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Common fixed points for discontinuous mappings in fuzzy metric spaces

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Abstract. In this paper we prove some common fixed point theorems for fuzzy contraction respect to a mapping, which satisfies a condition of weak compatibility. We deduce also fixed point results for fuzzy contractive mappings in the sense of Gregori and Sapena.

Keywords Fuzzy metric spaces · Discontinuous mappings · Common fixed points · Points of coincidence

Mathematics Subject Classification (2000) 54A40 · 54H25

1 Introduction

The notion of fuzzy metric space was introduced in different ways [1, 4, 11, 14]. Recently the notion of fuzzy metric space, introduced by Kramosil and Michalek [11], was modified by George and Veeramani [5, 6] that obtained a Hausdorff topology for this class of fuzzy metric spaces. In [7, 9] it was proved that the topology induced by a fuzzy metric space in the sense of George and Veeramani is metrizable. In this type of spaces, for some classes of fuzzy contractive mappings, it was proved that the *contraction Theorem of Banach* is valid [2, 8, 10]. Grabiec ([8]) proved the contraction principle in

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the setting of fuzzy metric spaces introduced by Kramosil and Michalek. Gregori and Sapena ([10]) obtained fixed point results for complete fuzzy metric space in the sense of George and Veeramani, and also for Kramosil and Michalek's fuzzy metric spaces which are complete in Grabiec's sense (we say *G-complete* fuzzy metric space). Vasuki ([15]), Di Bari and Vetro ([3]) proved results regarding common fixed point for a family of mappings defined on a G-complete fuzzy metric space. Recently Pant ([12]), Pathak, Cho and Kang ([13]) obtained results of fixed point for discontinuous mappings in metric spaces by the notion of *R-weak commutativity*. Vasuki ([16]) defined R-weak commutativity of mappings in fuzzy metric spaces and prove the fuzzy version of Pant's Theorem for G-complete fuzzy metric space. In this paper we use a notion of weak compatibility to obtain results on fixed point for discontinuous mappings. Finally we deduce as particular case a result of Gregori and Sapena.

2 Preliminaries on the fuzzy metric spaces

In this section we recall the basic notion of fuzzy metric space in the sense of George and Veeramani. We denote with $\mathbb{N}$ the set of all positive integers and with $\mathbb{R}$ the set of all real numbers.

Definition 1 (Schweizer and Sklar [14]) A binary operation $*: [0,1]\times[0,1]\to [0,1]$ is a continuous t-norm if satisfies the following conditions:

(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a*1 = a$ for every $a \in [0,1]$,
(iv) $a*b \le c*d$ if $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Definition 2 (George and Veeramani [5]) A tern $(X,M,*)$ is a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X\times X\times]0,+\infty[$ satisfying, for every $x,y,z \in X$ and $s,t > 0$, the following conditions:

(i) $M(x,y,t) > 0$,
(ii) $M(x,y,t) = 1$ iff $x = y$,
(iii) $M(x,y,t) = M(y,x,t)$,
(iv) $M(x,y,t)*M(y,z,s) \le M(x,z,t+s)$,
(v) $M(x,y,\cdot):]0,+\infty[\to [0,1]$ is continuous.

George and Veeramani proved that every fuzzy metric M on X induces a Hausdorff first countable topology τ_M which has as base the family of open set

$$\{B(x,r,t):\ \ x \in X,\ \ 0 < r < 1,\ \ t > 0\},$$

where

$$B(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\}.$$

Let $(X,M,*)$ be a fuzzy metric space. A sequence $(x_n) \subset X$ is a Cauchy sequence if for every $0 < r < 1$ and for every $t > 0$, there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - r$ for every $n, m \geq n_0$. A sequence $(x_n) \subset X$ is a G-sequence, that is a Cauchy sequence in the sense of Grabiec [8], if $M(x_n, x_{n+p}, t) \to 1$ as $n \to +\infty$ for every $p \in \mathbb{N}$ and every $t > 0$.

Theorem 1 (George and Veeramani [5]) *A sequence (x_n) in a fuzzy metric space $(X,M,*)$ converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to +\infty$.*

A fuzzy metric space $(X,M,*)$ is complete (respectively G-complete) if every Cauchy sequence (respectively G-sequence) is convergent. If $(X,M,*)$ is a complete (respectively G-complete) fuzzy metric space, then M is a complete (respectively G-complete) fuzzy metric on X. George and Veeramani [5] have pointed that the definition of G-Cauchy sequence is weaker than the Cauchy sequence.

Remark 1 (Result 2.9 of [6]) Let (X,d) be a metric space. We define $a * b = ab$ for all $a, b \in [0,1]$ and $M_d(x,y,t) = t/[t + |x-y|]$ for every $(x,y,t) \in X \times X \times]0,+\infty[$, then $(X, M_d, *)$ is a fuzzy metric space. The fuzzy metric space $(X, M_d, *)$ is complete if and only if the metric space (X,d) is complete.

3 Contractive mappings with respect to a map

In this section we introduce the notion of fuzzy contractive mapping with respect to a map. Let $(X,M,*)$ be a fuzzy metric space and let $f, g : X \to X$ be maps. For all $x, y \in X$ and each $t > 0$ we define

$$m(f,g;x,y,t) = \min\{M(g(x),g(y),t), M(f(x),g(x),t), M(f(y),g(y),t)\}.$$

Definition 3 Let $(X,M,*)$ be a fuzzy metric space and let $f, g : X \to X$ be maps. The map f is a fuzzy contraction with respect to g if there exists an upper semicontinuous function $r : [0,+\infty[\to [0,+\infty[$, with $r(\tau) < \tau$ for every $\tau > 0$, such that

$$\frac{1}{M(f(x),f(y),t)} - 1 \leq r\left(\frac{1}{m(f,g;x,y,t)} - 1\right) \tag{1}$$

for every $x, y \in X$ and each $t > 0$.

Let $f, g : X \to X$ be such that $f(X) \subset g(X)$ and $x_0 \in X$. The sequence $(f(x_n))$ defined by $f(x_n) = g(x_{n+1})$ for every $n \in \mathbb{N} \cup \{0\}$ is an f-g-sequence with initial point x_0. Such a sequence exists in virtue of the fact that $f(X) \subset g(X)$.

Lemma 1 *Let $(X, M, *)$ be a fuzzy metric space and let $f, g : X \to X$ be such that $f(X) \subset g(X)$. If f is a fuzzy contraction with respect to g, then every f-g-sequence $(f(x_n))$ with initial point $x_0 \in X$ is a G-sequence.*

Proof (i). We suppose that $f(x_{n-1}) \neq f(x_n)$ for all n. Then by virtue of (1) and $r(\tau) < \tau$ for every $\tau > 0$, we obtain

$$\begin{aligned} m(f, g; x_n, x_{n+1}, t) &= \min\{M(f(x_{n-1}), f(x_n), t), M(f(x_n), f(x_{n+1}), t)\} \\ &= M(f(x_{n-1}), f(x_n), t). \end{aligned}$$

It follows

$$\begin{aligned} \frac{1}{M(f(x_n), f(x_{n+1}), t)} - 1 &\leq r\left(\frac{1}{M(f(x_{n-1}), f(x_n), t)} - 1\right) \\ &< \frac{1}{M(f(x_{n-1}), f(x_n), t)} - 1. \end{aligned}$$

Consequently $M(f(x_n), f(x_{n+1}), t) > M(f(x_{n-1}), f(x_n), t)$ for all n and thus $(M(f(x_{n-1}), f(x_n), t))$ is an increasing sequence of positive real numbers in [0,1]. Let

$$S(t) = \lim_{n \to +\infty} M(f(x_{n-1}), f(x_n), t)$$

and we show that $S(t) = 1$ for all $t > 0$. We suppose that there is $t > 0$ such that $S(t) < 1$, then from

$$\frac{1}{M(f(x_n), f(x_{n+1}), t)} - 1 \leq r\left(\frac{1}{m(f, g; x_n, x_{n+1}, t)} - 1\right)$$

as $n \to +\infty$, we deduce

$$\frac{1}{S(t)} - 1 \leq r\left(\frac{1}{S(t)} - 1\right) < \frac{1}{S(t)} - 1$$

and this is a contradiction.

Now for each positive integer p, we have

$$M(f(x_n), f(x_{n+p}), t) \geq M(f(x_n), f(x_{n+1}), \frac{t}{p}) * \cdots * M(f(x_{n+p-1}), f(x_{n+p}), \frac{t}{p}).$$

It follows that

$$\lim_{n \to +\infty} M(f(x_n), f(x_{n+p}), t) \geq \underbrace{1 * \cdots * 1}_{p} = 1$$

and $(f(x_n))$ is a G-sequence.

(ii). If there exists $n_0 \in \mathbb{N}$ such that $f(x_{n_0}) = f(x_{n_0+1})$, then $f(x_n) = f(x_{n+1})$ for all $n \geq n_0$. Consequently $(f(x_n))$ is a Cauchy sequence and therefore a G-sequence. □

4 Common fixed points

Let f be a fuzzy contraction with respect to g. In this section we prove that if f, g verify a condition of weak compatibility, then f and g have a unique common fixed point.

Definition 4 Let $f,g : X \to X$ be mappings. If $z = f(u) = g(u)$ for some $u \in X$, then u is called a coincidence point of f and g, and z is called a point of coincidence of f and g.

Definition 5 The mappings $f,g : X \to X$ are weakly compatible if, for every $x \in X$, holds:

$$f(g(x)) = g(g(x)) \quad \text{whenever} \quad f(x) = g(x). \tag{2}$$

Proposition 1 *Let $(X,M,*)$ be a fuzzy metric space and let $f,g : X \to X$ be weakly compatible. If f and g have a unique point of coincidence $z = f(u) = g(u)$, then z is the unique common fixed point of f and g.*

Proof Since $z = f(u) = g(u)$ and f and g are weakly compatible, we deduce that $f(z) = f(g(u)) = g(g(u)) = g(z)$, that is, $w = f(z) = g(z)$ is a point of coincidence of f and g. But we have supposed that z is the only point of coincidence of f and g, thus $z = w = f(z) = g(z)$. Being every common fixed point of f and g a point of coincidence, we deduce that z is the unique fixed point of f and g. □

Theorem 2 *Let $(X,M,*)$ be a fuzzy metric space and let $f,g : X \to X$ be such that $f(X) \subset g(X)$. If f is a fuzzy contraction with respect to g, $f(X)$ or $g(X)$ is G-complete, then f and g have a unique point of coincidence. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.*

Proof Let x_0 be any point in X. By Lemma 1, every f-g-sequence $(f(x_n))$ with initial point x_0 is a G-sequence and if $f(X)$ is G-complete then there is $z \in f(X)$ such that $f(x_n) = g(x_{n+1}) \to z$. (This holds also if $g(X)$ is G-complete with $z \in g(X)$.)

Let $u \in X$ be such that $g(u) = z$. We show that z is a point of coincidence of f and g. If $f(u) \neq g(u)$, from

$$M(f(u), g(u), t) < 1$$

and

$$\lim_{n\to+\infty} M(g(u), g(x_n), t) = \lim_{n\to+\infty} M(f(x_n), g(x_n), t) = 1,$$

for large value of n, we have

$$m(f,g;u,x_n,t) = M(f(u), g(u), t).$$

Then, for large value of n, we have

$$\begin{aligned}\frac{1}{M(f(u),f(x_n),t)}-1 &\leq r(\frac{1}{m(f,g;u,x_n,t)}-1)\\ &= r(\frac{1}{M(f(u),g(u),t)}-1)\end{aligned}$$

Now, by Corollary 7 of [8], $\lim_{n\to+\infty} M(f(u),f(x_n),t)=M(f(u),g(u),t)$ and consequently

$$\begin{aligned}\frac{1}{M(f(u),g(u),t)}-1 &= \lim_{n\to+\infty}(\frac{1}{M(f(u),f(x_n),t)}-1)\\ &\leq r(\frac{1}{M(f(u),g(u),t)}-1)\\ &< \frac{1}{M(f(u),g(u),t)}-1\end{aligned}$$

for every $t>0$. It follows that $f(u)=g(u)=z$. Consequently z is a point of coincidence and u a coincidence point of f and g.

Finally if $w\in X$ is another point of coincidence, that is $w=f(y)=g(y)$ for some $y\in X$, and $w\neq z$, then

$$\begin{aligned}\frac{1}{M(f(u),f(y),t)}-1 &\leq r(\frac{1}{m(f,g;u,y,t)}-1)\\ &< \frac{1}{M(f(u),f(y),t)}-1\end{aligned}$$

and this is a contradiction and so $z=w$.

Finally, Proposition 1 assures that f and g have a unique fixed point. □

If we suppose that the *f*-*g*-sequence are of Cauchy, then we deduce the following result:

Theorem 3 *Let $(X,M,*)$ be a fuzzy metric space and let $f,g:X\to X$ be such that $f(X)\subset g(X)$. If f is a fuzzy contraction with respect to g, $f(X)$ or $g(X)$ is complete, every f-g-sequence $(f(x_n))$ is of Cauchy, then f and g have a unique point of coincidence. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.*

Proof Fix $x_0\in X$, by hypothesis every *f*-*g*-sequence $(f(x_n))$ with initial point x_0 is of Cauchy and being $f(X)$ or $g(X)$ complete there is $z\in g(X)$ such that $f(x_n)\to z$. Proceeding as in Theorem 2, we prove Theorem 3. □

Example 1 Let $X=[-1,2]$ and $a*b=ab$ for all $a,b\in[0,1]$. For every $x,y\in X$ and $t>0$, let

$$M(x,y,t)=\frac{t}{t+|x-y|},$$

then $(X,M,*)$ is a complete fuzzy metric space. Let $f,g: X\to X$ be such that $f(x)=1$ for each $x\in[-1,1]$, $f(x)=6/5$ for each $x\in]1,2]$ and $g(x)=1+x^2/4$ for each $x\in[-1,1[$, $g(1)=c$ and $g(x)=2$ if $x\in]1,2]$. Define $r:[0,+\infty[\to[0,+\infty[$ by $r(\tau)=\tau/2$ for all $\tau>0$. The map f is a fuzzy contraction with respect to g. But if $c\neq 1$, the mappings f and g do not commute in the coincidence point 0, and therefore are not weakly compatible. The mappings f and g do not have common fixed point. This example underlines the crucial role of weak compatibility.

5 Fixed points for fuzzy contractive mappings

In this section we prove that in fuzzy metric spaces with triangular fuzzy metric there is a class of fuzzy contraction with respect to a map such that each f-g-sequence is of Cauchy. So we obtain result of common fixed point in complete fuzzy metric spaces.

Definition 6 (Di Bari and Vetro [2]) Let $(X,M,*)$ be a fuzzy metric space. The fuzzy metric M is triangular if it satisfies the condition

$$\frac{1}{M(x,y,t)}-1\leq\frac{1}{M(x,z,t)}-1+\frac{1}{M(z,y,t)}-1$$

for every x, y, $z\in X$ and every $t>0$.

Definition 7 Let $(X,M,*)$ be a fuzzy metric space and let $f,g: X\to X$ be maps. The map f is a fuzzy k-contraction with respect to g if there exists $k\in]0,1[$, such that

$$\frac{1}{M(f(x),f(y),t)}-1\leq k\left(\frac{1}{m(f,g;x,y,t)}-1\right)\tag{3}$$

for every x, $y\in X$ and every $t>0$.

Lemma 2 *Let $(X,M,*)$ be a fuzzy metric space, with M triangular, and let $f,g: X\to X$ be such that $f(X)\subset g(X)$. If f is a fuzzy k-contraction with respect to g, then every f-g-sequence $(f(x_n))$ with initial point $x_0\in X$ is a Cauchy sequence.*

Proof Let $(f(x_n))$ be an f-g-sequence of initial point x_0. If there exists $n_0\in\mathbb{N}$ such that $f(x_{n_0})=f(x_{n_0+1})$, then $f(x_n)=f(x_{n+1})$ for all $n\geq n_0$. Consequently $(f(x_n))$ is a Cauchy sequence. Assume that $f(x_n)\neq f(x_{n-1})$. By (3) of Definition 7, for every $n\in\mathbb{N}$ we have

$$\frac{1}{M(f(x_n),f(x_{n+1}),t)}-1\leq k^n\left(\frac{1}{M(f(x_0),f(x_1),t)}-1\right).$$

Being M triangular, we deduce

$$\frac{1}{M(f(x_n),f(x_{n+p}),t)}-1 \le (\frac{1}{M(f(x_0),f(x_1),t)}-1)\sum_{m=1}^{p} k^{n+m-1}$$
$$\le (\frac{1}{M(f(x_0),f(x_1),t)}-1)\frac{k^n}{1-k}.$$

It follows that $(f(x_n))$ is a Cauchy sequence. □

Theorem 4 *Let $(X,M,*)$ be a fuzzy metric space and let $f: X \to X$ be a fuzzy contraction with respect to the identity mapping I_X on X. If $f(X)$ or X is G-complete, then f has a unique fixed point.*

Proof It follows from Theorem 2 being f and I_X weakly compatible. □

From Theorem 4 and Lemma 2 we obtain the following

Corollary 1 *Let $(X,M,*)$ be a fuzzy metric space with M triangular and let $f: X \to X$ be a fuzzy k-contraction with respect to the identity mapping I_X on X. If $f(X)$ or X is complete, then f has a unique fixed point.*

Definition 8 ([10], Definition 3.5) Let $(X,M,*)$ be a fuzzy metric space and let $f: X \to X$ a map. The map f is a fuzzy contraction if there exists $k \in]0,1[$ such that

$$\frac{1}{M(f(x),f(y),t)}-1 \le k(\frac{1}{M(x,y,t)}-1)$$

for every x, $y \in X$ and every $t > 0$.

In a fuzzy metric space $(X,M,*)$ a sequence (x_n) is said to be fuzzy contractive ([10], Definition 3.8) if there exists $0 < k < 1$ such that

$$\frac{1}{M(x_{n+1},x_{n+2},t)}-1 \le k(\frac{1}{M(x_n,x_{n+1},t)}-1)$$

for every $n \in \mathbb{N}$. From $m(f,I_X;x,y,t) \le M(x,y,t)$ which implies $1/M(x,y,t) \le 1/m(f,I_X;x,y,t)$, we deduce that every fuzzy contraction is a fuzzy k-contraction with respect to identity mapping. From Corollary 1 it follows the result of Gregori and Sapena.

Corollary 2 ([10], Theorem 4.4) *Let $(X,M,*)$ be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy. Let $f: X \to X$ be a fuzzy contractive mapping, then f has a unique fixed point.*

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