E$ is defined by $F_{a,b}^{z,w}(t,s) := F_{a,b}(t+z,s+w)$ for every $(t,s)\in G\times G$, then $F_{a,b}^{z,w}\in B(G\times G,S(E))$ and $M[F_{a,b}] = M[F_{a,b}^{z,w}]$.

Define the function $K : G\times G\longrightarrow E$ by $K(x,y) := M[F_{x,y}]$ for $(x,y)\in G\times G$. We know that $B(G\times G,S(E))$ contains constant functions. By using property (i) of M it is easy to verify that if $f : G\times G\longrightarrow E$ is constant, i.e., $f(x,y)=c$ for $(x,y)\in G\times G$, where $c\in E$, then $M[f]=c$. We now show that $f-K$ is $\frac{1}{2}$ $\mathbf{d}$-Lipschitz. Let for any $(x,y)\in G\times G$ the constant function $R_{x,y} : G\times G\longrightarrow E$ be the function $R_{x,y}(z,w) := f(x,y)$ for all $(z,w)\in G\times G$. We have

$$\begin{aligned}&(f(x,y)-K(x,y))-(f(z,w)-K(z,w))\\ &\quad = \big(M[R_{x,y}]-M[F_{x,y}]\big)-\big(M[R_{z,w}]-M[F_{z,w}]\big)\\ &\quad = M[R_{x,y}-F_{x,y}]-M[R_{z,w}-F_{z,w}]\\ &\quad = M\left[\frac{1}{2}Qf(\cdot,\cdot,x,y)-\frac{1}{2}Qf(\cdot,\cdot,z,w)\right]\end{aligned}$$

for all $(x,y),(z,w)\in G\times G$. On the other hand

$$\frac{1}{2}Qf(t,s,x,y)-\frac{1}{2}Qf(t,s,z,w)\in\frac{1}{2}\mathbf{d}\left((x,y),(z,w)\right) \tag{2.1}$$

for all $(x,y),(z,w)\in G\times G$. From this we deduce that

$$\mathtt{Im}\left(\frac{1}{2}Qf(\cdot,\cdot,x,y)-\frac{1}{2}Qf(\cdot,\cdot,z,w)\right)\subseteq\frac{1}{2}\mathbf{d}\left((x,y),(z,w)\right).$$

In view of property (i) of M we conclude that

$$M\left[\frac{1}{2}Qf(\cdot,\cdot,x,y)-\frac{1}{2}Qf(\cdot,\cdot,z,w)\right]\in\frac{1}{2}\mathbf{d}\left((x,y),(z,w)\right)$$

for all $(x, y), (z, w) \in G \times G$. This shows that

$$(f(x, y) - K(x, y)) - (f(z, w) - K(z, w)) \in \frac{1}{2}\mathbf{d}((x, y), (z, w))$$

for all $(x, y), (z, w) \in G \times G$, i.e., $f - K$ is a $\frac{1}{2}$ $\mathbf{d}$-Lipschitz function. We now have

$$2K(x, y) + 2K(z, w) = 2M[F_{x,y}(t, s)] + 2M[F_{z,w}(t, s)].$$

Furthermore, applying property (ii) of M, one gets

$$M[F_{x,y}] = M\left[F_{x,y}^{z,w}\right]$$
$$M[F_{x,y}] = M\left[F_{x,y}^{-z,-w}\right]$$

for $(z, w) \in G \times G$. Consequently, we have

$$\begin{aligned} 2K(x, y) + 2K(z, w) &= 2M[F_{x,y}] + 2M[F_{z,w}] \\ &= M\left[F_{x,y}^{z,w}\right] + M\left[F_{x,y}^{-z,-w}\right] + 2M[F_{z,w}]. \end{aligned}$$

On the other hand we have

$$\begin{aligned} &M[F_{x,y}^{z,w}] + M[F_{x,y}^{-z,-w}] + 2M[F_{z,w}] \\ &= M\left[\frac{1}{2}f(t + x + z, s + y + w) + \frac{1}{2}f(t - x + z, s - y + w) - f(t + z, s + w)\right] \\ &\quad + M\left[\frac{1}{2}f(t + x - z, s + y - w) + \frac{1}{2}f(t - x - z, s - y - w) - f(t - z, s - w)\right] \\ &\quad + M[f(t + z, s + w) + f(t - z, s - w) - 2f(t, s)] \\ &= K(x + z, y + w) + K(x - z, y - w). \end{aligned}$$

This shows that K is 2-variable quadratic. Define $Q : G \longrightarrow E$ by $Q(x) := K(x, x)$. We have

$$f_{|\Delta(G)} - Q = f_{|\Delta(G)} - K_{|\Delta(G)} = (f - K)_{|\Delta(G)}.$$

The function $f - K$ is $\frac{1}{2}$ $\mathbf{d}$-Lipschitz and so is $f_{|\Delta(G)} - Q$. The following equality entails that Q is quartic.

$$\begin{aligned} &4(Q(x + y) + Q(x - y)) + 24Q(x) - 6Q(y) \\ &\quad = 4(K(x + y, x + y) + K(x - y, x - y)) + 24K(x, x) - 6K(y, y) \\ &\quad = 32K(x, x) + 2K(y, y) \\ &\quad = 2K(2x, 2x) + 2K(y, y) \\ &\quad = K(2x + y, 2x + y) + K(2x - y, 2x - y) \\ &\quad = Q(2x + y) + Q(2x - y). \end{aligned}$$

□

Remark 2.2 Assuming the hypotheses of Theorem 2.1 and $\mathtt{Im}Qf \subset A$ for some $A \in S(E)$, we then obtain $\mathtt{Im}(f_{|\Delta(G)} - Q) \subset \frac{1}{2}A$. In fact,

$$\mathtt{Im}\left(\frac{1}{2}Qf(x, y, \cdot, \cdot)\right) \subset \mathtt{Im}\left(\frac{1}{2}Qf\right) \subset \frac{1}{2}A$$

and so $\frac{1}{2}Qf(x, y, \cdot, \cdot) \in B(G \times G, S(E))$ for all $(x, y) \in G \times G$. Thus, property (i) of M implies

$$f(x, y) - K(x, y) = M\left[\frac{1}{2}Qf(x, y, \cdot, \cdot)\right] \in \frac{1}{2}A$$

for all $(x, y) \in G \times G$. Therefore, $\mathtt{Im}(f_{|\Delta(G)} - Q) \subset \mathtt{Im}(f - K) \subset \frac{1}{2}A$.

3 Approximation with Lipschitz norm

Consider an Abelian group $(G \times G, +)$ with a metric d invariant under translation, i.e., satisfying the condition

$$\begin{aligned} d\left((x+a, y+b), (w+a, z+b)\right) &= d\left((a+x, b+y), (a+w, b+z)\right) \\ &= d((x, y), (w, z)) \end{aligned}$$

for all $(a, b), (x, y), (w, z) \in G \times G$. We say that a metric D on $G \times G \times G \times G$ is a product metric if it is an invariant metric and the following condition holds

$$\begin{aligned} D\left((a, b, x, y), (a, b, w, z)\right) &= D\left((x, y, a, b), (w, z, a, b)\right) \\ &= d((x, y), (w, z)) \ \ (a, b), (x, y), (w, z) \in G \times G. \end{aligned}$$

Theorem 3.1 *Let $(G \times G, +, d, D)$ be a product metric, and let E be a normed space such that $B(G \times G, CB(E))$ admits LIM. Assume that $f : G \times G \longrightarrow E$ be a function. If $Qf \in Lip(G \times G \times G \times G, E)$, then there exists a quartic function Q such that*

$$||f_{|\Delta(G)} - Q||_{Lip} \leq \frac{1}{2}||Qf||_{Lip}.$$

Proof Assume that $m_{Qf} : \mathbb{R}^+ \to \mathbb{R}^+$ is the module of continuity of $Qf : G\times G\times G\times G \to E$ with the product metric D on $G \times G \times G \times G$. It is immediate that

$$\begin{aligned} ||Qf(t, s, x, y) - Qf(t, s, w, z)|| &\leq \inf_{D((t,s,x,y),(t,s,w,z))\leq\delta} m_{Qf}(\delta) \\ &= \inf_{d((x,y),(w,z))\leq\delta} m_{Qf}(\delta) \end{aligned}$$

for all $(t, s), (x, y), (w, z) \in G \times G$. Define the set-valued function $\mathbf{d} : G \times G \longrightarrow CB(E)$ by

$$\mathbf{d}((x, y), (w, z)) := \inf_{d((x,y),(w,z))\leq\delta} m_{Qf}(\delta)B(0, 1),$$

where $B(0, 1)$ is the closed unit ball with center at zero. We now conclude that $Qf(t, s, \cdot, \cdot)$ is $\mathbf{d}$-Lipschitz and so Theorem 2.1 implies there exists a quartic function Q such that $f_{|\Delta(G)} - Q$ is $\frac{1}{2}$ $\mathbf{d}$-Lipschitz. Hence,

$$||(f(x, x) - Q(x)) - (f(z, z) - Q(z))|| \leq \inf_{d((x,x),(z,z))\leq\delta} \frac{1}{2}m_{Qf}(\delta),$$

which shows that $m_{f_{|\Delta(G)}-Q} = \frac{1}{2}m_{Qf}$. Moreover, $||Qf||_{\sup} < \infty$ and clearly $\mathtt{Im}Qf \subset ||Qf||_{\sup}B(0, 1)$. Using Remark 2.2 we get

$$||f_{|\Delta(G)} - Q||_{\sup} \leq \frac{1}{2}||Qf||_{\sup}. \tag{3.1}$$

We may also prove that $m_{Qf} = \mathtt{lip}(Qf)$ and so $\mathtt{lip}(f_{|\Delta(G)} - Q) \leq \frac{1}{2}\mathtt{lip}(Qf)$. Applying the inequality (3.1) we get

$$\begin{aligned}||f_{|\Delta(G)} - Q||_{Lip} &= ||f_{|\Delta(G)} - Q||_{\sup} + \mathtt{lip}(f_{|\Delta(G)} - Q) \\ &\leq \frac{1}{2}||Qf||_{\sup} + \frac{1}{2}\mathtt{lip}(Qf) \\ &= \frac{1}{2}||Qf||_{Lip}.\end{aligned}$$

□

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