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Positive solutions to periodic boundary value problems involving nonlinear operators of Meir-Keeler-type

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Abstract We establish fixed point theorems for mixed monotone operators of Meir-Keeler type on ordered Banach spaces through projective metric. As applications, we use the fixed point theorems obtained in this paper to study the existence and uniqueness of positive solutions for different classes of nonlinear problems which include two-order two-point boundary value problems and fourth-order two-point boundary value problems for elastic beam equations.

Keywords Fixed point · Ordered Banach space · Cone · Meir-Keeler-type · Thompson metric · ε-chainable · Periodic boundary value problem · Elastic beam equation

Mathematics Subject Classification 47H07 · 47H10 · 34B15

1 Introduction and preliminaries

One of the most known results in Fixed Point Theory is the Banach contraction principle [4], which assures that every contraction from a complete metric space into itself has a unique fixed point. Many generalizations of this famous theorem and other important fixed point theorems exist in the literature (see, for example, [1, 5, 7, 8, 10, 11, 14, 15, 18, 19, 24, 25, 27, 30]).

In [18], Meir and Keeler generalized the Banach fixed point theorem as follows.

Theorem 1.1 (Meir and Keeler [18]) *Let (X,d) be a complete metric space and $T:X\to X$. Suppose that for all $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that*

$$x,y\in X,\quad \varepsilon\le d(x,y)<\varepsilon+\delta(\varepsilon)\quad \textit{implies}\quad d(Tx,Ty)<\varepsilon.$$

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Then, T admits a unique fixed point $\xi \in X$ and for all $x \in X$, the sequence $\{T^n x\}$ converges to ξ.

Edelstein [11] proposed an extension of the Banach Contraction Principle in another direction.

Definition 1.1 Let (X,d) be a metric space and $\varepsilon > 0$ be fixed. We say that (X,d) is ε-chainable if for any $x, y \in X$, there exist $x_0 = x, x_1, \dots, x_{n-1}, x_n = y \in X$ (n may depend on both x and y) such that $d(x_i, x_{i+1}) < \varepsilon$ for all $i = 0, 1, \dots, n-1$.

Theorem 1.2 (Edelstein [11]) *Let (X,d) be a complete ε-chainable metric space and $T : X \to X$. Suppose that there exists $\lambda \in (0,1)$ such that*

$$x, y \in X, \quad d(x,y) < \varepsilon \quad \textit{implies} \quad d(Tx, Ty) \le \lambda d(x,y).$$

Then T has a unique fixed point.

Blending the arguments of Meir-Keeler [18] and Edelstein [11], Jachymski [14] established the following theorem.

Theorem 1.3 (Jachymski [14]) *Let (X,d) be a complete metric space and $T : X \to X$. Suppose there exists $\varepsilon > 0$ such that*

(i) *$d(Tx, Ty) < d(x,y)$ for any $x, y \in X$ with $0 < d(x,y) < \varepsilon$;*
(ii) *for any $\eta \in (0,\varepsilon)$, there exists $\delta \in (0, \varepsilon - \eta)$ such that, for any $x, y \in X$, $d(x,y) < \eta + \delta$ implies $d(Tx, Ty) \le \eta$.*

Then, we have

(a) *If there exists $x_0 \in X$ such that $d(x_0, Tx_0) < \varepsilon$, then $\{T^n x_0\}$ converges to a fixed point of T.*
(a) *If (X,d) is ε-chainable, then T has a unique fixed point $\xi \in X$ with $\{T^n x\}$ converges to ξ for all $x \in X$.*

From Theorem 1.3, we can deduce the following result (see [6]).

Theorem 1.4 *Let (X,d) be a complete ε-chainable metric space and $T : X \to X$. Suppose that for any $a \in (0,\varepsilon)$, there exists $b \in (a,\varepsilon)$ such that*

$$x, y \in X, \quad a \le d(x,y) < b \quad \textit{implies} \quad d(Tx, Ty) < a.$$

Then T has a unique fixed point $\xi \in X$ with $\{T^n x\}$ converges to ξ for all $x \in X$.

Now, we briefly recall various basic definitions and facts.

Let $(E, \|\cdot\|)$ be a real Banach space and P a nonempty convex closed subset of E. We say that P is a cone in E if P satisfies the following two conditions:

i. $\lambda x \in P$ for any $x \in P$ and $\lambda \ge 0$;
ii. $x, -x \in P$ implies $x = 0_E$, where 0_E is the zero vector of E.

The Banach space E can be partially ordered by a cone P, that is,

$$x, y \in E, \quad x \preceq y \quad \text{if and only if} \quad y - x \in P.$$

In the remainder of the paper, we always assume that the norm is monotone, i.e.,

$$x, y \in E, \quad 0_E \preceq x \preceq y \quad \text{implies} \quad \|x\| \leq \|y\|.$$

We denote by $\mathring{P}$ the interior set of P. Two elements $x, y \in P\backslash\{0_E\}$ are said to be comparable if there exist positive numbers λ, μ such that $\mu y \preceq x \preceq \lambda y$. This equivalent relation divides $P\backslash\{0_E\}$ into disjoint components of P. If $\mathring{P} \neq \emptyset$ (solid cone), then it is a component of P. Through this paper, C will always stand for a component of P. For $x, y \in C$, if $y - x \in C$, then we denote $x \ll y$. Since $\mathring{P}$ is a component of P, $x \ll y$ means $y - x \in \mathring{P}$ if $x, y \in \mathring{P}$.

We say that $A : C \times C \to C$ is a mixed monotone operator if

$$(x, y), (u, v) \in C \times C, \quad x \preceq u, \quad y \succeq v \implies A(x, y) \preceq A(u, v).$$

For more details about ordered Banach spaces, we refer the reader to [13].

For $x, y \in C$, let

$$M\left(\frac{x}{y}\right) = \inf\{\lambda > 0 \,|\, x \preceq \lambda y\}.$$

Thompson's metric [26] is defined by

$$d(x, y) = \ln\left[\max\left\{M\left(\frac{x}{y}\right), M\left(\frac{y}{x}\right)\right\}\right].$$

We recall some standard results form the literature.

Lemma 1.1 (see [26]) *Each component C of P is complete with Thompson's metric.*

Lemma 1.2 (see [26]) *Let C be a component of P and $\{x_n\}$ be a Cauchy sequence in C with respect to Thompson's metric d. Then $\{x_n\}$ is also a Cauchy sequence in $(C, \|\cdot\|)$. Furthermore*, $\lim_{n\to\infty} d(x_n, u) = 0$, $u \in C$, *implies* $\lim_{n\to\infty} \|x_n - u\| = 0$.

In [22], Nussbaum proved that $(\mathring{P}, d)$ is ε-chainable, $\varepsilon > 0$. Using elementary arguments, Y.Z. Chen [6] proved that each component C is ε-chainable under Thompson's metric.

Theorem 1.5 (Y.Z. Chen [6]) *Let $\varepsilon > 0$. Then (C, d) is ε-chainable.*

Based on Theorems 1.4 and 1.5, Y.Z. Chen established some fixed point results for increasing mappings on ordered Banach spaces.

In this paper, using Thompson's metric, we study the existence and uniqueness of fixed points for mixed monotone operators of Meir-Keeler-type on ordered Banach spaces. Our obtained results are illustrated by some applications to the study of existence and uniqueness of positive solutions to periodic boundary value problems which include two-order two-point boundary value problems and fourth-order two-point boundary value problems for elastic beam equations.

2 Main results

Our first result is the following.

Theorem 2.1 *Let $A : C \times C \to C$ be a mixed monotone operator. Suppose that there exists $\varepsilon \in (0,1)$ such that for any $\alpha \in (\varepsilon, 1)$, we have $\beta \in (\varepsilon, \alpha)$ so that*

$$t \in (\beta, \alpha] \quad \textit{implies} \quad A\big(tx, t^{-1}x\big) \gg \alpha A(x,x). \tag{1}$$

Then A has a unique fixed point $x^* \in C$ and, for any $z_0 \in C$, $z_n \to x^*$ as $n \to \infty$, where

$$z_n = A(z_{n-1}, z_{n-1}), \quad n = 1, 2, 3, \ldots. \tag{2}$$

Proof Let $\varepsilon' = -\ln \varepsilon$. Then (C, d) is ε'-chainable (see Theorem 1.5). Given $a \in (0, \varepsilon')$, there exists $\alpha \in (0,1)$ such that $\alpha = e^{-a}$. For this α, there exists $\beta \in (\varepsilon, \alpha)$ such that (1) is satisfied. Put $b = -\ln \beta$. Then we have $a < b < \varepsilon'$.

Let $x, y \in C$ with

$$a = -\ln \alpha \le d(x,y) < b = -\ln \beta. \tag{3}$$

Without loss of generality, we can assume that

$$M\left(\frac{x}{y}\right) \ge M\left(\frac{y}{x}\right).$$

Then we have

$$d(x,y) = \ln M\left(\frac{x}{y}\right). \tag{4}$$

From (3) and (4), we get that

$$\frac{1}{\alpha} \le M\left(\frac{x}{y}\right) < \frac{1}{\beta}. \tag{5}$$

Now, using the mixed monotone property of A, we have

$$\begin{aligned} Bx = A(x,x) &\succeq A\left(\left[M\left(\frac{y}{x}\right)\right]^{-1} y, M\left(\frac{x}{y}\right) y\right) \\ &\succeq A\left(\left[M\left(\frac{x}{y}\right)\right]^{-1} y, M\left(\frac{x}{y}\right) y\right) \\ &\gg \alpha A(y,y) = \alpha By \quad \text{(from (1) and (5))}. \end{aligned}$$

Since $Bx - \alpha By \in C$ and $By \in C$, there exists $h > 0$ such that $Bx - \alpha By \succeq hBy$. Then $Bx \succeq (\alpha + h) By$, which implies that

$$M\left(\frac{By}{Bx}\right) \le \frac{1}{\alpha + h} < \frac{1}{\alpha}. \tag{6}$$

Similarly, we have

$$By = A(y,y) \succeq A\left(\left[M\left(\frac{x}{y}\right)\right]^{-1} x, M\left(\frac{y}{x}\right) x\right)$$

$$\succeq A\left(\left[M\left(\frac{x}{y}\right)\right]^{-1}x, M\left(\frac{x}{y}\right)x\right)$$
$$\gg \alpha A(x,x) = \alpha Bx.$$

By a similar argument, we can see that

$$M\left(\frac{Bx}{By}\right) < \frac{1}{\alpha}. \tag{7}$$

Now, from (6) and (7), we obtain that

$$d(Bx, By) < -\ln\alpha = a.$$

Thus, we proved that for any $a \in (0, \varepsilon')$ there exists $b \in (a, \varepsilon')$ such that

$$a \le d(x,y) < b \quad \text{implies} \quad d(Bx, By) < a.$$

By Theorem 1.4, B has a unique fixed point $x^* \in C$, and for any $z_0 \in C$, the sequence $\{z_n\}$ defined by (2) converges to x^* with respect to d, that is, $\lim_{n\to\infty} d(z_n, x^*) = 0$, which implies from Lemma 1.2 that $\lim_{n\to\infty}\|z_n - x^*\| = 0$. □

Remark 2.1 Theorem 2.1 extends and generalizes Theorem 2.6 in [6].

From Theorem 2.1, we can deduce the following result.

Theorem 2.2 *Let $A : C \times C \to C$ be a mixed monotone operator. Suppose that there exists $\varepsilon \in (0,1)$ such that for any $t \in (\varepsilon, 1)$, we have*

$$A\left(tx, t^{-1}x\right) \succeq \varphi(t)A(x,x), \tag{8}$$

where $\varphi : (\varepsilon, 1) \to (\varepsilon, 1]$ is left-continuous and $\varphi(t) > t$. Then A has a unique fixed point $x^* \in C$ and, for any $z_0 \in C$, $z_n \to x^*$ as $n \to \infty$, where

$$z_n = A(z_{n-1}, z_{n-1}), \quad n = 1, 2, 3, \ldots$$

Proof Let $t_0 \in (\varepsilon, 1)$ be given and denote

$$\delta = \varphi(t_0) - t_0 > 0.$$

Since φ is left-continuous, we have $\lim_{s\to t_0^-}\varphi(s) = \varphi(t_0)$, that is, there exists $\eta > 0$ such that

$$0 \le t_0 - s < \eta \implies \left|\varphi(s) - \varphi(t_0)\right| < \delta. \tag{9}$$

Without loss of generality, we can suppose that $t_0 - \eta > \varepsilon$. Then, from (8), for any $s \in (t_0 - \eta, t_0]$, we have

$$A\left(sx, s^{-1}x\right) - t_0A(x,x) \succeq \varphi(s)A(x,x) - t_0A(x,x) = \left(\varphi(s) - t_0\right)A(x,x). \tag{10}$$

Note that from (9), we have

$$\varphi(s) - t_0 > \varphi(t_0) - \delta - t_0 = 0.$$

On the other hand, since $A(sx, s^{-1}x) \in C$ and $A(x, x) \in C$, there exists $\lambda > 0$ such that

$$A\big(sx, s^{-1}x\big) - t_0 A(x, x) \preceq A\big(sx, s^{-1}x\big) \preceq \lambda A(x, x). \tag{11}$$

From (10) and (11), we have

$$\big(\varphi(s) - t_0\big)A(x, x) \preceq A\big(sx, s^{-1}x\big) - t_0 A(x, x) \preceq \lambda A(x, x),$$

which implies that $A(sx, s^{-1}x) - t_0 A(x, x) \in C$, that is,

$$A\big(sx, s^{-1}x\big) \gg t_0 A(x, x).$$

Hence A satisfies (1). Applying Theorem 2.1, we get the desired result. □

Theorem 2.3 *Let $A : C \times C \to C$ be a mixed monotone operator and $x_0 \in C$. Suppose that there exists $\varepsilon \in (0, 1)$ such that for any $t \in (\varepsilon, 1)$, we have*

$$A\big(tx, t^{-1}x\big) \succeq \varphi(t)A(x, x), \tag{12}$$

where $\varphi : (\varepsilon, 1) \to (\varepsilon, 1]$ is left-continuous and $\varphi(t) > t$. Then there exists a unique $x^* \in C$ solution to the operator equation:

$$x = A(x, x) + x_0. \tag{13}$$

Moreover, for any $z_0 \in C$, $z_n \to x^*$ as $n \to \infty$, where

$$z_n = A(z_{n-1}, z_{n-1}) + x_0, \quad n = 1, 2, 3, \ldots$$

Proof Consider the operator $B : C \times C \to C$ given by

$$B(x, y) = A(x, y) + x_0$$

for all $x, y \in C$. Clearly B is a mixed monotone operator. From (12), for any $t \in (\varepsilon, 1)$,

$$B\big(tx, t^{-1}x\big) = A\big(tx, t^{-1}x\big) + x_0 \succeq \varphi(t)A(x, x) + x_0 \succeq \varphi(t)\big(A(x, x) + x_0\big) = \varphi(t)B(x, x),$$

for any $x \in C$. Now, by Theorem 2.2, the operator B has a unique fixed point $x^* \in C$, that is, $x^* \in C$ is the unique solution to (13). Also, we have for any $z_0 \in C$, $z_n \to x^*$ as $n \to \infty$, where

$$z_n = B(z_{n-1}, z_{n-1}) = A(z_{n-1}, z_{n-1}) + x_0, \quad n = 1, 2, 3, \ldots$$

Thus we proved our theorem. □

Now, we shall prove the following result.

Theorem 2.4 *Let $A : C \times C \to C$ be a mixed monotone operator. Suppose that for any $t \in (0, 1)$, there exists $r = r(t) \in (0, 1)$ such that*

$$A\big(tx, t^{-1}x\big) \gg t^r A(x, x). \tag{14}$$

Then A has a unique fixed point $x^* \in C$ and, for any $z_0 \in C$, $z_n \to x^*$ as $n \to \infty$, where

$$z_n = A(z_{n-1}, z_{n-1}), \quad n = 1, 2, 3, \ldots \tag{15}$$

Proof Let $\varepsilon > 0$, then there exists $\alpha \in (0,1)$ such that $\varepsilon = -\ln\alpha$. Denote $\delta(\varepsilon) = (r(\alpha)^{-1} - 1)\varepsilon$. Let $x, y \in C$ such that

$$d(x,y) < \varepsilon + \delta(\varepsilon) = -r(\alpha)^{-1}\ln\alpha. \tag{16}$$

Without loss of generality, we can assume that

$$M\left(\frac{x}{y}\right) \geq M\left(\frac{y}{x}\right),$$

which implies that

$$d(x,y) = \ln M\left(\frac{x}{y}\right).$$

From (16), we have

$$\alpha^{1/r(\alpha)} < \left[M\left(\frac{x}{y}\right)\right]^{-1}. \tag{17}$$

Now, using (14), (17) and the mixed monotone property of A, we have

$$\begin{aligned} Bx = A(x,x) &\succeq A\left(\left[M\left(\frac{y}{x}\right)\right]^{-1} y, M\left(\frac{x}{y}\right) y\right) \\ &\succeq A\left(\left[M\left(\frac{x}{y}\right)\right]^{-1} y, M\left(\frac{x}{y}\right) y\right) \\ &\succeq A\left(\alpha^{1/r(\alpha)} y, \alpha^{-1/r(\alpha)} y\right) \\ &\gg \alpha A(y,y) = \alpha By. \end{aligned}$$

Since $Bx - \alpha By \in C$ and $By \in C$, there exists $h > 0$ such that $Bx - \alpha By \succeq hBy$. Then $Bx \succeq (\alpha + h)By$, which implies that

$$M\left(\frac{By}{Bx}\right) \leq \frac{1}{\alpha + h} < \frac{1}{\alpha}. \tag{18}$$

Similarly, we have

$$\begin{aligned} By = A(y,y) &\succeq A\left(\left[M\left(\frac{x}{y}\right)\right]^{-1} x, M\left(\frac{y}{x}\right) x\right) \\ &\succeq A\left(\left[M\left(\frac{x}{y}\right)\right]^{-1} x, M\left(\frac{x}{y}\right) x\right) \\ &\succeq A\left(\alpha^{1/r(\alpha)} x, \alpha^{-1/r(\alpha)} x\right) \\ &\gg \alpha A(x,x) = \alpha Bx. \end{aligned}$$

By a similar argument, we can see that

$$M\left(\frac{Bx}{By}\right) < \frac{1}{\alpha}. \tag{19}$$

From (18) and (19), we obtain that

$$d(Bx, By) < -\ln\alpha = \varepsilon.$$

Thus, we proved that for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$d(x, y) < \varepsilon + \delta(\varepsilon) \quad \text{implies} \quad d(Bx, By) < \varepsilon.$$

By Theorem 1.1, B has a unique fixed point $x^* \in C$, and for any $z_0 \in C$, the sequence $\{z_n\}$ defined by (15) converges to x^* with respect to d, that is, $\lim_{n\to\infty} d(z_n, x^*) = 0$, which implies from Lemma 1.2 that $\lim_{n\to\infty} \|z_n - x^*\| = 0$. □

Theorem 2.5 *Let $F, G, H : C \times C \to C$ be mixed monotone operators. Suppose that*

(i) *there exists $\alpha \in (0, 1)$ such that for all $t \in (0, 1)$, for all $x \in C$,*

$$F\big(tx, t^{-1}x\big) \succeq t^{\alpha} F(x, x);$$

(ii) *for all $t \in (0, 1)$, for all $x \in C$,*

$$G\big(tx, t^{-1}x\big) \succeq tG(x, x);$$

(iii) *for all $\lambda > 0$, for all $x \in C$,*

$$H\big(\lambda x, \lambda^{-1}x\big) = \lambda H(x, x);$$

(iv) *there exist $\delta_1, \delta_2 > 0$ such that for all $x \in C$,*

$$F(x, x) \gg \delta_1 G(x, x) + \delta_2 H(x, x).$$

Then there exists a unique $x^* \in C$ solution to

$$x = F(x, x) + G(x, x) + H(x, x).$$

Moreover, for any $z_0 \in C$, $z_n \to x^*$ as $n \to \infty$, where

$$z_n = F(z_{n-1}, z_{n-1}) + G(z_{n-1}, z_{n-1}) + H(z_{n-1}, z_{n-1}), \quad n = 1, 2, 3, \ldots$$

Proof Define an operator $A : C \times C \to C$ by

$$A(x, y) = F(x, y) + G(x, y) + H(x, y), \quad \text{for all } x, y \in C.$$

Clearly, A is a mixed monotone operator.

Let $\delta_0 = \min\{\delta_1, \delta_2\}$. From (iv), we have

$$F(x, x) \gg \delta_0\big(G(x, x) + H(x, x)\big), \quad \text{for all } x \in C. \tag{20}$$

We claim that for all $t \in (0, 1)$, there exists $r = r(t) \in (\alpha, 1)$ such that

$$0 < \frac{t^r - t}{t^{\alpha} - t^r} \le \delta_0. \tag{21}$$

Suppose that there exists $t \in (0,1)$ such that

$$\frac{t^r - t}{t^\alpha - t^r} > \delta_0, \quad \text{for all } r \in (\alpha, 1).$$

Letting $r \to 1^-$ in the above inequality, we get that

$$0 = \lim_{r \to 1^-} \frac{t^r - t}{t^\alpha - t^r} \geq \delta_0,$$

which is a contradiction with $\delta_0 > 0$. Then our claim (21) holds.

From (20) and (21), we get that

$$F(x,x) \gg \delta_0\big(G(x,x) + H(x,x)\big) \gg \frac{t^{r(t)} - t}{t^\alpha - t^{r(t)}}\big(G(x,x) + H(x,x)\big),$$
$$\text{for all } t \in (0,1),\ x \in C.$$

Then we obtain

$$t^\alpha F(x,x) + tG(x,x) + tH(x,x) \gg t^{r(t)}\big(F(x,x) + G(x,x) + H(x,x)\big),$$
$$\text{for all } t \in (0,1),\ x \in C.$$

Consequently, from (i)–(iii), for any $t \in (0,1)$ and $x \in C$, we have

$$\begin{aligned} A\big(tx, t^{-1}x\big) &= F\big(tx, t^{-1}x\big) + G\big(tx, t^{-1}x\big) + H\big(tx, t^{-1}x\big) \\ &\succeq t^\alpha F(x,x) + tG(x,x) + tH(x,x) \\ &\gg t^{r(t)}\big(F(x,x) + G(x,x) + H(x,x)\big) \\ &= t^{r(t)} A(x,x). \end{aligned}$$

Now the desired result follows immediately from Theorem 2.4. □

An immediate consequence of Theorem 2.5 is the following result.

Corollary 2.1 *Let $F, G, H : C \times C \to C$ be mixed monotone operators. Suppose that*

(i) *there exists $\alpha \in (0,1)$ such that for all $t \in (0,1)$, for all $x \in C$,*

$$F\big(tx, t^{-1}x\big) \succeq t^\alpha F(x,x);$$

(ii) *for all $t \in (0,1)$, for all $x \in C$,*

$$G\big(tx, t^{-1}x\big) \succeq tG(x,x);$$

(iii) *for all $\lambda > 0$, for all $x \in C$,*

$$H(\lambda x, \lambda^{-1}x) = \lambda H(x,x);$$

(iv) *there exist $\delta_1, \delta_2 > 0$ such that for all $x \in C$,*

$$F(x,x) \gg \delta_1 G(x,x) + \delta_2 H(x,x).$$

Then the operator equation $F(x,x)+G(x,x)+H(x,x)=\lambda x$ *has a unique solution* $x_\lambda^* \in C$ *for any given* $\lambda > 0$. *Moreover, constructing successively the sequence* $z_n = \frac{1}{\lambda}(F(z_{n-1},z_{n-1})+G(z_{n-1},z_{n-1})+H(z_{n-1},z_{n-1})), n=1,2,3,\dots$ *for any initial value* $z_0 \in C$, *we have* $z_n \to x_\lambda^*$ *as* $n \to \infty$.

3 Applications

Boundary value problems for ordinary differential equations play a very important role in both theory and applications. They are used to describe a large number of physical, biological and chemical phenomena. Here, we illustrate how the results of Sect. 2 can be used in the study of certain boundary value problems.

3.1 Periodic boundary value problem of second order

Consider the periodic boundary value problem (PBVP),

$$\begin{cases} -x''(t) = f(t,x(t),x(t)), & 0<t<1 \\ x(0)=x(1), & x'(0)=x'(1), \end{cases} \tag{22}$$

where $I=[0,1]$ and $f : I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous; $f \in C(I\times\mathbb{R}\times\mathbb{R},\mathbb{R})$.

The study of PBVP (22) has been discussed in several papers by many authors for different aspects of the solutions. See for example, Lakshmikantham and Leela [16], Leela [17], Nieto [20, 21], Yao [29], and the references therein. In this section, we apply our main results to study the existence and uniqueness of solutions to (22).

Suppose that there exists a constant $M>0$ such that

(i) $f(s,u,v)+\frac{M}{2}(u+v)>0$ for all $s\in I$, $u,v>0$;
(ii) $f(s,u,v)-f(s,w,z)\geq \frac{M}{2}[(u-w)+(z-v)]$ for all $s\in I$, $u\geq w$, $v\leq z$;
(iii) there exists $\varepsilon\in(0,1)$ such that for any $\alpha\in(\varepsilon,1)$, we have $\beta\in(\varepsilon,\alpha)$, so that $k\in(\beta,\alpha]$ implies

$$f\big(s,ku,k^{-1}u\big)-\alpha f(s,u,u)+M(k-\alpha)u>0 \quad \text{for all } u>0.$$

We have the following result.

Theorem 3.1 *Under the assumptions* (i)–(iii), *PBVP* (22) *has a unique solution* $x^*\in C(I,(0,\infty))$. *Moreover, for any* $z_0\in C(I,(0,\infty))$, *the sequence* $\{z_n\}$ *defined by*

$$z_n(t)=\int_0^1 G^*(t,s)\big[f\big(s,z_{n-1}(s),z_{n-1}(s)\big)+Mz_{n-1}(s)\big]\,ds, \quad t\in I,\ n=1,2,3,\dots$$

converges to x^*, *that is*, $\|z_n-x^*\|_\infty\to 0$ *as* $n\to\infty$, *where* G^* *is given by* (24).

Proof It is well known that $x\in C^2(I,\mathbb{R})$ is a solution of (22) iff $x\in C(I,\mathbb{R})$ is a solution of the nonlinear Fredholm integral equation

$$x(t)=\int_0^1 G^*(t,s)\big[f\big(s,x(s),x(s)\big)+Mx(s)\big]\,ds, \quad t\in I, \tag{23}$$

where

$$G^*(t,s)=\begin{cases}\mu^{-1}[e^{\sqrt{M}(s-t)}+e^{\sqrt{M}(1-s+t)}], & t\le s,\\ \mu^{-1}[e^{\sqrt{M}(t-s)}+e^{\sqrt{M}(1-t+s)}], & t>s,\end{cases} \tag{24}$$

and $\mu=2\sqrt{M}(e^{\sqrt{M}}-1)$.

Let us consider the operator A defined by

$$A(x,y)(t)=\int_0^1 G^*(t,s)\big[f\big(s,x(s),y(s)\big)+(M/2)x(s)+(M/2)y(s)\big]\,ds, \quad t\in I.$$

Condition (i) implies that $A:C(I,(0,\infty))\times C(I,(0,\infty))\to C(I,(0,\infty))$. Condition (ii) implies that A is a mixed monotone operator with respect to the partial ordered in $C(I,\mathbb{R})$ induced by the solid cone $C(I,[0,\infty))$. Note that $C(I,\mathbb{R})$ is a Banach space with respect to $\|\cdot\|_\infty$. From condition (iii), for all $k\in(\beta,\alpha]$, for all $u\in$, for all $t\in I$, we have

$$\begin{aligned}A\big(kx,k^{-1}x\big)(t)&=\int_0^1 G^*(t,s)\big[f\big(s,kx(s),k^{-1}x(s)\big)+(M/2)kx(s)+(M/2)k^{-1}x(s)\big]\,ds\\ &>\int_0^1 G^*(t,s)\big[f\big(s,kx(s),k^{-1}x(s)\big)+Mkx(s)\big]\,ds\\ &>\int_0^1 G^*(t,s)\big[\alpha f\big(s,x(s),x(s)\big)+M(\alpha-k)x(s)+Mkx(s)\big]\,ds\\ &=\alpha\int_0^1 G^*(t,s)\big[f\big(s,x(s),x(s)\big)+Mx(s)\big]\,ds\\ &=\alpha A(x,x)(t).\end{aligned}$$

Then for all $k\in(\beta,\alpha]$, for all $u\in C(I,(0,\infty))$, we have $A(kx,k^{-1}x)-\alpha A(x,x)\in C(I,(0,\infty))$, that is, $A(kx,k^{-1}x)\gg\alpha A(x,x)$.

Now, Applying Theorem 2.1, we obtain that the operator equation $x=A(x,x)$ has a unique solution $x^*\in C(I,(0,\infty))$, that is, (23) has a unique solution $x^*\in C(I,(0,\infty))$. Moreover, for any $z_0\in C(I,(0,\infty))$, the sequence $\{z_n\}$ defined by

$$z_n(t)=A(z_{n-1},z_{n-1})(t), \quad t\in I,\ n=1,2,3,\ldots$$

converges to x^*, that is, $\|z_n-x^*\|_\infty\to 0$ as $n\to\infty$. □

3.2 Nonlinear elastic beam equations

We shall study the existence and uniqueness of positive solutions for the following fourth-order two-point boundary value problem for elastic beam equations

$$\begin{cases}x''''(t)=f(t,x(t),x(t)), & 0<t<1\\ x(0)=x'(0)=0, \quad x''(1)=0, \quad x'''(1)=g(x(1)),\end{cases} \tag{25}$$

where $I=[0,1]$, $f\in C(I\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $g\in C(\mathbb{R},\mathbb{R})$.

The problem (25) models an elastic beam of length $L=1$ subject to a nonlinear foundation given by the function f. The first boundary condition $x(0)=x'(0)=0$ means that the left end of the beam is fixed. The boundary condition $x''(1) = 0$, $x'''(1) = g(x(1))$ means that the right end of the beam is attached to a bearing device, given by the function g.

Graef and Yang in [12] established some results on the existence and nonexistence of positive solutions to the problem consisting of the differential equation $x''''(t) = \lambda a(t) f(x(t))$, $0 < t < 1$ with $g \equiv 0$, via the famous Guo-Krasnosel'skii fixed point theorem [13], where λ is a parameter. In [3], Bai obtained some existence results via the lower and upper solution method. By using the Leggett-Williams fixed point theorem, Yang [28] established an existence criterion for triple positive solutions of the nonlinear problem (25). Some results on the existence and multiplicity of positive solutions are obtained in [23] by using Leray-Schauder nonlinear alternate [9], Leray-Schauder fixed point theorem and a fixed point theorem due to Avery and Peterson [2]. In all theses works, only existing results were considered. Very recently, Zhai and Anderson [31] established an existence and uniqueness result of positive solution to (25) by supposing that the source term $f(t,x,x)$ is increasing with respect to x. Inspired by this work, we consider here the case where $f(t,\cdot,\cdot)$ is mixed monotone.

We shall prove the following result.

Theorem 3.2 *Assume that*

(i) $f(t,x,y) : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ *is continuous and* $g : [0,\infty) \to (-\infty,0)$ *is continuous, decreasing*;

(ii) *for fixed* $t \in [0,1]$, $f(t,x,y)$ *is increasing in* $x \in [0,\infty)$, $f(t,x,y)$ *is decreasing in* $y \in [0,\infty)$, *and there exists a constant* $\alpha \in (0,1)$ *such that*

$$f\big(t, ku, k^{-1}u\big) \geq k^{\alpha} f(t,u,u), \quad \forall t \in [0,1],\ k \in (0,1),\ u \in [0,\infty); \tag{26}$$

(iii) *for any* $k \in (0,1)$ *and* $u \in [0,\infty)$, $g(ku) \leq kg(u)$;

(iv) *there exists a constant* $\rho > 0$ *such that*

$$0 < -g(u) \leq \rho < \frac{1}{2}\int_0^1 s^2 f(s,0,\lambda)\,ds, \quad u \geq 0,\ \lambda > 0. \tag{27}$$

Then (25) *has a unique solution* $x^* \in C$, *where*

$$C = \big\{x \in C\big([0,1],\mathbb{R}\big) \,|\, \exists \lambda, \mu > 0,\ \mu t^2 \leq x(t) \leq \lambda t^2,\ t \in [0,1]\big\}.$$

Moreover, for any $z_0 \in C$, *the sequence* $\{z_n\}$ *defined by*

$$z_n(t) = \int_0^1 G(t,s) f\big(s, z_{n-1}(s), z_{n-1}(s)\big)\,ds - g\big(z_{n-1}(1)\big)\left(\frac{1}{2}t^2 - \frac{1}{6}t^3\right),$$
$$t \in [0,1],\ n = 1,2,3,\ldots$$

converges to x^*, *that is,* $\|z_n - x^*\|_\infty \to 0$ *as* $n \to \infty$, *where* G *is given by* (28).

Proof Set $E = C([0,1],\mathbb{R})$ the Banach space of continuous functions on $[0,1]$ with the norm $\|\cdot\|_\infty$. Set $P = \{x \in E \,|\, x(t) \geq 0, \quad t \in [0,1]\}$. Clearly P is a normal cone of which the normality constant is 1. Now, let $h \in E$ given by $h(t) = t^2$ for all $t \in [0,1]$. Then

$$C = \big\{x \in C\big([0,1],\mathbb{R}\big) \,|\, \exists\, \lambda, \mu > 0,\ \mu t^2 \leq x(t) \leq \lambda t^2,\ t \in [0,1]\big\}$$

is a component of P associated to h.

The integral formulation of (25) is given by

$$x(t)=\int_0^1 G(t,s)f\big(s,x(s),x(s)\big)\,ds-g\big(x(1)\big)\left(\frac{1}{2}t^2-\frac{1}{6}t^3\right),\quad t\in[0,1],$$

where the green function $G(t,s)$ is given by

$$G(t,s)=\frac{1}{6}\begin{cases}s^2(3t-s), & 0\le s\le t\le 1,\\ t^2(3s-t), & 0\le t\le s\le 1.\end{cases}\tag{28}$$

It is easy to show that

$$\frac{1}{3}t^2s^2\le G(t,s)\le\frac{1}{2}t^2s,\quad \text{for all } t,s\in[0,1].\tag{29}$$

Define the operators $A,B:P\times P\to E$ by

$$A(x,y)(t)=\int_0^1 G(t,s)f\big(s,x(s),y(s)\big)\,ds,\qquad B(x,y)(t)=-g\big(x(1)\big)\left(\frac{1}{2}t^2-\frac{1}{6}t^3\right)$$

for all $x,y\in P$. Then x is the solution of (25) if and only if $x=A(x,x)+B(x,x)$.

Now, let $x,y\in C$. Then there exist $\lambda,\mu>0$ such that

$$\mu t^2\le x(t),y(t)\le\lambda t^2,\quad \text{for all } t\in[0,1].\tag{30}$$

For all $t\in[0,1]$, we have

$$\frac{-g(x(1))}{3}t^2\le B(x,y)(t)=-g\big(x(1)\big)\left(\frac{1}{2}t^2-\frac{1}{6}t^3\right)\le\frac{-g(x(1))}{2}t^2.\tag{31}$$

Since $g(x(1))<0$, the above inequality implies that $B(x,y)\in C$.

Using (29), (30) and condition (ii), we have

$$\begin{aligned}A(x,y)(t)&=\int_0^1 G(t,s)f\big(s,x(s),y(s)\big)\,ds\\&\le\frac{1}{2}t^2\int_0^1 sf\big(s,\lambda s^2,\mu s^2\big)\,ds\\&\le\frac{1}{2}t^2\int_0^1 sf(s,\lambda,0)\,ds.\end{aligned}$$

On the other hand, from (27), we have

$$\begin{aligned}\int_0^1 sf(s,\lambda,0)\,ds&\ge\int_0^1 s^2f(s,\lambda,0)\,ds\\&\ge\int_0^1 s^2f(s,0,0)\,ds\\&\ge\int_0^1 s^2f(s,0,\lambda)\,ds>0.\end{aligned}$$

Thus we get that

$$A(x, y)(t) \le \theta t^2, \quad \theta = \frac{1}{2}\int_0^1 sf(s, \lambda, 0)\, ds > 0, \quad t \in [0, 1]. \tag{32}$$

Also, we have

$$\begin{aligned} A(x, y)(t) &= \int_0^1 G(t, s) f\big(s, x(s), y(s)\big)\, ds \\ &\ge \frac{1}{3}t^2 \int_0^1 s^2 f\big(s, \mu s^2, \lambda s^2\big)\, ds \\ &\ge \frac{1}{3}t^2 \int_0^1 s^2 f(s, 0, \lambda)\, ds. \end{aligned}$$

Thus we get that

$$A(x, y)(t) \ge \beta t^2, \quad \beta = \frac{1}{3}\int_0^1 s^2 f(s, 0, \lambda)\, ds > 0, \ t \in [0, 1]. \tag{33}$$

Combining (32) and (33), we obtain

$$\beta t^2 \le A(x, y)(t) \le \theta t^2, \quad t \in [0, 1]. \tag{34}$$

This implies that $A(x, y) \in C$.

Thus we proved that the operators $A, B : C \times C \to C$ are well defined.

Let $x, y, u, v \in C$ such that $x(t) \le u(t)$ and $y(t) \ge v(t)$ for all $t \in [0, 1]$. Using (ii), that is, for fixed $t \in [0, 1]$, $f(t, x, y)$ is increasing in $x \in [0, \infty)$, $f(t, x, y)$ is decreasing in $y \in [0, \infty)$, we have

$$A(x, y)(t) = \int_0^1 G(t, s) f\big(s, x(s), y(s)\big)\, ds \le \int_0^1 G(t, s) f\big(s, u(s), v(s)\big)\, ds = A(u, v)(t)$$

for all $t \in [0, 1]$. Using (i), that is, $g : [0, \infty) \to (-\infty, 0)$ is decreasing, we have

$$B(x, y)(t) = -g\big(x(1)\big)\left(\frac{1}{2}t^2 - \frac{1}{6}t^3\right) \le -g\big(u(1)\big)\left(\frac{1}{2}t^2 - \frac{1}{6}t^3\right) = B(u, v)(t)$$

for all $t \in [0, 1]$. Thus we proved that $A, B : C \times C \to C$ are mixed monotone operators.

Let $k \in (0, 1)$ and $x \in C$. For all $t \in [0, 1]$, using (26), we get that

$$\begin{aligned} A\big(kx, k^{-1}x\big)(t) &= \int_0^1 G(t, s) f\big(s, kx(s), k^{-1}x(s)\big)\, ds \\ &\ge k^\alpha \int_0^1 G(t, s) f\big(s, x(s), x(s)\big)\, ds \\ &= k^\alpha A(x, x)(t). \end{aligned}$$

Using (iii), for all $t \in [0, 1]$, we have

$$B\big(kx, k^{-1}x\big)(t) = -g\big(kx(1)\big)\left(\frac{1}{2}t^2 - \frac{1}{6}t^3\right)$$

$$
\begin{aligned}
&\geq -kg\big(x(1)\big)\left(\frac{1}{2}t^2 - \frac{1}{6}t^3\right) \\
&= kB(x,x)(t).
\end{aligned}
$$

Thus we proved that for all $x \in C$, for all $k \in (0,1)$, we have

$$A\big(kx, k^{-1}x\big) \succeq k^{\alpha} A(x,x) \quad \text{and} \quad B\big(kx, k^{-1}x\big) \succeq kB(x,x).$$

Let $x \in C$. Then there exist $\lambda, \mu > 0$ such that

$$\mu t^2 \leq x(t) \leq \lambda t^2, \quad \text{for all } t \in [0,1]. \tag{35}$$

Using (31) and (34), for all $t \in [0,1]$, we have

$$Mt^2 \leq A(x,x)(t) - B(x,x)(t) \leq Nt^2, \tag{36}$$

where

$$N = \frac{g(x(1))}{3} + \frac{1}{2}\int_0^1 sf(s,\lambda,0)\,ds \quad \text{and} \quad M = \frac{g(x(1))}{2} + \frac{1}{3}\int_0^1 s^2 f(s,0,\lambda)\,ds.$$

We claim that

$$N > 0 \quad \text{and} \quad M > 0. \tag{37}$$

Using (27), we have

$$
\begin{aligned}
M &= \frac{g(x(1))}{2} + \frac{1}{3}\int_0^1 s^2 f(s,0,\lambda)\,ds \\
&> \frac{g(x(1))}{2} + \frac{2}{3}\rho \\
&\geq \frac{g(x(1))}{2} + \frac{2}{3}\big(-g\big(x(1)\big)\big) \\
&= \frac{1}{6}\big(-g\big(x(1)\big)\big) > 0.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
N &= \frac{g(x(1))}{3} + \frac{1}{2}\int_0^1 sf(s,\lambda,0)\,ds \\
&\geq \frac{g(x(1))}{3} + \frac{1}{2}\int_0^1 s^2 f(s,\lambda,0)\,ds \\
&\geq \frac{g(x(1))}{3} + \frac{1}{2}\int_0^1 s^2 f(s,0,\lambda)\,ds \\
&> \frac{g(x(1))}{3} + \rho \\
&\geq \frac{g(x(1))}{3} - g\big(x(1)\big)
\end{aligned}
$$

$$= \frac{2}{3}\big(-g\big(x(1)\big)\big) > 0.$$

Thus we proved that our claim (37) holds.

From (36) and (37), we have

$$A(x,x) - B(x,x) \in C, \quad \text{for all } x \in C,$$

that is

$$A(x,x) \gg \delta_1 B(x,x), \quad \text{for all } x \in C,$$

with $\delta_1 = 1$.

Now, the desired result follows immediately from Theorem 2.5. □

We give now an example to illustrate our Theorem 3.2.

Example 3.1 Consider the elastic beam equations

$$\begin{cases} x''''(t) = 2 + \frac{t-1}{1+[x(t)]^{1/3}}, & 0 < t < 1 \\ x(0) = x'(0) = 0, \quad x''(1) = 0, & x'''(1) = -1/9. \end{cases} \tag{38}$$

Problem (38) is equivalent to

$$\begin{cases} x''''(t) = f\big(t, x(t), x(t)\big), & 0 < t < 1 \\ x(0) = x'(0) = 0, \quad x''(1) = 0, & x'''(1) = g\big(x(1)\big), \end{cases} \tag{39}$$

where

$$f(t,u,v) = \frac{t + [u]^{1/3}}{1 + [v]^{1/3}} + 1, \quad \text{for all } t \in [0,1],\ u \geq 0,\ v \geq 0$$

and

$$g(u) = -\frac{1}{9}, \quad \text{for all } u \geq 0.$$

We shall check that conditions (26) and (27) of Theorem 3.2 are satisfied.

Let $t \in [0,1]$, $k \in (0,1)$ and $u \geq 0$. We have

$$\begin{aligned} f\big(t, ku, k^{-1}u\big) &= \frac{t + [ku]^{1/3}}{1 + [k^{-1}u]^{1/3}} + 1 \\ &\geq \frac{t[k]^{1/3} + [ku]^{1/3}}{1 + [k]^{-1/3}[u]^{1/3}} + [k]^{2/3} \\ &\geq \frac{[k]^{1/3}(t + [u]^{1/3})}{[k]^{-1/3}(1 + [u]^{1/3})} + [k]^{2/3} \\ &= [k]^{2/3} f(t,u,u). \end{aligned}$$

Thus we proved that (26) is satisfied with $\alpha = 2/3$.

Let $u \geq 0$ and $\lambda > 0$. We have

$$\frac{1}{2}\int_0^1 s^2 f(s,0,\lambda)\,ds = \frac{1}{2}\int_0^1 \left(\frac{s^3}{1 + [\lambda]^{1/3}} + s^2\right) ds$$

$$= \frac{1}{2}\left(\frac{1}{4}\frac{1}{1+[\lambda]^{1/3}} + \frac{1}{3}\right)$$
$$> \rho = \frac{1}{6} > \frac{1}{9} = -g(u).$$

Then condition (27) is satisfied.

Finally from Theorem 3.2, Problem (38) has a unique solution $x^* \in C$. Moreover, for any $z_0 \in C$, the sequence $\{z_n\}$ defined by

$$z_n(t) = \int_0^1 G(t,s) f\big(s, z_{n-1}(s), z_{n-1}(s)\big)\, ds - g\big(z_{n-1}(1)\big)\left(\frac{1}{2}t^2 - \frac{1}{6}t^3\right),$$
$$t \in [0,1],\ n = 1, 2, 3, \ldots \tag{40}$$

converges to x^*.

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