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On the asymptotic behavior of the solutions of third order delay differential equations

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Abstract By constructing a Lyapunov functional, we obtain some sufficient conditions which guarantee the stability and boundedness of solutions for some nonlinear differential equations of third order with delay. Our result improve and generalize existing results in the relevant literature of nonlinear third order differential equations.

Keywords Stability · Lyapunov functional · Non-autonomous differential equations of third order with delay

Mathematics Subject Classification 34C11

1 Introduction

We consider nonlinear third order delay differential equation of the form

$$[\Psi(x)x']'' + a(t)x'' + b(t)\Phi(x)x' + c(t)f(x(t-r)) = e(t), \tag{1.1}$$

where $r > 0$, and the functions $a(t),\ b(t),\ c(t),\ e(t),\ f(x),\ \ \Psi(x)$, and $\Phi(x)$ are continuous in their respective arguments and $f'(x),\ \ \Psi'(x)$, $\Phi'(x)$ exist and are continuous for all x.

The asymptotic property of solutions of third order differential equations has received a considerable amount of attention. In numerous places in the literature, for example [1–21], the authors dealt with the problems by considering Lyapunov functions or functionals and obtained the criteria for the stability.

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In 1974, Hara [8] investigated the asymptotic behavior of solutions of the differential equation without delay of the form

$$x''' + a(t)x'' + b(t)x' + c(t)f(x) = e(t), \tag{1.2}$$

and showed that all solutions of the Eq. (1.2) are uniformly bounded and satisfy $x(t) \to 0$, $x'(t) \to 0$ and $x''(t) \to 0$. More recently in 2005, Sadek in [13] establishes conditions under which all solutions of third order differential equation with delay of the form,

$$x''' + a(t)x'' + b(t)x' + c(t)f(x(t-r)) = 0$$

tend to the zero solution as $t \to \infty$. Our objective in this paper is to extend the results verified by Sadek [13] to obtain sufficient conditions for the stability and the boundedness of solutions of delay differential equation (1.1) for the cases $e(t) \equiv 0$ and $e(t) \neq 0$. Clearly the equation discussed in Sadek [13] is a special case of Eq. (1.1) when $\Psi(x) = \Phi(x) = 1$. We shall use appropriate Lyapounov function and impose suitable conditions on the functions $f(x)$, $\Psi(x)$ and $\Phi(x)$. On the other hand, we can find the same result for the Eq. (1.1) without delay by putting $r = 0$, witch is generalization of Hara [8] results.

2 Preliminaries

First, we will give some basic definitions and important stability criteria for the general non-autonomous delay differential system. We consider

$$x' = f(t, x_t), \quad x_t(\theta) = x(t+\theta), \quad -r \leq \theta \leq 0, \ t \geq 0, \tag{2.1}$$

where $f : I \times C_H \to \mathbb{R}^n$ is a continuous mapping, $f(t,0) = 0$, $C_H := \{\phi \in (C[-r,0], \mathbb{R}^n) : \|\phi\| \leq H\}$, and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(t,\phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Definition 2.1 [5] An element $\psi \in C$ is in the $\omega - limit$ set of ϕ, say $\Omega(\phi)$, if $x(t,0,\phi)$ is defined on $[0,+\infty)$ and there is a sequence $\{t_n\}$, $t_n \to \infty$, as $n \to \infty$, with $\|x_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$ where $x_{t_n}(\phi) = x(t_n + \theta, 0, \phi)\ for\ -r \leq \theta \leq 0$.

Definition 2.2 [5] A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution of (2.1), $x(t,0,\phi)$, is defined on $[0,\infty)$ and $x_t(\phi) \in Q$ for $t \in [0,\infty)$.

Lemma 2.3 [3] *If $\phi \in C_H$ is such that the solution $x_t(\phi)$ of (2,1) with $x_0(\phi) = \phi$ is defined on $[0,\infty)$ and $\|x_t(\phi)\| \leq H_1 < H$ for $t \in [0,\infty)$, then $\Omega(\phi)$ is a non-empty, compact, invariant set and*

$$dist(x_t(\phi), \Omega(\phi)) \to 0 \quad as \quad t \to \infty.$$

Lemma 2.4 [3] *let $V(t,\phi) : I \times C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition. $V(t,0) = 0$, and such that:*

(i) $W_1(|\phi(0)|) \leq V(t,\phi) \leq W_2(\|\phi\|)$ *where $W_1(r)$, $W_2(r)$ are wedges.*
(ii) $V'_{(2,1)}(t,\phi) \leq 0$, *for $\phi \in C_H$.*

Then the zero solution of (2.1) is uniformly stable.

If $Z = \{\phi \in C_H : V'_{(2,1)}(t,\phi) = 0\}$, then the zero solution of (2.1) is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

3 Assumptions and main results

First, we state some assumptions on the functions that appeared in (1.1). Suppose that there are positive constants a_0, b_0, c_0, ψ_0, ψ_1, ϕ_0, ϕ_1, A, B, C, N_1, δ_0, and δ_1 such that the followings conditions are satisfied

(i) $0 < a_0 \le a(t) \le A$; $0 < b_0 \le b(t) \le B$; $0 < c_0 \le c(t) \le C, t \ge 0,$
(ii) $0 < \psi_0 \le \Psi(x) \le \psi_1$ and $0 < \phi_0 \le \Phi(x) \le \phi_1$ for all x,
(iii) $f(0) = 0,\ \dfrac{f(x)}{x} \ge \delta_0 > 0\ (x \ne 0)$, and $|f'(x)| \le \delta_1$, for all x,
(iv) $\displaystyle\int_{-\infty}^{+\infty} \left|\Psi'(u)\right| du < \infty$ and $\displaystyle\int_{-\infty}^{+\infty} \left|\Phi'(u)\right| du < \infty$,
(v) $\displaystyle\int_{0}^{\infty} \left|c'(s)\right| ds \le N_1 < \infty$ and $c'(t) \to 0$ as $t \to \infty$.

For the case $e(t) \equiv 0$, the following result is introduced.

Theorem 3.1 *In addition to conditions (i)–(v) being satisfied, suppose that the following conditions hold*

(H1) $\dfrac{\psi_1 C}{b_0\phi_0}\delta_1 < \mu < a_0,$
(H2) $\mu a'(t) + \Psi(x)\Phi(x)b'(t) - \Psi^2(x)\dfrac{\delta_1}{\mu}c'(t) < \mu b_0\phi_0 - \psi_1 C\delta_1.$

Then every solution of (1.1) is uniformly asymptotically stable, provided that

$$r < \min\left\{\frac{2(a_0-\mu)}{\psi_1 C\delta_1}, \frac{\psi_0^3(\mu b_0\phi_0 - \psi_1 C\delta_1)}{\psi_1^2 C\delta_1(\mu + \mu\psi_0^2 + \psi_0)}\right\}.$$

Proof We use the following differential system which is equivalent to Eq. (1.1)

$$\begin{aligned} x' &= \frac{1}{\Psi(x)}y, \\ y' &= z, \\ z' &= -\frac{a(t)}{\Psi(x)}z + \frac{a(t)\Psi'(x)}{\Psi^3(x)}y^2 - \frac{b(t)\Phi(x)y}{\Psi(x)} - c(t)f(x) \\ &\quad + \int_{t-r}^{t} y(s)\frac{f'(x(s))}{\Psi(x(s))}. \end{aligned} \tag{3.1}$$

The proof depend on some fundamental properties of a continuously differentiable Lyapunov functional $V = V(t, x, y, z)$ defined as

$$\begin{aligned} V(t, x_t, y_t, z_t) = {} & \mu c(t)F(x) + c(t)f(x)y + \frac{1}{2}\frac{b(t)\Phi(x)}{\Psi(x)}y^2 + \frac{\mu a(t)}{2\Psi^2(x)}y^2 \\ & + \frac{\mu}{\Psi(x)}yz + \frac{1}{2}z^2 + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds, \end{aligned}$$

such that $F(x) = \int_0^x f(u)du$, and λ is a positive constant which will be determined later in the proof. To show that V is a positive function, we rewrite V above thus

$$V(t, x_t, y_t, z_t) = \mu c(t)G(x, y) + V_1 + V_2 + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds, \tag{3.2}$$

where

$$G(x,y)=F(x)+\frac{1}{\mu}yf(x)+\frac{\delta_1}{2\mu^2}y^2,$$
$$V_1=V_1(t,x_t,y_t,z_t)=\frac{1}{2}\left[-\frac{c(t)\delta_1}{\mu}+\frac{b(t)\Phi(x)}{\Psi(x)}\right]y^2,$$
$$V_2=V_2(t,x_t,y_t,z_t)=\frac{\mu a(t)}{2\Psi^2(x)}y^2+\frac{\mu}{\Psi(x)}yz+\frac{1}{2}z^2.$$

By using hypotheses, we obtain

$$\begin{aligned}\mu c(t)G(x,y)&=\mu c(t)\left[F(x)+\frac{\delta_1}{2\mu^2}\left(y+\frac{\mu}{\delta_1}f(x)\right)^2-\frac{1}{2\delta_1}f^2(x)\right]\\&\geq \mu c(t)\left[\int_0^x\left(1-\frac{f'(u)}{\delta_1}\right)f(u)du\right]\geq 0.\end{aligned}$$

V_2 can be rearranged as the following

$$\begin{aligned}V_2(t,x_t,y_t,z_t)&=\frac{1}{2}\frac{\mu a(t)}{\Psi^2(x)}y^2+\frac{\mu}{\Psi(x)}yz+\frac{1}{2}z^2\\&=\frac{1}{2}\left(z+\frac{\mu}{\Psi(x)}y\right)^2-\frac{1}{2}\frac{\mu^2}{\Psi^2(x)}y^2+\frac{1}{2}\frac{\mu a(t)}{\Psi^2(x)},\end{aligned}$$

from hypothesis (H_1), $a_0-\mu>0$, then $\frac{a(t)\mu}{\Psi^2(x)}-\frac{\mu^2}{\Psi^2(x)}>0$, it follows that there is a positive constant k_1 such that

$$V_2(t,x_t,y_t,z_t)\geq k_1(y^2+z^2),$$

from which we deduce that V_2 is positive definite. Furthermore, from hypotheses (i) and (ii), we obtain

$$V_1(t,x_t,y_t,z_t)\geq\frac{1}{2}\left[\frac{b_0\phi_0\mu-\psi_1C\delta_1}{\mu\psi_1}\right]y^2.$$

Hence, it is evident from (H1) and the terms contained in the last inequality, that there exist sufficiently small positive constant k_2, such that

$$V_1+V_2\geq k_2(y^2+z^2).$$

Using (3.2) we get

$$V\geq\mu c_0G(x,y)+k_2(y^2+z^2). \tag{3.3}$$

Therefore we can find a continuous function $W_1(|\varphi(0)|)$ with

$$W_1(|\varphi(0)|)\geq 0\quad\text{and}\quad W_1(|\varphi(0)|)\leq V(t,\varphi).$$

The existence of a continuous function $W_2(\|\varphi\|)$ which satisfies the inequality $V(t,\varphi)\leq W_2(\|\varphi\|)$, is easily verified.

The derivative of the Lyapunov functional $V(t,x_t,y_t,z_t)$, along a solution $(x(t),y(t),z(t))$ of the system (3.1), with respect to t is after simplifying

$$
\begin{aligned}
V'_{(3.1)} &= \mu c'(t)F(x) + c'(t)yf(x) + \frac{c'(t)\delta_1}{2\mu}y^2 + (a(t)-\mu)\alpha(t)zy \\
&\quad + \frac{b(t)}{2}\beta(t)y^2 + \left(\frac{c(t)f'(x)}{\Psi(x)} - \frac{\mu b(t)\Phi(x)}{\Psi^2(x)}\right)y^2 \\
&\quad + \left(\frac{1}{2}\frac{\mu a'(t)}{\Psi^2(x)} + \frac{b'(t)\Phi(x)}{2\Psi(x)} - \frac{c'(t)\delta_1}{2\mu}\right)y^2 + \left(\frac{\mu - a(t)}{\Psi(x)}\right)z^2 + \lambda r y^2 \\
&\quad + c(t)\left(z + \frac{\mu}{\Psi(x)}y\right)\int_{t-r}^{t} y(s)\frac{f'(x(s))}{\Psi(x(s))}ds - \lambda\int_{t-r}^{t} y^2(\xi)d\xi,
\end{aligned}
$$

where

$$
\alpha(t) = \frac{\Psi'(x(t))}{\Psi^2(x(t))}x'(t), \quad \beta(t) = \frac{\Psi(x)\Phi'(x) - \Phi(x)\Psi'(x)}{\Psi^2(x)}x'(t).
$$

By the assumptions (i)–(iii), (H1)–(H2), and using the Schwartz inequality $2|uv| \le u^2 + v^2$ we find

$$
\begin{aligned}
V'_{(3.1)} &\le \mu c'(t)\left[F(x) + \frac{1}{\mu}yf(x) + \frac{\delta_1}{2\mu^2}y^2\right] \\
&\quad + \frac{1}{\psi_1}(\mu - a_0)z^2 + \left[\frac{\psi_1 C\delta_1 - \mu b_0\phi_0}{\psi_1^2} + \lambda r\right]y^2 \\
&\quad + \frac{1}{2}\left((A-\mu)\,|\alpha(t)| + B\,|\beta(t)|\right)(y^2 + z^2) \\
&\quad + \frac{1}{2\psi_1^2}\left[\mu a'(t) + b'(t)\Phi(x)\Psi(x) - \Psi^2(x)\frac{c'(t)\delta_1}{\mu}\right]y^2 \\
&\quad + c(t)\left(z + \frac{\mu}{\Psi(x)}y\right)\int_{t-r}^{t} y(s)\frac{f'(x(s))}{\Psi(x(s))}ds - \lambda\int_{t-r}^{t} y^2(\xi)d\xi.
\end{aligned}
$$

Taking $k_3 = \frac{1}{2}\max\{A-\mu, B\}$ then

$$
\begin{aligned}
V'_{(3.1)} &\le \mu c'(t)G(x,y) + \left[\frac{\psi_1 C\delta_1 - \mu b_0\phi_0}{\Psi^2(x)} + \lambda r\right]y^2 \\
&\quad + \frac{1}{2\psi_1^2}\left[\mu a'(t) + b'(t)\Phi(x)\Psi(x) - \Psi^2(x)\frac{\delta_1}{\mu}c'(t)\right]y^2 \\
&\quad + \frac{1}{\psi_1}(\mu - a_0)z^2 + k_3(|\alpha(t)| + |\beta(t)|)(y^2 + z^2) \\
&\quad + c(t)\left(z + \frac{\mu}{\Psi(x)}y\right)\int_{t-r}^{t} y(s)\frac{f'(x(s))}{\Psi(x(s))}ds - \lambda\int_{t-r}^{t} y^2(\xi)d\xi.
\end{aligned}
$$

From (iii) $|f'(x)| \le \delta_1$, and using the Schwartz inequality again we have

$$
\frac{\mu c(t)}{\Psi(x)}y\int_{t-r}^{t}\frac{y(s)}{\Psi(x)}f'(x(s))ds \le \frac{C\delta_1\mu r}{2\psi_0}y^2 + \frac{C\mu\delta_1}{2\psi_0^3}\int_{t-r}^{t} y^2(\xi)d\xi,
$$

and

$$
c(t)z\int_{t-r}^{t}\frac{y(s)}{\Psi(x)}f'(x(s))ds \le \frac{C\delta_1 r}{2}z^2 + \frac{C\delta_1}{2\psi_0^2}\int_{t-r}^{t} y^2(\xi)d\xi,
$$

from which we deduce that

$$V'_{(3.1)} \le \mu c'(t)G(x,y) + \frac{1}{2\psi_1^2}\left[\mu a'(t) + b'(t)\Phi(x)\Psi(x) - \Psi^2(x)\frac{\delta_1}{\mu}c'(t)\right]y^2$$
$$+\left[\frac{\psi_1 C\delta_1 - \mu b_0\phi_0}{\psi_1^2} + \lambda r + \frac{C\delta_1\mu r}{2\psi_0}\right]y^2 + \left[\frac{1}{\psi_1}(\mu - a_0) + \frac{C\delta_1 r}{2}\right]z^2$$
$$+k_3(|\alpha(t)| + |\beta(t)|)(y^2+z^2) + \left[\frac{C\delta_1}{2\psi_0^2}\left(1+\frac{\mu}{\psi_0}\right) - \lambda\right]\int_{t-r}^{t} y^2(\xi)d\xi.$$

Choosing $\frac{C\delta_1}{2\psi_0^2}\left(1+\frac{\mu}{\psi_0}\right) = \lambda$, and using condition (H1) we get

$$V'_{(3.1)} \le \mu c'(t)G(x,y) - \left[\frac{\mu b_0\phi_0 - \psi_1 C\delta_1}{2\psi_1^2} - \frac{C\delta_1}{2\psi_0}\left(\mu + \frac{1}{\psi_1} + \frac{\mu}{\psi_0^2}\right)r\right]y^2$$
$$-\left[\frac{a_0-\mu}{\psi_1} - \frac{C\delta_1 r}{2}\right]z^2 + k_3(|\alpha(t)| + |\beta(t)|)(y^2+z^2).$$

We define the Lyapounov functional $W = W(t, x_t, y_t, z_t)$ as

$$W(t, x_t, y_t, z_t) = (\exp -\eta(t))V(t, x_t, y_t, z_t) = (\exp -\eta(t))V,$$

where

$$\eta(t) = \int_0^t \left[\frac{1}{\gamma}(|\alpha(s)| + |\beta(s)|) + \frac{1}{c_0}|c'(s)|\right]ds,$$

and γ is a positive constant which will be determined later in the proof. It is easily verified that

$$W'_{(3.1)}(t, x_t, y_t, z_t) = (\exp -\eta(t))\left[V'_{(3.1)} - \left(\frac{1}{\gamma}(|\alpha(t)| + |\beta(t)|) + \frac{1}{c_0}|c'(t)|\right)V\right],$$

from conditions (ii) and (iv) we obtain

$$\int_0^t |\alpha(s)ds| = \int_0^t \left|\frac{\Psi'(x(s))}{\Psi^2(x(s))}x'(s)\right|ds$$
$$= \int_{\omega_1(t)}^{\omega_2(t)} \left|\frac{\Psi'(u)}{\Psi^2(u)}\right|du \le \frac{1}{\psi_0^2}\int_{\omega_1(t)}^{\omega_2(t)} |\Psi'(u)|du$$
$$< \frac{1}{\psi_0^2}\int_{-\infty}^{+\infty} |\Psi'(u)|du \le N_2 < \infty,$$

where $\omega_1(t) = \min\{x(0), x(t)\}$, $\omega_2(t) = \max\{x(0), x(t)\}$. We get also

$$\int_0^t |\beta(s)ds| \le \int_0^t \left|\Phi(x(s))\frac{\Psi'(x(s))}{\Psi^2(x(s))}x'(s)\right|ds + \int_0^t \left|\frac{\Phi'(x(s))x'(s)}{\Psi(x(s))}\right|ds$$
$$= \int_{\omega_1(t)}^{\omega_2(t)} \left|\Phi(u)\frac{\Psi'(u)}{\Psi^2(u)}\right|du + \int_{\omega_1(t)}^{\omega_2(t)} \left|\frac{\Phi'(u)}{\Psi(u)}\right|du$$
$$\le \frac{\phi_1}{\psi_0^2}\int_{\omega_1(t)}^{\omega_2(t)} |\Psi'(u)|du + \frac{1}{\psi_0}\int_{\omega_1(t)}^{\omega_2(t)} |\Phi'(u)|du$$
$$< \frac{\phi_1}{\psi_0^2}\int_{-\infty}^{+\infty} |\Psi'(u)|du + \frac{1}{\psi_0}\int_{-\infty}^{+\infty} |\Phi'(u)|du \le N_3 < \infty.$$

Using the inequality (3.3) we have

$$
\begin{aligned}
&V'_{(3.1)} - \left(\frac{1}{\gamma}(|\alpha(t)| + |\beta(t)|) + \frac{1}{c_0}|c'(t)|\right) V \\
&\quad \le -\left[\frac{\mu b_0 \phi_0 - \psi_1 C\delta_1}{2\psi_1^2} - \frac{C\delta_1}{2\psi_0}\left(\mu + \frac{1}{\psi_0} + \frac{\mu}{\psi_0^2}\right) r\right] y^2 - \left[\frac{a_0 - \mu}{\psi_1} - \frac{C\delta_1 r}{2}\right] z^2 \\
&\qquad + \left[\left(k_3\,|\alpha(t)| - \frac{k_2}{\gamma}\,|\alpha(t)|\right) + \left(k_3\,|\beta(t)| - \frac{k_2}{\gamma}\,|\beta(t)|\right)\right](y^2 + z^2).
\end{aligned}
$$

Putting $\gamma = \dfrac{k_2}{k_3}$ we obtain

$$
W'_{(3.1)} \le -K\left(\left[\frac{\mu b_0 \phi_0 - \psi_1 C\delta_1}{2\psi_1^2} - \frac{C\delta_1}{2\psi_0}\left(\mu + \frac{1}{\psi_0} + \frac{\mu}{\psi_0^2}\right) r\right] y^2 - \left[\frac{a_0 - \mu}{\psi_1} - \frac{C\delta_1 r}{2}\right] z^2\right)
$$

where $K = \exp - \left(\dfrac{k_3(N_2 + N_3)}{k_2} + \dfrac{N_1}{c_0}\right)$. If we take

$$
r < \min\left\{\frac{2(a_0 - \mu)}{\psi_1 C\delta_1}, \frac{\psi_0^3(\mu b_0 \phi_0 - \psi_1 C\delta_1)}{\psi_1^2 C\delta_1(\mu + \mu\psi_0^2 + \psi_0)}\right\}
$$

then

$$
W'_{(3.1)}(t, x_t, y_t, z_t) \le -L(y^2 + z^2), \quad \text{for some} \quad L > 0.
$$

It can also be followed that the largest invariant set in Z is $Q = \{0\}$, where

$$
Z = \{\phi \in C_H : W'_{(3.1)}(\phi) = 0\}.
$$

That is, the only solution of system (3.1) for which $W'_{(3.1)}(t, x_t, y_t, z_t) = 0$ is the solution $x = y = z = 0$. The above discussion guarantees that the null solution of Eq. (1.1) is uniformly asymptotically stable.

The proof of the theorem is now completed. □

Example We consider the following third order delay differential equation

$$
\begin{aligned}
&\left[\left(\frac{\cos(x)}{1 + x^2} + 4\right) x'(t)\right]'' + (\cos t + 15)x''(t) \\
&\quad + \left(\frac{5}{2} - \frac{1}{2}e^{-2t}\right)\left(\frac{\sin(x)}{1 + x^2} + 11\right) x'(t) \\
&\quad + \left(\sin\frac{t}{2} + 3\right)\left[x(t - r) + \frac{x(t - r)}{1 + x^2(t - r)}\right] = 0.
\end{aligned}
\tag{3.4}
$$

It can be seen that

$$
\begin{aligned}
&14 = a_0 \le a(t) = \cos t + 15 \le 16, \quad -1 \le a'(t) = -\sin t \le 1,\ t \ge 0, \\
&2 = b_0 \le b(t) = \frac{5}{2} - \frac{1}{2}e^{-2t} \le \frac{5}{2},\ 0 \le b'(t) = e^{-2t} \le 1,\ t \ge 0, \\
&2 \le c(t) = \sin\frac{t}{2} + 3 \le 4 = C,\ -\frac{1}{2} \le c'(t) = \frac{1}{2}\cos\frac{t}{2} \le \frac{1}{2},\ t \ge 0, \\
&1 \le \frac{f(x)}{x} = 1 + \frac{1}{1 + x^2} \text{ with } x \ne 0,\ \|f'(x)| \le \delta_1 = 2 \text{ and } \mu = 8,
\end{aligned}
$$

$$3 \leq \Psi(x) = \frac{\cos(x)}{1+x^2} + 4 \leq 5,$$
$$10 = \phi_0 \leq \Phi(x) = \frac{\sin(x)}{1+x^2} + 11 \leq 12.$$

An easy computations show that conditions (H1) and (H2) are satisfied. Indeed,

$$\frac{\psi_1 C}{b_0 \phi_0}\delta_1 = 2 < \mu < a_0 = 14.$$

We have also

$$\mu a'(t) + \Psi(x)\Phi(x)b'(t) - \Psi^2(x)\frac{\delta_1}{\mu}c'(t) \leq \mu + 60 + \frac{25}{\mu} = 71.12$$
$$< \mu b_0\phi_0 - \psi_1 C\delta_1 = 120.$$

It is straightforward to verify that

$$\int_{-\infty}^{+\infty} \left|\Psi'(u)\right| du \leq \int_{-\infty}^{+\infty} \left[\left|\frac{\sin u}{1+u^2}\right| + \left|\frac{2u\cos u}{(1+u^2)^2}\right|\right] du$$
$$\leq \pi + 2.$$

Similarly,

$$\int_{-\infty}^{+\infty} \left|\Phi'(u)\right| du \leq \int_{-\infty}^{+\infty} \left[\left|\frac{\cos u}{1+u^2}\right| + \left|\frac{2u\sin u}{(1+u^2)^2}\right|\right] du$$
$$\leq \pi + 2.$$

Thus all the assumptions of Theorem 3.1. hold, this shows that every solution of (3.4) is uniformly asymptotically stable.

In the case $e(t) \neq 0$ we have the following result:

Theorem 3.2 *If the assumptions of Theorem 3.1 hold true, and in addition*

$$\int_0^t e(s)ds \leq e_0 < \infty \textit{ for all } t \geq 0,$$

then all solutions of the Eq. (1.1) are bounded.

Proof The remaining of this proof follows the strategy indicated in the proof of Theorem 2 in [12] and hence it omitted. □

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