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Rosaria Di Nardo · Adamaria Perrotta

An approach via symmetrization methods to nonlinear elliptic problems with a lower order term

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Abstract. In this paper we consider a class of nonlinear elliptic problems of the type

$$\begin{cases} -\operatorname{div}(\mathbf{a}(x,\nabla u)) - \operatorname{div}(\Phi(x,u)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of R^N, $N \geq 2$, f is a $L^1(\Omega)$ function or a Radon measure with bounded total variation. We fix some structural conditions on a and Φ to prove uniqueness results when $f \in L^1(\Omega)$.

Keywords Rearrangements · Schwarz symmetrization · Isoperimetric inequalities · Measure data

Mathematics Subject Classification (2000) 35J25 · 35J60

1 Introduction

In this paper we consider a class of nonlinear elliptic problems of the type

$$\begin{cases} -\operatorname{div}(\mathbf{a}(x,\nabla u)) - \operatorname{div}(\Phi(x,u)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

R. Di Nardo (✉)
Dipartimento di Matematica e Applicazioni "Renato Caccioppoli", Università degli Studi di Napoli "Federico II", Complesso Monte S.Angelo, via Cintia - 80126 Napoli (Italy)
Tel.: +39-081-675703, Fax: +39-081675636
E-mail: rosaria.dinardo@unina.it

A. Perrotta
Dipartimento di Matematica, Seconda Università di Napoli, Via Vivaldi - 81100 Caserta (Italy)
Tel.: +39-081675703, Fax: +39-081675636
E-mail: adamaria.perrotta@unina.it

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$ and f is a function belonging to $L^1(\Omega)$. Here $\mathbf{a} : (x,z) \in \Omega \times \mathbb{R}^N \to \mathbf{a}(x,z) = (a_i(x,z)) \in \mathbb{R}^N$ is a Carathéodory function satisfying the following conditions. First, constants λ, p, Λ and C exist such that $\lambda > 0$, $2 - \frac{1}{N} < p < N$,

$$\mathbf{a}(x,\xi) \cdot \xi \geq \lambda |\xi|^p, \tag{2}$$

$$|\mathbf{a}(x,\xi)| \leq \Lambda |\xi|^{p-1} + C, \tag{3}$$

for almost every $x \in \mathbb{R}^N$ and every $\xi \in \mathbb{R}^N$. Second,

$$(\mathbf{a}(x,\xi) - \mathbf{a}(x,\eta)) \cdot (\xi - \eta) > 0, \tag{4}$$

for almost every $x \in \mathbb{R}^N$ and for every ξ, $\eta \in \mathbb{R}^N$ such that $\xi \neq \eta$.

Furthermore $\boldsymbol{\Phi} : (x,s) \in \Omega \times \mathbb{R}^N \to \boldsymbol{\Phi}(x,s) = (\Phi_i(x,s)) \in \mathbb{R}^N$ is a Carathéodory function, differentiable with respect to s, satisfying the following conditions

$$|\Phi_s(x,s)| \leq c(x)(1+|s|)^{\gamma-1}, \qquad c(x) > 0, \tag{5}$$

with

$$c(x) \in L^r(\Omega), \quad r > \frac{N}{p-1}, \tag{6}$$

and

$$1 \leq \gamma \leq p-1, \tag{7}$$

when $p \geq 2$, while

$$c(x) \in L^r(\Omega), \quad r > \frac{N(p-1)}{1+N(p-2)}, \tag{8}$$

and

$$1 - \frac{1}{N} < \gamma < p-1, \tag{9}$$

when $2 - \frac{1}{N} < p < 2$.

Under these hypotheses it can be proved that there exists a weak solution u to problems of the type (1) with $\Phi = 0$ (see [11], [12]); such a solution is found by a natural approximation method and is known as SOLA, that is, Solution Obtained as Limit of Approximation (see [15], [16] and [18]). Existence and uniqueness for SOLA to problem (1) has been obtained also in [2] with $\Phi = 0$ and in [3] when the lower order term is of the type $b(x)|\nabla u|^{p-1}$ (see also [8]).

Further notions of solutions, for which both existence and uniqueness results have been proved, are well-known: we recall the entropy solutions ([6], [13]) and the renormalized solutions ([28], [30], [17]). In particular, uniqueness results for renormalized and entropy solutions to (1) are well known when f is an L^1 function and Φ *is a function which does not depends on x* (see [32] and [20]). Other uniqueness results for elliptic equations with L^1 data can be found in [9], [5].

In this paper we prove the uniqueness of a weak solution to (1) when f is an L^1 function and Φ is locally Lipschitz. For $f \in L^1(\Omega)$, we say that u is a weak solution to (1) if $u \in W_0^{1,1}(\Omega)$, $a(x,\nabla u) \in L^1(\Omega)$ and

$$\int_\Omega \mathbf{a}(x,\nabla u)\cdot\nabla v dx + \int_\Omega \Phi(x,u)\cdot\nabla v dx = \int_\Omega f v dx \qquad \text{for } v \in C_0^\infty(\Omega).$$

The novelty respect to the previous literature consists in allowing Φ *to possibly depend on x*.

The uniqueness follows immediately from a continuity result with respect to the data. To this aim, we assume that the function **a** satisfies stronger monotoniticy conditions

$$(\mathbf{a}(x,\xi)-\mathbf{a}(x,\eta))\cdot(\xi-\eta) \geq \beta\frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}}, \quad \beta>0, \quad \xi\neq\eta, \tag{10}$$

if $2-\frac{1}{N}<p<2$, or

$$(\mathbf{a}(x,\xi)-\mathbf{a}(x,\eta))\cdot(\xi-\eta) \geq \delta(1+|\xi|+|\eta|)^{p-2}|\xi-\eta|^2, \quad \delta>0, \quad \xi\neq\eta. \tag{11}$$

if $p \geq 2$.

The uniqueness results are stated in the following theorems.

Theorem 1 *Let $p \geq 2$. Let us assume conditions* (2)*,* (3)*,* (5)–(7) *and* (11) *and $f \in L^1(\Omega)$. Then problem* (1) *has one weak solution at most.*

Theorem 2 *Let $2-\frac{1}{N}<p<2$. Let us assume conditions* (2)*,* (3)*,* (5) *and* (8)–(10) *and $f \in L^1(\Omega)$. Then problem* (1) *has one weak solution at most.*

We remark that the uniqueness is obtained under the hypotheses (7) and (9) on γ for which the existence is already known (see [10] and [5]; see also [19], [21] and [22]).

2 Preliminaries and a technical result

In this section we first recall some definitions and properties which will be used throughout, then we prove a result which gives an estimate of the difference of the two gradients in terms of the difference of two solutions.

Let Ω be a measurable subset of $\mathbb{R}^N$ and $u:\Omega\to\mathbb{R}$ a measurable function in Ω. The distribution function of u is the decreasing map μ from $[0,+\infty[$ into $[0,+\infty[$ defined at any point $t \geq 0$ as the measure of a level set of u, $\{x\in\Omega : |u(x)|>t\}$. The decreasing rearrangement u^* of u is the distribution function of μ, that is

$$u^*(s) = \sup\{t \geq 0 : \mu(t) > s\}, \qquad s \in (0, |\Omega|).$$

The main property of rearrangements is the fact that the distribution of u^* is μ, in other words u and u^* are equidistribuited. For an exhaustive treatment of rearrangements see [14] and [25].

Now we recall a comparison result between the solution of the nonlinear elliptic problem (1) with regular datum and the solution of a suitable problem with radially symmetric data; it will be useful throughout the paper. The result, contained in [7], is

$$u^*(s) \leq \frac{2^{p-1}}{(N\omega_N^{\frac{1}{N}})^{p'}} \int_s^{|\Omega|} \frac{1}{t^{p'(1-\frac{1}{N})}} \left(\int_0^t f^*(\tau)\, d\tau \right)^{\frac{1}{p-1}} \exp\left(\int_s^t \frac{C(r)^{\frac{1}{p-1}}}{N\omega_N^{\frac{1}{N}} r^{1-\frac{1}{N}}} dr \right) dt, \tag{12}$$

for $a.e.\ s \in (0, |\Omega|]$, where ω_N is the measure of the unit ball of $\mathbb{R}^N$.
Here the function $C(r)$ is defined by

$$\int_{|u|>t} c(x)^{p'} dx = \int_0^{\mu(t)} C(r)^{p'} dr, \quad \text{for every } t \in (0, +\infty).$$

In [4] and [33] it is shown that $C(r)$ can be obtained as weak limit of functions having the same rearrangement of $c(x)$. As consequence of this any Lebesgue or Lorentz norm of $C(r)$ can be estimated from above with the same norm of $c(x)$. This implies that $C(r)$ and $c(x)$ have the same sommability and so (12) implies

$$u^*(s) \leq K \|f\|_{L^1}^{\frac{1}{p-1}} s^{-\frac{N-p}{N(p-1)}}, \qquad a.e.s \in (0, |\Omega|], \tag{13}$$

where K is a constant depending on $|\Omega|$, N, p, $\|c\|_{L^{p'}}$.

We explicitly remark that analogous inequalities have been proved in the linear case in [29] and [36].

For $1 < \delta < +\infty$ and $1 \leq r \leq +\infty$, the Lorentz space $L^{\delta,r}(\Omega)$ is the class of the measurable function u such that:

$$\|u\|_{\delta,r}^* = \left(\int_0^{+\infty} \left[u^*(s) s^{\frac{1}{\delta}} \right]^r \frac{ds}{s} \right)^{\frac{1}{r}} < \infty, \tag{14}$$

$$\|u\|_{\delta,\infty}^* = \sup_{s>0} u^*(s) s^{\frac{1}{\delta}} < \infty. \tag{15}$$

In order to prove the uniqueness of a weak solution stated in Theorem 1 and Theorem 2 we need a result of continuity with respect to the data. To this aim we get the following preliminary result.

Proposition 1 *Let us assume conditions* (2), (3), (5)–(7) *and* (11) *with when* $p \geq 2$, (2), (3), (5) *and* (8)–(10) *when* $2 - \frac{1}{N} < p < 2$. *Let* u, v *be weak solutions to problem (1) with data* $f,\ g \in C^{\infty}(\Omega)$ *respectively and assume* $q < \frac{N(p-1)}{N-1}$ *and* $m < q^* = \frac{Nq}{N-q}$.

Then if $p \geq 2$

$$\| |\nabla (u-v)| \|_{L^q} \leq K \|u-v\|_{L^m}^{\frac{1}{p}} \left(1 + \|f\|_{L^1} + \|g\|_{L^1}\right)^{\frac{\gamma}{p(p-1)}}, \tag{16}$$

or if $2 - \frac{1}{N} < p < 2$

$$\| |\nabla (u-v)| \|_{L^q} \leq K \|u-v\|_{L^m}^{\frac{1}{2}} \left(\|f\|_{L^1}^{\frac{1}{2}} + \|g\|_{L^1}^{\frac{1}{2}}\right) \left(\|f\|_{L^1}^{\frac{1}{p-1}} + \|g\|_{L^1}^{\frac{1}{p-1}}\right)^{1-\frac{p}{2}}. \tag{17}$$

The constant K *depends on* $N, p, q, \gamma, |\Omega|, \|c\|_{L^r}$.

Proof Denoted by μ the distribution function of $|u-v|$, let us consider the test function

$$\varphi(x) = \operatorname{sign}(u-v) \int_0^{|u-v|(x)} [\mu(t)]^{\alpha}\, dt,$$

with $\alpha > 0$. Taking φ in appropriate equations defining weak solutions with data f and g, and subtracting we get

$$\begin{aligned} &\int_{\Omega} [(\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla v)) \cdot \nabla (u-v)] [\mu(|u-v|(x))]^{\alpha}\, dx \\ &\quad = \int_{\Omega} [\Phi(x,u) - \Phi(x,v)] \cdot \nabla (u-v) [\mu(|u-v|(x))]^{\alpha}\, dx \\ &\qquad + \int_{\Omega} (f-g)\, \varphi dx. \end{aligned} \tag{18}$$

We have to distinguish the cases $p \geq 2$ and $2 - \frac{1}{N} < p < 2$.

Case 1 $(p \geq 2)$ Firstly we estimate from below the right-hand side of (18). From (5) and Hölder inequality we have

$$\begin{aligned} I &= \int_{\Omega} |\Phi(x,u) - \Phi(x,v)| \cdot |\nabla (u-v)| [\mu(|u-v|(x))]^{\alpha}\, dx \\ &\leq \|c\|_{L^r} \|1 + |u| + |v|\|_{L^{q^*}}^{\gamma-1} \| |\nabla u| + |\nabla v| \|_{L^q} \\ &\quad \times \left(\int_{\Omega} |u-v|^{\sigma} [\mu(|u-v|(x))]^{\alpha\sigma}\, dx\right)^{\frac{1}{\sigma}}, \end{aligned}$$

with σ such that

$$\frac{1}{\sigma}+\frac{\gamma-1}{q^{*}}+\frac{1}{r}+\frac{1}{q}=1.$$

Since u belongs to $W_0^{1,q}(\Omega)$ with $q<\frac{N(p-1)}{N-1}$, the following a priori estimate of the gradient of u (see [11], [12] and [19]) holds

$$\| |\nabla u| \|_{L^q} \le K \|f\|_{L^1}^{\frac{1}{p-1}}; \tag{19}$$

for the rest of the paper K is a constant which can vary from line to line, but depends only on the data of the problem.

So, from the Hardy-Littlewood inequality, the coarea Formula, the comparison result (13) applied to u^* and v^* and (19), we finally have

$$I \le K \|c\|_{L^r} \left(1+\|f\|_{L^1}+\|g\|_{L^1}\right)^{\frac{\gamma}{p-1}} \|u-v\|_{\frac{\sigma}{\alpha\sigma+1},\sigma}. \tag{20}$$

We assume that

$$\alpha > \frac{N-p}{N(p-1)}. \tag{21}$$

If $\frac{1}{\alpha}<m<q^{*}$, then $\frac{\sigma}{\alpha\sigma+1}<m$ and consequently (20) becomes

$$I \le K \left(1+\|f\|_{L^1}+\|g\|_{L^1}\right)^{\frac{\gamma}{p-1}} \|u-v\|_{L^m}. \tag{22}$$

Now, let us consider the last term in (18); we have

$$\int_\Omega |f-g|\,|\varphi|\,dx \le \|f-g\|_{L^1} \|\varphi\|_{L^\infty}. \tag{23}$$

From (21) we deduce

$$\begin{aligned}\sup_\Omega |\varphi(x)| &= \int_0^{+\infty} [\mu(t)]^\alpha\,dt \\ &= \alpha \int_0^{|\Omega|} s^{\alpha-1}(u-v)^*(s)\,ds = \alpha \|u-v\|_{\frac{1}{\alpha},1}.\end{aligned} \tag{24}$$

So from (22) and (24) we have the following estimate of the left-hand side of (18)

$$\begin{aligned}&\int_\Omega [\Phi(x,u)-\Phi(x,v)]\cdot\nabla(u-v)\,[\mu(|u-v|(x))]^\alpha\,dx + \int_\Omega (f-g)\,\varphi dx \\ &\quad \le K\left(1+\|f\|_{L^1}+\|g\|_{L^1}\right)^{\frac{\gamma}{p-1}} \|u-v\|_{L^m}.\end{aligned} \tag{25}$$

Now we have to estimate from below the left-hand side of (18). From (11) we have

$$\begin{aligned}&\int_\Omega [(\mathbf{a}(x,\nabla u)-\mathbf{a}(x,\nabla v))\cdot\nabla(u-v)]\,[\mu(|u-v|(x))]^\alpha\,dx \\ &\quad \ge \beta \int_\Omega |\nabla(u-v)|^p\,[\mu(|u-v|(x))]^\alpha\,dx.\end{aligned} \tag{26}$$

So using (25) and (26) we obtain

$$\int_\Omega |\nabla (u-v)|^p [\mu (|u-v|(x))]^\alpha dx \\ \leq K(1+\|f\|_{L^1}+\|g\|_{L^1})^{\frac{\gamma}{p-1}} \|u-v\|_{L^m}. \quad (27)$$

From Hardy-Littlewood inequality, and choosing once again α so that $q < \frac{p}{1+\alpha}$, we get

$$\int_\Omega |\nabla (u-v)|^p [\mu (|u-v|(x))]^\alpha dx \\ \geq K \||\nabla (u-v)|\|^p_{\frac{p}{1+\alpha},p} \geq K\||\nabla (u-v)|\|^p_{L^q}.$$

Therefore, by (27) and by last inequality we obtain (16).

Case 2 $(2-\frac{1}{N} < p < 2)$ As for the case $p \geq 2$, firstly we evaluate the right-hand side of (18). Since $\gamma < 1$, from (5) and Hölder inequality we have

$$I = \int_\Omega |\Phi(x,u) - \Phi(x,v)| \cdot |\nabla (u-v)| [\mu (|u-v|(x))]^\alpha dx \leq \\ \leq \|c\|_{L^r} \||\nabla u| + |\nabla v|\|_{L^q} \left(\int_\Omega |u-v|^\sigma [\mu (|u-v|(x))]^{\alpha\sigma} dx \right)^{\frac{1}{\sigma}},$$

with σ such that

$$\frac{1}{\sigma} + \frac{1}{r} + \frac{1}{q} = 1,$$

and so, proceeding as before,

$$I \leq K\|c\|_{L^r} (\|f\|_{L^1} + \|g\|_{L^1})^{\frac{1}{p-1}} \|u-v\|_{\frac{\sigma}{\alpha\sigma+1},\sigma}. \quad (28)$$

If $\frac{1}{\alpha} < m < q^*$, then $\frac{\sigma}{\alpha\sigma+1} < m$, and consequently (28) becomes

$$I \leq K(\|f\|_{L^1} + \|g\|_{L^1})^{\frac{1}{p-1}} \|u-v\|_{L^m}. \quad (29)$$

From (29) and (24) we have the following estimate of the left-hand side of (18)

$$\int_\Omega [\Phi(x,u) - \Phi(x,v)] \cdot \nabla (u-v) [\mu (|u-v|(x))]^\alpha dx + \int_\Omega (f-g)\varphi dx \\ \leq K(\|f\|_{L^1} + \|g\|_{L^1})^{\frac{1}{p-1}} \|u-v\|_{L^m}. \quad (30)$$

Let us consider the function

$$G(x) = \frac{|\nabla (u-v)|^{\frac{2}{p}}}{(|\nabla u| + |\nabla v|)^{\frac{2-p}{p}}}.$$

Coming back to (18), by (10) and (25) we get

$$\int_{\Omega} G(x)^p \left[\mu\left(|u-v|(x)\right)\right]^{\alpha} dx \le K\left(\|f\|_{L^1}+\|g\|_{L^1}\right)^{\frac{1}{p-1}} \|u-v\|_{L^m}. \tag{31}$$

Now we estimate from below the left-hand side of (31). By Hardy-Littlewood inequality we have

$$\int_{\Omega} G(x)^p \left[\mu(|u-v|)(x))\right]^{\alpha} dx \ge \|G\|^p_{\frac{p}{1+\alpha},p}.$$

If we choose once again α in way that $q < \frac{p}{1+\alpha}$, we obtain

$$\|G\|^p_{L^q} \le K\left(\|f\|_{L^1}+\|g\|_{L^1}\right)^{\frac{1}{p-1}} \|u-v\|_{L^m}. \tag{32}$$

Since (see [2])

$$\|\nabla(u-v)\|_{L^q} \le K\|G\|^{p/2}_{L^q}\left(\|f\|^{\frac{1}{p-1}}_{L^1}+\|g\|^{\frac{1}{p-1}}_{L^1}\right)^{1-\frac{p}{2}}, \tag{33}$$

then from (32) and (33) we have

$$\||\nabla(u-v)|\|_{L^q} \le K\|u-v\|^{\frac{1}{2}}_{L^m}\left(\|f\|^{\frac{1}{2}}_{L^1}+\|g\|^{\frac{1}{2}}_{L^1}\right)\left(\|f\|^{\frac{1}{p-1}}_{L^1}+\|g\|^{\frac{1}{p-1}}_{L^1}\right)^{1-\frac{p}{2}},$$

where K depends on N, p, q, $|\Omega|$, $\|c\|_{L^r}$, γ. □

3 Uniqueness results

In this section we prove the uniqueness of a weak solution to problem (1) stated in Theorem 1 and Theorem 2. Proposition 1 is not enough to achieve the continuity with respect to the data. We need, in fact, also an estimate of rearrangements of the difference of two solutions in term of L^1 norm of the difference of data.

Proposition 2 *Let $p \ge 2$ and let us assume* (2), (3), (5)–(7) *and* (11). *Let u and v be weak solutions to problem* (1) *with data f, $g \in L^1(\Omega)$ respectively, then*

$$(u-v)^*(s) \le K\|f-g\|_{L^1} s^{-\frac{N-2}{N}}, \qquad a.e. s \in (0,|\Omega|], \tag{34}$$

where K depends on N, $|\Omega|$, δ, p, $\|f\|_{L^1}$, $\|g\|_{L^1}$, r, γ; furthermore K is bounded when f, g belong to bounded subset of L^1.

Proof Set $w = u - v$ and $h = f - g$. For any positive constants t and k, we consider the function

$$\Psi = \begin{cases} k\ \mathrm{sign} w & \text{if } |w| > t+k \\ w - t\,\mathrm{sign} w & \text{if } t < |w| \le t+k \\ 0 & \text{otherwise.} \end{cases}$$

Taking Ψ in appropriate equations defining weak solutions with f and g as data, subtracting and dividing by k, we have

$$\begin{aligned} &\frac{1}{k} \int\limits_{t<|w|\le t+k} [\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla v)] \cdot \nabla w dx \\ &= \frac{1}{k} \int\limits_{t<|w|\le t+k} [\Phi(x,u) - \Phi(x,v)] \cdot \nabla w\, dx + \\ &+ \int\limits_{|w|>t+k} h\, signw\, dx + \frac{1}{k} \int\limits_{t<|w|\le t+k} h[(w-t)\ signw]dx. \end{aligned} \tag{35}$$

We set

$$\nu(x) = (1 + |\nabla u| + |\nabla v|)^{p-2}. \tag{36}$$

On using assumptions (11), (5) and the definition of $\nu(x)$, (35) becomes

$$\begin{aligned} \frac{\delta}{k} \int\limits_{t<|w|\le t+k} \nu(x)\, |\nabla w|^2\, dx \le \frac{(t+k)}{k} \int\limits_{t<|w|\le t+k} c(x)(1+|u|+|v|)^{\gamma-1}\, |\nabla w|\, dx + \\ + \int\limits_{|w|>t+k} |h|\, dx + \int\limits_{t<|w|\le t+k} |h|\, dx. \end{aligned}$$

Since $\nu(x) \ge 1$, by applying Hölder inequality and letting k go to zero in the previous inequality, we obtain

$$\begin{aligned} -\frac{d}{dt} \int_{|w|>t} \nu(x)\, |\nabla w|^2\, dx \le \frac{1}{\delta} t \left(-\frac{d}{dt} \int_{|w|>t} \nu(x)\, |\nabla w|^2\, dx \right)^{\frac{1}{2}} \\ \times \left[\left(-\frac{d}{dt} \int_{|w|>t} c(x)^2 (1+|u|+|v|)^{2(\gamma-1)} dx \right)^{\frac{1}{2}} \right] + \frac{1}{\delta} \int_{|w|>t} |h|\, dx. \end{aligned}$$

If μ denotes the distribution function of w, proceeding as in [4] allows one to define a function T such that

$$T(\mu(t))\, |\mu'(t)| = -\frac{d}{dt} \left(\int_{|w|>t} c(x)^2 (1+|u|+|v|)^{2(\gamma-1)} dx \right). \tag{37}$$

The function defined in (37) is a weak limit of functions having the same rearrangement of $c(x)^2(1+|u|+|v|)^{2(\gamma-1)}$.

By Hardy-Littlewood inequality and by the definition of T, we obtain

$$-\frac{d}{dt}\int_{|w|>t}\nu(x)\,|\nabla w|^2\,dx$$
$$\le \frac{t}{\delta}\left|\mu'(t)\right|^{\frac{1}{2}}\left[T(\mu(t))\right]^{\frac{1}{2}}\left(-\frac{d}{dt}\int_{|w|>t}\nu(x)\,|\nabla w|^2\,dx\right)^{\frac{1}{2}}+\frac{1}{\delta}\int_0^{\mu(t)}h^*(s)ds. \tag{38}$$

On the other hand, denoted by $k_N=\omega_N^{1/N}N$, the isoperimetric and Schwarz inequalities and the inequality $\nu(x)\ge 1$ give (see [35])

$$k_N\mu(t)^{1-\frac{1}{N}}\le -\frac{d}{dt}\int_{|w|>t}|\nabla w|\,dx$$
$$\le\left(-\frac{d}{dt}\int_{|w|>t}\nu(x)\,|\nabla w|^2\,dx\right)^{\frac{1}{2}}\left|\mu'(t)\right|^{\frac{1}{2}}. \tag{39}$$

By (38), (39) we obtain

$$-\frac{d}{dt}\int_{|w|>t}\nu(x)\,|\nabla w|^2\,dx\le\frac{t}{\delta}\left(-\frac{d}{dt}\int_{|w|>t}\nu(x)\,|\nabla w|^2\,dx\right)^{\frac{1}{2}}\left|\mu'(t)\right|^{\frac{1}{2}}\left[T(\mu(t))\right]^{\frac{1}{2}}$$
$$+\frac{1}{\delta}k_N^{-1}\mu(t)^{\frac{1}{N}-1}\left(-\frac{d}{dt}\int_{|w|>t}\nu(x)\,|\nabla w|^2\,dx\right)^{\frac{1}{2}}\left|\mu'(t)\right|^{\frac{1}{2}}\int_0^{\mu(t)}h^*(s)ds. \tag{40}$$

Using again (39), (40) becomes

$$1\le k_N^{-1}t\mu(t)^{\frac{1}{N}-1}\left|\mu'(t)\right|\left[T(\mu(t))\right]^{\frac{1}{2}}+k_N^{-2}\mu(t)^{2(\frac{1}{N}-1)}\left|\mu'(t)\right|\int_0^{\mu(t)}h^*(s)ds.$$

Integrating the previous inequality between 0 and t, and using the definition of $w^*(s)$, we have

$$w^*(s)\le K\int_s^{|\Omega|}w^*(t)t^{\frac{1}{N}-1}T(t)^{\frac{1}{2}}dt+K\int_s^{|\Omega|}t^{2(\frac{1}{N}-1)}\left(\int_0^t h^*(\tau)d\tau\right)dt.$$

Gronwall's Lemma yields

$$w^*(s)\le K\int_s^{|\Omega|}\left\{t^{2(\frac{1}{N}-1)}\left(\int_0^t h^*(\tau)d\tau\right)\left(\exp\int_s^t\tau^{\frac{1}{N}-1}T(\tau)^{\frac{1}{2}}d\tau\right)\right\}dt. \tag{41}$$

Now we have to impose conditions on γ which ensure that

$$\int_0^{|\Omega|}\tau^{\frac{1}{N}-1}T(\tau)^{\frac{1}{2}}d\tau<+\infty.$$

This happens if the function T belongs to some space L^{ϑ} with $\vartheta > \frac{N}{2}$. But we have already observed that T has the same summability of $c(x)^2(|u| + |v|)^{2(\gamma-1)}$, that is $\frac{1}{\vartheta} = \frac{2(\gamma-1)}{q^*} + \frac{2}{r}$. Hence we have to impose that

$$\vartheta > \frac{N}{2} \quad \text{and} \quad q^* < \frac{N(p-1)}{N-p}.$$

These conditions are satisfied if we choose $\gamma < \frac{N-1}{N-p} - \frac{N(p-1)}{r(N-p)}$; since γ satisfies (7) and $p-1 < \frac{N-1}{N-p} - \frac{N(p-1)}{r(N-p)}$, then, (41) becomes

$$(u-v)^*(s) \leq K \int_s^{|\Omega|} t^{2(\frac{1}{N}-1)} \left(\int_0^t h^*(\tau) d\tau \right) dt \leq K \|f-g\|_{L^1} s^{-\frac{N-2}{N}},$$

where K depends on N, $|\Omega|$, δ, p, $\|f\|_{L^1}$, $\|g\|_{L^1}$, r, γ. □

Proposition 3 *Let* $2 - \frac{1}{N} < p < 2$ *and let us assume* (2), (3), (5) *and* (8)–(10). *If u and v are weak solutions to problem* (1) *with data* f, $g \in L^1(\Omega)$ *respectively, then*

$$(u-v)^*(s) \leq K \|f-g\|_{L^1} s^{-\frac{N-2}{N}-(2-p)\zeta}, \qquad a.e.\, s \in (0, |\Omega|], \tag{42}$$

for some $\zeta > \frac{N-1}{N(p-1)}$. *Here K depends on* N, β, $|\Omega|$, p, $\|f\|_{L^1}$, $\|g\|_{L^1}$, r, q, γ; *furthermore K is bounded when* f, g *belong to bounded subset of* L^1.

Proof As in the proof of Proposition 2, we consider (35) and (36). By (10), (5), we get

$$\frac{\beta}{k} \int_{t<|w|\leq t+k} |\nabla w|^2 (|\nabla u| + |\nabla v|)^{p-2} dx \leq \frac{(t+k)}{k} \int_{t<|w|\leq t+k} c(x)(1+|u|+|v|)^{\gamma-1} |\nabla w| \, dx + \int_{|w|>t+k} |h| \, dx + \int_{t<|w|\leq t+k} |h| \, dx.$$

Since $p < 2$, we have $(|\nabla u| + |\nabla v|)^{\frac{2-p}{2}} \leq (1 + |\nabla u| + |\nabla v|)^{\frac{2-p}{2}}$, and so

$$\frac{\beta}{k} \int_{t<|w|\leq t+k} |\nabla w|^2 (|\nabla u| + |\nabla v|)^{p-2} dx \leq \frac{(t+k)}{k} \int_{t<|w|\leq t+k} c(x) \frac{(1+|u|+|v|)^{\gamma-1} |\nabla w|}{(|\nabla u| + |\nabla v|)^{\frac{2-p}{2}}} (1 + |\nabla u| + |\nabla v|)^{\frac{2-p}{2}} dx + \int_{|w|>t+k} |h| \, dx + \int_{t<|w|\leq t+k} |h| \, dx. \tag{43}$$

Using Hölder inequality in (43) and letting k go to 0, we have

$$\begin{aligned}
&-\frac{d}{dt}\int_{|w|>t}|\nabla w|^2\left(|\nabla u|+|\nabla v|\right)^{p-2}dx\\
&\leq Kt\left(-\frac{d}{dt}\int_{|w|>t}c(x)^2\frac{(1+|u|+|v|)^{2(\gamma-1)}}{\nu(x)}dx\right)^{\frac{1}{2}}\\
&\times\left(-\frac{d}{dt}\int_{|w|>t}|\nabla w|^2\left(|\nabla u|+|\nabla v|\right)^{p-2}dx\right)^{\frac{1}{2}}+K\int_{|w|>t}|h|\,dx. \qquad (44)
\end{aligned}$$

Arguing as in the previous proof, we introduce two functions $\bar{T}$ and $\bar{\nu}$ such that

$$\bar{T}(\mu(t))\left|\mu'(t)\right|=-\frac{d}{dt}\int\limits_{|w|>t}c(x)^2\frac{(1+|u|+|v|)^{2(\gamma-1)}}{\nu(x)}dt, \qquad (45)$$

$$\bar{\nu}(\mu(t))\left|\mu'(t)\right|=-\frac{d}{dt}\int\limits_{|w|>t}\frac{1}{\nu(x)}dt. \qquad (46)$$

From (39) , we obtain

$$\begin{aligned}
&k_N\mu(t)^{1-\frac{1}{N}}\\
&\leq\left(-\frac{d}{dt}\int\limits_{|w|>t}|\nabla w|^2\left(|\nabla u|+|\nabla v|\right)^{p-2}dx\right)^{\frac{1}{2}}\bar{\nu}(\mu(t))^{\frac{1}{2}}\left|\mu'(t)\right|^{\frac{1}{2}}. \qquad (47)
\end{aligned}$$

Therefore, by (47), (44) becomes

$$\begin{aligned}
&-\frac{d}{dt}\int_{|w|>t}|\nabla w|^2\left(|\nabla u|+|\nabla v|\right)^{p-2}dx\\
&\leq Kt\bar{T}(\mu(t))^{\frac{1}{2}}\left|\mu'(t)\right|^{\frac{1}{2}}\left(-\frac{d}{dt}\int_{|w|>t}|\nabla w|^2\left(|\nabla u|+|\nabla v|\right)^{p-2}dx\right)^{\frac{1}{2}}\\
&+Kk_N^{-1}\mu(t)^{\frac{1}{N}-1}\bar{\nu}(\mu(t))^{\frac{1}{2}}\left|\mu'(t)\right|^{\frac{1}{2}}\\
&\times\left(-\frac{d}{dt}\int_{|w|>t}|\nabla w|^2\left(|\nabla u|+|\nabla v|\right)^{p-2}dx\right)^{\frac{1}{2}}\int_0^{\mu(t)}h^*(\tau)d\tau. \qquad (48)
\end{aligned}$$

We proceed as in the case $p\geq 2$, use the definition of $w^*(s)$, Gronwall's Lemma and integration by parts, we get

$$\begin{aligned}
w^*(s)\leq K\int_s^{|\Omega|}&\left\{t^{2(\frac{1}{N}-1)}\bar{\nu}(t)\left(\int_0^t h^*(\tau)d\tau\right)\right.\\
&\left.\left(\exp\int_s^t\tau^{\frac{1}{N}-1}\bar{T}(\tau)^{\frac{1}{2}}\bar{\nu}(\tau)^{\frac{1}{2}}d\tau\right)\right\}dt. \qquad (49)
\end{aligned}$$

The assumptions on γ ensure that

$$\int_0^{|\Omega|} \tau^{\frac{1}{N}-1}\overline{T}(\tau)^{\frac{1}{2}}\overline{\nu}(\tau)^{\frac{1}{2}}d\tau < +\infty.$$

$\overline{T}$ has the same summability of $c(x)^2\frac{(1+|u|+|v|)^{2(\gamma-1)}}{\nu(x)}$ and $\overline{\nu}$ has the same summability of $\frac{1}{\nu(x)}$; recalling the expression of ν and the estimate (19), we deduce that $\left(\overline{T}\overline{\nu}\right)^{\frac{1}{2}}$ belongs to $L^{\vartheta}(\Omega)$ with $\frac{1}{\vartheta} = \frac{\gamma-1}{q^*} + \frac{2-p}{q} + \frac{1}{r}$. So the integral is finite for every γ such that

$$\vartheta > N, \quad q^* < \frac{N(p-1)}{N-p} \quad \text{and} \quad q < \frac{N(p-1)}{N-1},$$

which holds true if condition $\gamma < \frac{(N-1)(p-1)}{N-p} - \frac{N(p-1)}{r(N-p)}$ is satisfied; since γ satisfies (9) and $p-1 < \frac{(N-1)(p-1)}{N-p} - \frac{N(p-1)}{r(N-p)}$, then coming back to (49), we arrive at

$$w^*(s) \le K\,\|h\|_{L^1} \int_s^{|\Omega|} t^{2(\frac{1}{N}-1)}\overline{\nu}(t)dt.$$

Using the summability of $\overline{\nu}(t)$ and the Hölder inequality, we have

$$w^*(s) \le K\,\|h\|_{L^1} \left\|\overline{\nu}\right\|_{L^{\frac{q}{2-p}}} \left[\int_s^{|\Omega|} t^{2(\frac{1}{N}-1)\frac{q}{q-2+p}}dt\right]^{\frac{q-2+p}{q}},$$

and then

$$(u-v)^*(s) \le K\,\|f-g\|_{L^1}\, s^{-\frac{N-2}{N}-(2-p)\zeta},$$

for some $\zeta > \frac{N-1}{N(p-1)}$; here K depends on N, β, $|\Omega|$, p, $\|f\|_{L^1}$, $\|g\|_{L^1}$, r, q, γ. □

From Proposition 2 and 3 we immediately obtain $\|u-v\|_{L^m} \le K\,\|f-g\|_{L^1}$ $\forall\, m < q^*$. Therefore, from Proposition 1 we easily deduce the following continuity result with respect to the data which implies the uniqueness results given by Theorems 1 and 2.

Theorem 3 *Let us assume hypotheses* (2), (3), (5)–(7) *and* (11) *when* $p \ge 2$ *or* (3), (5) *and* (8)–(10) *when* $2-\frac{1}{N} < p < 2$. *If u and v are weak solutions to problem (1) with data f, g respectively, then we have*

$$\|\nabla(u-v)\|_{L^q} \le K\,\|f-g\|_{L^1}^{\frac{1}{p}}, \qquad if\ p \ge 2,$$

$$\|\nabla(u-v)\|_{L^q} \le K\,\|f-g\|_{L^1}^{\frac{1}{2}}, \qquad if\ 2-\frac{1}{N} < p < 2.$$

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