KSME

Springer

Journal of Mechanical Science and Technology 37 (2) 2023

Original Article
DOI 10.1007/s12206-023-0124-3

Keywords:
· Cantilever nano-beam
· Nonlocal theory of elasticity
· Nonlinear Winkler-Pasternak elastic foundation
· Nonlinear/linear curvature
· Bending slope angle
· Bending deflection

Correspondence to:
Ahmad Mamandi
ahmad.mamandi@iau.ac.ir

Citation:
Mamandi, A. (2023). Nonlocal large deflection analysis of a cantilever nano-beam on a nonlinear Winkler-Pasternak elastic foundation and under uniformly distributed lateral load. Journal of Mechanical Science and Technology 37 (2) (2023) 813~824.
http://doi.org/10.1007/s12206-023-0124-3

Received June 5th, 2021
Revised May 3rd, 2022
Accepted October 24th, 2022

† Recommended by Editor Hyun-Gyu Kim

Nonlocal large deflection analysis of a cantilever nanobeam on a nonlinear Winkler-Pasternak elastic foundation and under uniformly distributed lateral load

Ahmad Mamandi

Department of Mechanical Engineering, Parand Branch, Islamic Azad University, Parand, Iran

Abstract In this paper, the PDEs and BCs governing the large deflection of an Euler-Bernoulli cantilever nano-beam on a nonlinear Winkler-Pasternak elastic foundation and under uniformly distributed lateral load have been derived using Eringen's nonlocal elasticity theory, considering the nonlinear and linear relationships of curvature-deformation, and then solved using finite difference method. The effect of changes of different parameters, including nonlocal parameter, load factor, linear/nonlinear and shear stiffness coefficients of the foundation on the deflection, bending slope angle of elastic curve and length change of the nano-beam, have been investigated. Results show that by increasing the nonlocal parameter, the bending slope angle and deflection of the free end of cantilever nano-beam are decreased and the dimensionless ratio of the final length of nano-beam is reduced. Also, the effect of nonlocal parameter on the nonlinear large deflection of the nano-beam is more significant at higher values of the applied lateral load.

1. Introduction

Significant improvements have been made in the application of micro and nano-scale systems in MEMS and NEMS electromechanical equipment. These devices are used as pressure and temperature sensors, accelerometer, micro-nozzles, gas detection sensors, biochemical sensors and atomic force microscopy nano-cantilevers in various fields including communications, medical, electronics, photonics, automotive, oil & gas and aerospace industries. Micro/nano-beams are the main components of small-scale structures. The recognition of the behavior of these structures is very important under different static or dynamic loads. Concerning the small scale of micro/nano structures, the use of formulation based on different classical and conventional theories for thin/thick beam such as Euler-Bernoulli beam, Timoshenko beam, Reddy beam and Levinson's beam models can cause errors in the beams bending analysis. Therefore, to achieve more appropriate and accurate results in investigating the nonlinear bending behavior of these micro/nano-beams under static/dynamic loads, the governing equations can be used based on novel theories such as Eringen's nonlocal elasticity theory. Note that since the nano-beams are mostly considered as thin beams in real and practical devices as atomic force microscopes and sensors/actuators there is no need to use a higher-order beam theory to accurately model bending of thicker beams. Briefly, the Euler-Bernoulli beam theory is sufficient to model and analyze these thin type structures as the thin cantilever nano-beams. Eringen presented the elasticity theory whereby the tension at a point is dependent not only on the strain at that point but to the strain at all parts of the body. In classical beam theory, the square of the first derivative of the beam deflection due to the bending moment is disregarded in the beam bending curvature relation. Therefore, the classical theory cannot be used when the deflection or the bending slope angle of deflected beam is so large. Also, the cantilever micro/nano beams, such as the cantilever AFMs, are bent in touching the surface of the specimen, or under loading effect, which can lead to bending with large deflection and a large bending angle at its free end. Moreover, these beams are mostly placed on a foundation or their contact with the surrounding environment is modeled with foundations that have different characteristics. In the following, some of works are reviewed that investigate beam deflection using the classical and non-classical theories.

The large deflection problem of a simply supported beam was investigated in Ref. [1], who observed that the load-deflection graph is not linear, but the deflection increases by increasing the lateral loading. Also, it was observed that the value of maximum deflection of a simply supported beam and under concentrated central load was slightly higher than that of classical theory. The differential equation governing the deflection problem of a cantilever beam of linear elastic material in which a vertical concentrated load was applied at the free end of the beam was analyzed in Ref. [2], and the numerical solution results compared with experimental results. The differential equations related to the large deflection of the beam were derived and then the governing equations were deduced for the small deflection of the cantilever beam. In Ref. [3], the modified nonlocal elasticity theory was applied to analyze and examine the nonlocal effects on the Euler-Bernoulli beam as an actuator in small-scale systems. It was concluded that the cantilever actuators do not exhibit nonlocal effects at micro-scale, whereas such effects can emerge in the nano-scale devices. In Ref. [4], the effect of the length scale parameter on the bending of micro/nano rods and tubes was investigated using the Euler-Bernoulli and Timoshenko beam theories; and explicit solutions for static deformation of such structures were obtained. Based on analysis of nano-scale rods and tubes with their associated boundary conditions and subjected to different loadings, it was concluded that the effect of the small scale parameter shows itself more at the location of the applied concentrated force. Different beam theories, including Euler-Bernoulli, Reddy, Timoshenko, and Levinson, were reformulated in Ref. [5] using the differential equations of Eringen’s nonlocal elasticity theory. Then, analytical solutions were presented to investigate the effects of nonlocal behavior on deflection, buckling load and natural frequencies of beams vibration. It was concluded that the nonlocal effect increases the slope value and reduces natural frequencies under bending lateral loads. The large deflection of cantilever beams was investigated by considering nonlinear geometry in Ref. [6]. In Ref. [7] the large deflection of a cantilever beam under a concentrated load at the free end of the beam using the homotopy analytical method (HAM) was investigated. Explicit analytical relations were obtained for the slope angle at the free end of the beam and also the vertical and horizontal deflections of the cantilever beam. In Ref. [8] the large-scale deflection of the cantilever beam to find a suitable and optimal mathematical model using the semi-analytical adomian decomposition method (ADM) was analytically studied. Then, by changing different parameters in the beam deflection, the vertical and horizontal deflections of a cantilever beam were investigated. Large deflection theory of nano-beams was studied in Ref. [9] to show that surface energy and large deflection may individually or jointly have notable effects. The solution of problems associated with the large deflection of the cantilever beam using the moment integral method was investigated in Ref. [10]. Also, for more complicated loading, the accuracy of the obtained results using that method was compared to obtain results of the experimental and numerical methods. In Refs. [11, 12] the large deflection problem of the cantilever beam under a concentrated load applied at the free end and also with an inclination relative to the beam axis was investigated. The nonlinear differential equations governing the beam deflection were solved by fourth order Runge-Kutta method. The effect of the size parameter (length scale) on the bending analysis of the micro-tubes using nonlocal elasticity theory for an Euler-Bernoulli beam was studied in Ref. [13]. A new method was presented to obtain the exact bending moment and displacement in the micro-tubes with uniformly distributed and concentrated loads by using the continuum nonlocal theory. In addition, the effect of nonlocal parameter on the static response of the micro-tube bending problem was investigated using the differential quadrature method (DQM). The numerical results showed that the nonlocal parameter have an important effect on the static behavior of the micro-tubes. Further researches on the modeling and derivation of the equations governing the large deflection of the beam were performed by applying the nonlocal elasticity theory to the Euler-Bernoulli nano-beams under a point load in Refs. [14-17]. The large deflection of an Euler-Bernoulli beam placed on a linear elastic foundation based on von-Karman’s nonlinear strain-displacement relations was investigated in Ref. [18]. Based on 2D differential equations of the nonlocal elasticity for plane stress, the governing equations of the cantilever nano-beam under different types of external transverse loads with simple assumptions were obtained in Ref. [19]. The results obtained from the analysis showed a significant nonlocal effect for bending deflection when the beam was exposed to distributed transverse loads or a combination of distributed loads and concentrated forces. Also, the bending deformation of the cantilever nano-beam subjected to a concentrated force does not exhibit any nonlocal effect. Moreover, the results showed that nonlocal equivalent stiffness of a nano-structure may be either increased or reduced depending on specific type of applied loads. the large deflection of the cantilever nano-beam based on the nonlocal elasticity theory was studied in Ref. [20]. The state of the art of potential application of nano-beams as nano-sensors can be referred in Ref. [21]. In Ref. [21] free vibration analysis of a fully clamped SWCNT was investigated applying a molecular mechanics (MM) formulation and a continuum mechanics (CM) analytical approximation. The MM method is based on representing the SWCNT as a 3D finite element frame of point masses and linear springs, while the CM one is grounded on the Euler–Bernoulli beam theory has been utilized. The effect of SWCNT relative natural frequency shifts due to the mass addition, regarding specific modes of vibration and for different mass values and position combinations was investigated.

In the large deflection condition for transversely loaded thin beams, applying linear bending theories does not yield the correct results and these classical theories predict the bending deflections and slope angles far from the actual values. So, in these issues the nonlinear bending parameters in the governing equations should be considered to model thin beams really in the large deflection condition. Thus, the assumption of small bending slope angle φ (i.e., for $\varphi \leq 5°$) of every point on the deflected beam (elastic curve of the deformed beam) is not further satisfied in the beam's governing equations, i.e., $\cos\varphi \neq 1$ and $\sin\varphi \neq 0$. Briefly, the value of bending slope angle φ in the beam's governing equations is mostly taken as high value based on the geometry of deformed beam in the large deflection condition. In this condition, $\sin\varphi \neq \varphi$ and $\tan\varphi \neq \varphi$ (φ in terms of radian unit).

From the review of the literature there is no independent research performed that involves the nonlinear analysis of the large deflection for an Euler-Bernoulli cantilever nano-beam on a nonlinear Winkler-Pasternak elastic foundation and under uniformly distributed lateral load based on the Eringen's nonlocal elasticity theory in conjecture with nonlinear relationship of curvature-deformation of the beam. This can be regarded as the main aspect of novelty in the current research relative to the previous studies. Besides, through validation of the obtained results with the ones available in the special case, the advantages of the used numerical finite difference method of solution and applied scheme concerning the accuracy for the better problem representation over similar methods are taken into account and discussed comprehensively.

In this study, the nonlinear partial differential equation governing the large deflection of an Euler-Bernoulli cantilever nano-beam on the nonlinear Winkler-Pasternak elastic foundation with assumption of uniformly distributed lateral load always perpendicular to the undeformed longitudinal axis of the beam is derived along with its associated boundary conditions based on Eringen's nonlocal elasticity theory, considering nonlinear and linear curvature-deformation relations of the nano-beam. Then, the governing PDE and BCs are solved using the finite difference method. The effects of variation of the nonlocal parameter, load parameter, linear and non-linear stiffness and shear stiffness coefficients of the nonlinear Winkler-Pasternak foundation on the deflection (elastic curve of the deformed nonlocal thin nano-beam), bending slope angle and final length of the nano-beam are investigated. Also, the obtained results of numerical solution with two different assumptions including nonlinear and linear curvature of the nano-beam are discussed.

2. Mathematical modeling

An isotropic and homogeneous elastic Euler-Bernoulli cantilever nano-beam of length L, cross-section A, second moment inertia of area I, density ρ and elastic modulus E is shown in Fig. 1. The nano-beam is placed on a nonlinear Winkler-Pasternak elastic foundation with linear stiffness of k_1, nonlinear stiffness of k_2 and foundation shear stiffness of k_s under uniformly distributed lateral load of $q(x) = q$. The x and y coordinate axes are considered along the longitudinal direction of the neutral axis of the nano-beam and the transverse direction of it (perpendicular to the longitudinal direction), respectively. After bending deformation of the nano-beam under the lateral loading, the deflection at each point of coordinates (x, y) on the neutral axis of the nano-beam with respect to to the nano-beam's undeformed state is indicated with w.

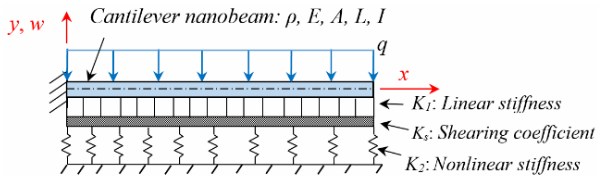


Fig. 1. A thin cantilever nano-beam on a nonlinear Winkler-Pasternak elastic foundation and under uniformly distributed lateral load.

In the beam's bending analysis the bending slope angle φ can be expressed as:

$$\tan\varphi = w_{,x} \ . \tag{1}$$

In which in the classical linear and small deflection theory of thin beams, the bending slope angle is defined by $\varphi = \partial w/\partial x$. From the geometry of deformed beam, one gets

$$\cos\varphi = dx / ds,\ \sin\varphi = dy / ds \tag{2}$$

where s is the arc length along the elastic curve of the deformed beam measured from any point of the deformed beam, in which

$$ds = dx\sqrt{1 + w_{,x}^{\ 2}} \ . \tag{3}$$

The curvature κ (equal to the reciprocal of the radius of curvature) of a planar curve is defined as the rate of slope angle of the curve with respect to the distance along the curve [22]. Therefore, the curvature κ along the curve length s of the neutral axis of the deformed elastic beam in the xy plane is defined as [20, 23],

$$\kappa(s) = (\sin\varphi)_{,s} = \varphi_{,s}\cos\varphi\,, \tag{4}$$

using Eqs. (1) and (2), noting that the relevant above-mentioned explanations and after doing some mathematics, Eq. (4) yields to [23]

$$\kappa(s) = w_{,xx} / (1 + w_{,x}^{\ 2}) \ . \tag{5}$$

Eq. (5) represents the nonlinear relationship for the planar curvature-deformation of a thin beam in the large deflection condition along the elastic curve of the deformed beam. Because of the assumption of large deflection for the elastic beam bending analysis, the square term of a slope angle (i.e., w_x^2)

may not be regarded as negligible quantity in comparison with unity, whereas in the classical linear and small deflection theory of thin beams it is conventionally ignored. The one-dimensional constitutive relation for the Euler-Bernoulli beam based on the Eringen's nonlocal elasticity theory, for instance in direction *s* along the deformed elastic curve of the beam, is expressed as follows [3, 4, 20]

$$\sigma_s - (e_0 a)^2 \sigma_{s,ss} = E\varepsilon_s \tag{6}$$

where σ_s and ε_s are the normal stress and normal strain along the elastic curve of the deformed beam, respectively, a is the internal characteristic length and e_0 is a scalar constant. Also, in the above equations the subscripts denoted by comma indicate the partial derivative of the term with respect to the variable comes after that; for instance $(\ )_{,s}$ means $\partial(\ )/\partial s$. Eq. (6) can be used if the bending of the beam is also stated in the system of coordinates *xy*. The strain ε in Eq. (6) depends on the theory used to analyze the nano-beam. Now, Eq. (6) can be rewritten as [5, 23, 24, 25]

$$\sigma_x - (e_0 a)^2 \sigma_{x,xx} = E\varepsilon_x = -EI\kappa_x \ , \tag{7}$$

in which σ_x is the axial stress, ε_x is the axial strain and κ_x is the curvature of the neutral axis of the deformed elastic beam. Based on the relations for the distribution of the shear force $V(x)$ and bending moment $M(x)$ at any cross section of distance x from the free end of beam where $V(x) = [M(x)]_{,x}$ in the nonlinear and linear curvature conditions, respectively, one can obtain [3, 4]:

$$V_{,x} = M_{,xx} = \frac{q(x) - K_1 w - K_2 w^3 + K_s w_{,xx}}{\sqrt{1 + {w_{,x}}^2}}, \tag{8a}$$

$$V_{,x} = M_{,xx} = q(x) - K_1 w - K_2 w^3 + K_s w_{,xx} \tag{8b}$$

Based on Eq. (5) for an Euler-Bernoulli beam in the nonlinear and linear curvature conditions, respectively, we get,

$$\kappa_x = w_{,xx} / (1 + {w_{,x}}^2) \ , \tag{9a}$$

$$\kappa_x = w_{,xx} \ . \tag{9b}$$

From Eq. (7) in the nonlinear and linear curvature conditions, respectively, it is obtained that [4, 19]

$$M - (e_0 a)^2 M_{,xx} = -EI \frac{w_{,xx}}{1 + {w_{,x}}^2} \ , \tag{10a}$$

$$M - (e_0 a)^2 M_{,xx} = -EI w_{,xx} \ . \tag{10b}$$

Doing twice differentiation of Eq. (10) with respect to x, in the nonlinear/linear curvature condition it is obtained that

$$M_{,xx} - (e_0 a)^2 (M_{,xx})_{,xx} = -EI\kappa_{x,xx} \ . \tag{11}$$

Using Eq. (9) into Eq. (11) in the nonlinear and linear curvature conditions, respectively, one gets

$$\begin{aligned} M_{,xx} - (e_0 a)^2 M_{,xxxx} = -EI[&\frac{w_{,xxxx}}{1 + {w_{,x}}^2} \\ &- \frac{6 w_{,xxx} w_{,xx} w_{,x}}{(1 + {w_{,x}}^2)^2} + \frac{8 {w_{,xx}}^3 {w_{,x}}^2}{(1 + {w_{,x}}^2)^3} - \frac{2 {w_{,xx}}^3}{(1 + {w_{,x}}^2)^2}] \ , \end{aligned} \tag{12a}$$

$$M_{,xx} - (e_0 a)^2 (M_{,xx})_{,xx} = -EI w_{,xxxx} \ . \tag{12b}$$

By substituting $M_{,xx}$ from Eq. (8) into Eq. (12), in the nonlinear and linear curvature conditions, respectively, we have

$$\begin{aligned} &\frac{q(x) - K_1 w - K_2 w^3 + K_s w_{,xx}}{\sqrt{1 + {w_{,x}}^2}} \\ &-(e_0 a)^2 [\frac{q(x) - K_1 w - K_2 w^3 + K_s w_{,xx}}{\sqrt{1 + {w_{,x}}^2}}]_{,xx} = \\ &-EI[\frac{w_{,xxxx}}{1 + {w_{,x}}^2} - \frac{6 w_{,xxx} w_{,xx} w_{,x}}{(1 + {w_{,x}}^2)^2} + \frac{8 {w_{,xx}}^3 {w_{,x}}^2}{(1 + {w_{,x}}^2)^3} - \frac{2 {w_{,xx}}^3}{(1 + {w_{,x}}^2)^2}] \ , \end{aligned} \tag{13a}$$

$$\begin{aligned} &q(x) - K_1 w - K_2 w^3 + K_s w_{,xx} \\ &-(e_0 a)^2 [q(x) - K_1 w - K_2 w^3 + K_s w_{,xx}]_{,xx} = -EI w_{,xxxx} \ . \end{aligned} \tag{13b}$$

Then by expanding the differentiation of Eq. (13) in the nonlinear and linear curvature conditions, respectively, yields to

$$\begin{aligned} &(1 + {w_{,x}}^2)^{-1/2} (-q + K_1 w + K_2 w^3 - K_s w_{,xx}) \\ &+(e_0 a)^2 \{(1 + {w_{,x}}^2)^{-1/2} (q_{,xx} - K_1 w_{,xx} - 3K_2 w_{,xx} w^2 - 6K_2 {w_{,x}}^2 w \\ &+K_s w_{,xxxx}) + (1 + {w_{,x}}^2)^{-3/2} [(-2 q_{,x} w_{,xx} w_{,x} + 2K_1 w_{,xx} {w_{,x}}^2 \\ &+6K_2 w_{,xx} {w_{,x}}^2 w^2 - 2K_s w_{,xxx} w_{,xx} w_{,x}) \\ &+(-q {w_{,xx}}^2 + K_1 {w_{,xx}}^2 w + K_2 {w_{,xx}}^2 w^3 - k_s {w_{,xx}}^3 - q w_{,xxx} w_{,x} \\ &+K_1 w_{,xxx} w_{,x} w + K_2 w_{,xxx} w_{,x} w^3 - K_s w_{,xxx} w_{,xx} w_{,x})] \\ &+(1 + {w_{,x}}^2)^{-5/2} (q {w_{,xx}}^2 {w_{,x}}^2 - K_1 {w_{,xx}}^2 {w_{,x}}^2 w \\ &-K_2 {w_{,xx}}^2 {w_{,x}}^2 w^3 + K_s {w_{,xx}}^3 {w_{,x}}^2)\} \\ &-EI[(1 + {w_{,x}}^2)^{-1} (w_{,xxxx}) - (1 + {w_{,x}}^2)^{-2} (6 w_{,xxx} w_{,xx} w_{,x}) \\ &+(1 + {w_{,x}}^2)^{-3} (8 {w_{,xx}}^3 {w_{,x}}^2) - (1 + {w_{,x}}^2)^{-2} (2 {w_{,xx}}^3)] = 0 \ , \end{aligned} \tag{14a}$$

$$\begin{aligned} &-q + K_1 w + K_2 w^3 - K_s w_{,xx} + (e_0 a)^2 [q_{,xx} - K_1 w_{,xx} \\ &-6K_2 w {w_{,x}}^2 - 3K_2 w^2 w_{,xx} + K_s w_{,xxxx}] - EI w_{,xxxx} = 0 \ . \end{aligned} \tag{14b}$$

Eqs. (14a) and (14b) are the nonlinear PDEs governing the large deflection of an Euler-Bernoulli nano-beam in conjecture with the nonlinear and linear curvatures, respectively, on the nonlinear Winkler-Pasternak elastic foundation based on the Eringen's nonlocal elasticity theory. The following dimensionless quantities are defined:

$$\begin{aligned} &\eta = \frac{x}{L}, \ \xi = \frac{w}{L}, \ \beta = (\frac{e_0 a}{L})^2, \ \alpha = \frac{qL^3}{EI} \ , \\ &k_1 = \frac{K_1 L^4}{EI}, \ k_2 = \frac{K_2 L^6}{EI}, \ k_s = \frac{K_s L^2}{EI}. \end{aligned} \tag{15}$$

The nondimensional PDEs governing the large deflection of the cantilever nano-beam in the nonlinear and linear curvature conditions, respectively, can be obtained as follows:

$$\begin{aligned}&(1+\xi_{,\eta}{}^2)^{-1/2}(-\alpha+k_1\xi+k_2\xi^3-k_s\xi_{,\eta\eta})+\beta(1+\xi_{,\eta}{}^2)^{-1/2}\\&.(\alpha_{,\eta\eta}-k_1\xi_{,\eta\eta}-6k_2\xi\xi_{,\eta\eta}-3k_2\xi^2\xi_{,\eta\eta}+k_s\xi_{,\eta\eta\eta\eta})\\&+\beta(1+\xi_{,\eta}{}^2)^{-3/2}(-2\alpha_{,\eta}\xi_{,\eta}\xi_{,\eta\eta}+2k_1\xi_{,\eta}{}^2\xi_{,\eta\eta}+6k_2\xi^2\xi_{,\eta}{}^2\xi_{,\eta\eta}\\&-2k_s\xi_{,\eta}\xi_{,\eta\eta}\xi_{,\eta\eta\eta})+\beta(1+\xi_{,\eta}{}^2)^{-5/2}(\alpha\xi_{,\eta}{}^2\xi_{,\eta\eta}{}^2-k_1\xi\xi_{,\eta}{}^2\xi_{,\eta\eta}{}^2\\&-k_2\xi^3\xi_{,\eta}{}^2\xi_{,\eta\eta}{}^2+k_s\xi_{,\eta}{}^2\xi_{,\eta\eta}{}^3)+\beta(1+\xi_{,\eta}{}^2)^{-3/2}(-\alpha\xi_{,\eta\eta}{}^2+k_1\xi\xi_{,\eta\eta}{}^2\\&+k_2\xi^3\xi_{,\eta\eta}{}^2-k_s\xi_{,\eta\eta}{}^3-\alpha\xi_{,\eta\eta\eta}\xi_{,\eta}+k_1\xi\xi_{,\eta\eta\eta}\xi_{,\eta}+k_2\xi^3\xi_{,\eta\eta\eta}\xi_{,\eta}\\&-k_s\xi_{,\eta\eta\eta}\xi_{,\eta\eta}\xi_{,\eta})-[(1+\xi_{,\eta}{}^2)^{-1}\xi_{,\eta\eta\eta\eta}-(1+\xi_{,\eta}{}^2)^{-2}(6\xi_{,\eta\eta\eta}\xi_{,\eta\eta}\xi_{,\eta})\\&+(1+\xi_{,\eta}{}^2)^{-3}(8\xi_{,\eta\eta}{}^3\xi_{,\eta}{}^2)-(1+\xi_{,\eta}{}^2)^{-2}(2\xi_{,\eta\eta}{}^3)]=0\,,\end{aligned}\tag{16a}$$

$$\begin{aligned}&-\alpha+k_1\xi+k_2\xi^3-k_s\xi_{,\eta\eta}-\beta\alpha_{,\eta\eta}-\beta k_1\xi_{,\eta\eta}\\&-3\beta k_2\xi^2\xi_{,\eta\eta}-6\beta k_2\xi\xi_{,\eta}{}^2+\beta k_s\xi_{,\eta\eta\eta\eta}-\xi_{,\eta\eta\eta\eta}=0.\end{aligned}\tag{16b}$$

The first to the fourth boundary conditions (BCs) of the cantilever beam are expressed as follows:

$$w(0)=0,\ \ w_{,x}\big|_{x=0}=0,\ \ M\big|_{x=L}=0,\ V\big|_{x=L}=0\,.\tag{17}$$

Using the relation Eq. (15), the non-dimensional forms of the first and second BCs are written as follows:

$$\xi(\eta=0)=0,\ \xi_{,\eta}\big|_{\eta=0}=0\,.\tag{18}$$

Substituting Eq. (8) into Eq. (10) in the nonlinear and linear curvature conditions, respectively, yields

$$M=(e_0a)^2\Big[\frac{q(x)-K_1w-K_2w^3+K_sw_{,xx}}{\sqrt{1+w_{,x}{}^2}}\Big]-\frac{EIw_{,xx}}{1+w_{,x}{}^2}\,,\tag{19a}$$

$$M=(e_0a)^2[q(x)-K_1w-K_2w^3+K_sw_{,xx}]-EIw_{,xx}\,.\tag{19b}$$

The third BC in the nonlinear and linear curvature conditions, respectively can be expressed as:

$$\begin{aligned}&M\big|_{x=L}=0\Rightarrow w_{,xx}\big|_{x=L}=\frac{-(e_0a)^2}{[EI-(e_0a)^2K_s\sqrt{1+w_{,x}{}^2}\,]_{x=L}}\\&\qquad.[-q(x)+K_1w+K_2w^3]_{x=L}(\sqrt{1+w_{,x}{}^2})_{x=L}\,,\end{aligned}\tag{20a}$$

$$\begin{aligned}&M\big|_{x=L}=0\Rightarrow\\&w_{,xx}\big|_{x=L}=\frac{-(e_0a)^2}{[EI-K_s(e_0a)^2]}[-q(x)+K_1w+K_2w^3]_{x=L}\,,\end{aligned}\tag{20b}$$

The nondimensional form of the third BC in the nonlinear and linear curvature conditions, respectively, is written as:

$$\begin{aligned}&\xi_{,\eta\eta}\big|_{\eta=1}=\Big(\frac{-\beta}{1-k_s\beta\sqrt{1+(\xi_{,\eta}\big|_{\eta=1})^2}}\Big).\\&\qquad[-\alpha+k_1\xi(1)+k_2\xi^3(1)].\sqrt{1+(\xi_{,\eta}\big|_{\eta=1})^2}\,,\end{aligned}\tag{21a}$$

$$\xi_{,\eta\eta}\big|_{\eta=1}=\Big(\frac{-\beta}{1-k_s\beta}\Big)[-\alpha+k_1\xi(1)+k_2\xi^3(1)]\,.\tag{21b}$$

By doing once differentiation of the third BC in Eq. (20) with respect to the x, the fourth boundary condition in the nonlinear and linear curvature conditions, respectively, can be obtained as:

$$\begin{aligned}&V\big|_{x=L}=0\Rightarrow M_{,x}\big|_{x=L}=0\Rightarrow\\&w_{,xxx}\big|_{x=L}=(e_0a)^2(-q_{,x}+K_1w_{,x}+3K_2w^2w_{,x}-K_sw_{,xxx})_{x=L}\\&.\sqrt{1+(w_{,x}\big|_{x=L})^2}+(EI[-q+K_1w+K_2w^3-K_sw_{,xx}]_{x=L}\\&.[w_{,x}w_{,xx}]_{x=L})/(\sqrt{1+(w_{,x}\big|_{x=L})^2})\,,\end{aligned}\tag{22a}$$

$$\begin{aligned}&V\big|_{x=L}=0\Rightarrow M_{,x}\big|_{x=L}=0\Rightarrow\\&-(e_0a)^2(-q_{,x}+K_1w_{,x}+3K_2w^2w_{,x})_{x=L}\\&=[EI-K_s(e_0a)^2](w_{,xxx})_{x=L}\,.\end{aligned}\tag{22b}$$

The nondimensional form of the fourth BC in the nonlinear and linear curvature conditions, respectively, can be written as:

$$\begin{aligned}&\xi_{,\eta\eta\eta}\big|_{\eta=1}=-\beta[-\alpha_{,\eta}(1)+k_1(\xi_{,\eta}\big|_{\eta=1})+3k_2\xi^2(1)(\xi_{,\eta}\big|_{\eta=1})^2\\&\qquad-k_s(\xi_{,\eta\eta}\big|_{\eta=1})]\sqrt{1+(\xi_{,\eta}\big|_{\eta=1})^2}\\&+\frac{-\beta[-\alpha(\eta)+k_1\xi(1)+k_2\xi^3(1)-k_s(\xi_{,\eta\eta}\big|_{\eta=1})](\xi_{,\eta}\big|_{\eta=1})(\xi_{,\eta\eta}\big|_{\eta=1})}{\sqrt{1+(\xi_{,\eta}\big|_{\eta=1})^2}}\,,\end{aligned}\tag{23a}$$

$$\xi_{,\eta\eta\eta}\big|_{\eta=1}=\Big(\frac{-\beta}{1-k_s\beta}\Big)[-\alpha_{,\eta}+k_1(\xi_{,\eta}\big|_{\eta=1})+3k_2\xi^2(1)(\xi_{,\eta}\big|_{\eta=1})^2]\,.\tag{23b}$$

Note that the third and fourth BCs are functions in terms of ξ, $\xi_{,\eta}$ and $\xi_{,\eta\eta}$ at $\eta = 1$, in which these are the unknown quantities of the problem to be calculated. The nonlinear governing PDE (i.e., Eq. (16)) along with its associated nonlinear BCs (i.e., Eqs. (18), (21) and (23)) are numerically solved by applying finite difference method (FDM) using the Maple mathematical programing software.

2.1 The length change of a nonlocal cantilever nano-beam in the large deflection condition

In Fig. 2, the deformed state of a cantilever beam subjected to a uniformly distributed lateral load is shown where the position of deflected free end of the cantilever beam in the horizontal and vertical direction is denoted with δx and δy, respectively. Concerning the nonlinear deflection of the beam, the effect of length change of the beam on the deflection cannot be neglected. The generated bending moment can be expressed as:

$$M(s)=-EI(\sin\varphi)_{,s}=-EI\varphi_{,s}\cos\varphi=-EI\kappa_x\tag{24}$$

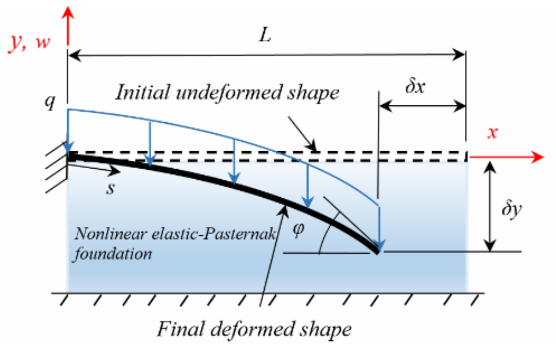


Fig. 2. The effect of length change of a thin nonlocal cantilever nano-beam considering the bending large deflection under uniformly distributed lateral load (in which is always perpendicular to the initial longitudinal direction of the beam) and on a nonlinear Winkler-Pasternak elastic foundation.

where the generated bending moment $M(s)$ at any cross section of the beam along the elastic curve of the beam due to general applied lateral load $f(x)$ can be obtained as follows [2]:

$$M(s)=\int f(x).xds\ . \tag{25}$$

The dimensionless parameter ψ is defined as:

$$\psi = s/L\ . \tag{26}$$

To obtain the location of any point on the deformed elastic curve of the beam in terms of the x-y coordinates, after solving the differential equations governing the bending large deflection of the beam and its associated BCs ((i.e., Eqs. (16), (18), (21) and (23)), the following relations are defined:

$$\begin{aligned} x&=\int_0^{L_{final}}\cos\varphi ds \Rightarrow \eta=\int_0^1 \cos\varphi d\psi = x/L_{final}\,,\\ y&=-\int_0^{L_{final}}\sin\varphi ds \Rightarrow \xi=-\int_0^1 \sin\varphi d\psi = y/L_{final} \end{aligned} \tag{27}$$

where L_{final} is the final length of the beam after its deformation. According to Eq. (10), it can be obtained that [10, 20]

$$M\big|_{s=0}=-EI\kappa_x\big|_{s=0}=-EI\varphi_{,s}\big|_{s=0}\ . \tag{28}$$

The shear force V caused by the applied lateral load f is expressed as:

$$V=M_{,s}=f.a_h \tag{29}$$

where a_h is the horizontal length of the beam after deformation. According to Eqs. (28) and (29), it can be obtained that

$$a_h=\frac{-EI\varphi_{,ss}\big|_{s=0}}{f}\ . \tag{30}$$

In the nondimensional form one can find

$$\bar{a}=\frac{-\varphi_{,\psi\psi}\big|_{\psi=0}}{\alpha_t}=\frac{\xi_{,\eta\eta\eta}\big|_{\eta=0}}{\alpha_t} \tag{31}$$

where α_t is the nondimensional applied lateral load on the beam and $\bar{a}$ is the dimensionless horizontal length of the beam in which its value is equal to the value of η at the free end of the beam. Therefore,

$$\bar{L}_{final}=\int_0^{\bar{a}_{final}}\frac{d\bar{a}}{\cos\varphi}\,, \tag{32}$$

in which $\bar{a}_{final}$ can be calculated from Eq. (31) and $\bar{L}_{final}$ (= L_{final}/L) is the nondimensional ratio of the final length of the beam after deformation with respect to the initial length of the beam.

2.2 Method of solution of governing PDEs

The non-dimensional governing linear/nonlinear PDE of the nonlocal cantilever nano-beam and its associated BCs are numerically solved using written codes in the mathematical programing environment of the Maple software by employing the finite difference method (FDM). The following step by step sequence of the numerical solution algorithm has been employed:

Step 1 - Solving numerically the nonlinear PDE of the thin cantilever nano-beam (Eq. (16)) to obtain the values of ξ, $d\xi/d\eta$ and $d^2\xi/d\eta^2$ at η = 1.

Step 2 - Putting the obtained values of ξ, $d\xi/d\eta$ and $d^2\xi/d\eta^2$ at η = 1 at step 1 into the third and fourth BCs relations (Eqs. (21) and (23)) solving numerically the nonlinear PDE of the thin cantilever nano-beam on the nonlinear Winkler-Pasternak elastic foundation.

Step 3 - Obtaining the new values of ξ, $d\xi/d\eta$ and $d^2\xi/d\eta^2$ at η = 1 (solving numerically the new obtained governing PDE at the step 2) to improve and modify the values of the third and fourth BCs at the step 2, accordingly.

Step 4 - Comparing the modified obtained values for the third and fourth BCs at step 3 with the desired accuracy (acceptable relative error criteria). If the difference between the obtained values for the third and fourth BCs at steps 2 and 3 meets the desired accuracy condition, then the solution is satisfactory and will be finished. Otherwise, the algorithm at the step 1 will be again repeated to obtain the desired numerical accuracy (for example, if the relative error between the steps 2 and 3 is less than the value of desired accuracy 0.001, the outcome can be considered satisfactory).

3. Results and discussion

In this section, at first the validity of obtained results is examined through comparison with those of the available reference. Then, the numerical obtained results are presented for the deflection and bending slope angle for the nonlocal cantilever

Table 1. Comparison of the results obtained from the large deflection of the beam by considering the nonlinear curvature in the present study and Ref. [4] for α = 1 and 3, β = 0, $k_1 = k_2 = k_s$ = 0.

Characteristics of the nano-beam and loading	Quantity	Results of the present study	Results based on Ref. [4] method	Relative difference in percent
α = 1, β = 0	η	0.9913	1.0 [4]	0
	ξ	-0.1243	-0.1250 [4]	1
α = 3, β = 0	η	0.9346	1.0	7
	ξ	-0.3602	-0.625	42

nano-beam, considering linear/nonlinear curvature under a uniformly distributed lateral load and on a nonlinear Winkler-Pasternak foundation with linear/nonlinear elastic stiffnesses and shear stiffness coefficients. Moreover, the effect of changes of different parameters on the beam bending behavior are studied comprehensively.

3.1 Validation of results

To verify the validity of the obtained results, a specific case of a classical thin beam (i.e., β = 0) is considered with no foundation ($k_1 = k_2 = k_s$ = 0). In this case the differential equation governing the beam bending deflection is

$$\alpha(\eta)+[\xi(\eta)]_{,\eta\eta\eta\eta}=0\ , \tag{33}$$

and the BCs for the beam are

$$\xi(\eta)\big|_{\eta=0}=[\xi(\eta)]_{,\eta}\big|_{\eta=0}=[\xi]_{,\eta\eta}\big|_{\eta=1}=[\xi(\eta)]_{,\eta\eta\eta}\big|_{\eta=1}=0\ . \tag{34}$$

In Table 1, the obtained results in the present study are compared with the results reported in the Ref. [4] for a classical thin beam without foundation. As can be seen, there is a good agreement between the results of the present study and results in the Ref. [4] in the case that the loading coefficient is small (i.e., α = 1). However, by increasing the loading coefficient (i.e., α = 3), the behavior of the system tends toward nonlinear trend and the difference between the results of two studies is noticeable. In this case, as expected the nonlinear deformation theory is required. Since the nature of the problem is nonlinear if small deflection theory of the beam is used, the obtained results will be greater than the actual values using nonlinear large deflection theory.

3.2 Analysis results for the large deflection of the nonlocal cantilever nano-beam

In Table 2, the obtained results of the numerical solution for a cantilever nano-beam with no foundation using the nonlinear deformation theory for different values of the load coefficient α (α = 1, 2, 3) and different values of the nonlocal parameter β are presented. Based on Table 2, it can be observed that for

Table 2. Numerical results for large deflection of the nano-beam without foundation considering nonlinear curvature in the present study for α = 1, 2, 3 and different values of β.

α	β	η_{end}	ξ_{end}	$\bar{L}_{final}$
1	0	0.9913	-0.1243	1.0091
	0.05	0.9948	-0.0998	1.0057
	0.1	0.9976	-0.0752	1.0032
2	0	0.9677	-0.2451	1.0350
	0.05	0.9798	-0.1991	1.0226
	0.1	0.9904	-0.1516	1.0128
3	0	0.9346	-0.3602	1.0742
	0.05	0.9573	-0.2970	1.0497
	0.1	0.9844	-0.2335	1.0301

the cantilever nano-beam with the same value of the loading coefficient α, the deflection of free end of the nano-beam (ξ_{end}) and the dimensionless ratio of final length of the nano-beam $\bar{L}_{final}$ (= L_{final}/L) decrease with increasing the nonlocal coefficient of β. On the other hand, as the value of the nonlocal parameter β increases, the value of η_{end} increases. It means that the position where the maximum deflection (ξ_{end}) occurs tends to move toward the free end of the nano-beam. The reason can be explained by considering the nonlinear curvature (which means an extensional condition of the neutral axis of the beam is prevailing). The stiffness of the structure increases and, on the other hand, an increase in the nonlocal parameter of β also increases the bending stiffness of the beam under lateral loading. The result of the interaction of these two phenomena in the structure is more evident by increasing the lateral loading value α and increasing the value of nonlocal parameter β. Note that in the nonlinear classical theories for thin beams, nonlinear effects of extension (stretching) emerge by considering the nonlinear strain-displacement relation (such as von-Karman strain relations for small finite deformations). From Table 2, it is also observed that at the same value of nonlocal parameter β, the free end deflection (ξ_{end}) of the nano-beam and its associated dimensionless final length $\bar{L}_{final}$ increases with increasing the loading coefficient α.

In Table 3, the obtained results of the numerical solution for a cantilever nano-beam with no foundation are presented using the linear curvature assumption for the load coefficients α = 1, 2, 3 and different values of the nonlocal parameter β. According to Table 3, for the cantilever nano-beam with the same value of the loading coefficient α, by increasing the nonlocal coefficient of β, the deflection at the free end of the beam (ξ_{end}) decreases. Also, it is observed that by increasing the value of nonlocal parameter, the value of η_{end} remains constant as that of the initial length of the nano-beam. It means that the position of the free end of the deflected nano-beam (η_{end}) where the maximum deflection occurs (ξ_{end}) has not changed. It can be explained that when the relationship of linear curvature in the bending equation of the beam is considered, the nonlinear bending stretching of the nano-beam neutral axis is not prevail-

Table 3. Numerical results for deflected nano-beam without foundation considering linear curvature in the present study for α = 1, 2, 3 and different values of β.

α	β	η_{end}	ξ_{end}	$\bar{L}_{final}$
1	0	1.0000	-0.1250	1.0092
	0.05	1.0000	-0.1000	1.0057
	0.1	1.0000	-0.0750	1.0031
2	0	1.0000	-0.2500	1.0363
	0.05	1.0000	-0.2000	1.0227
	0.1	1.0000	-0.1500	1.0125
3	0	1.0000	-0.3750	1.0795
	0.05	1.0000	-0.3000	1.0504
	0.1	1.0000	-0.2250	1.0279

Table 4. Numerical results for deflected cantilever nano-beam considering linear curvature in the present study for $k_2 = k_s = 0$ and different values of and α, β and k_1.

α	β	k_1	η_{end}	ξ_{end}	$\bar{L}_{final}$
1	0	0	1.0000	-0.1250	1.0092
		10	0.7218	-0.0684	1.0027
		20	0.6133	-0.0468	1.0013
	0.05	0	1.0000	-0.1000	1.0057
		10	0.7909	-0.0714	1.0030
		20	0.6792	-0.0584	1.0020
3	0	0	1.0000	-0.3750	1.0795
		10	0.7218	-0.2052	1.0243
		20	0.6133	-0.1403	1.0113
	0.05	0	1.0000	-0.3000	1.0504
		10	0.7915	-0.2196	1.0277
		20	0.6769	-0.1859	1.0203

Table 5. Numerical results for deflected cantilever nano-beam considering the linear curvature in the present study for $k_1 = k_s = 0$ and different values of α, β and k_2.

α	β	k_2	η_{end}	ξ_{end}	$\bar{L}_{final}$
1	0	0	1.0000	-0.1250	1.0092
		10	0.9964	-0.1241	1.0091
		20	0.9929	-0.1232	1.0089
	0.05	0	1.0000	-0.1000	1.0057
		10	0.9997	-0.0998	1.0057
		20	0.9994	-0.0996	1.0057
3	0	0	1.0000	-0.3750	1.0795
		10	0.9719	-0.3533	1.0708
		20	0.9511	-0.3372	1.0647
	0.05	0	1.0000	-0.3000	1.0504
		10	0.9924	-0.2969	1.0494
		20	0.9859	-0.2944	1.0486

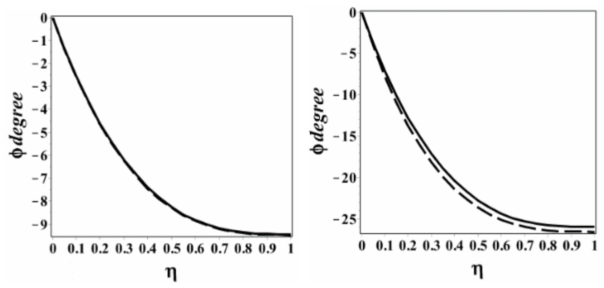


Fig. 3. Variation of the bending angle φ of the classical cantilever Euler-Bernoulli beam versus the dimensionless longitudinal distance η without foundation ($k_1 = k_2 = k_s = 0$): (a) α = 1; (b) α = 3, ------ linear curvature (dashed line), —— nonlinear curvature (solid line).

ing and thus there is linear deformation in the structure with in extensional condition. Also, by increasing the nonlocal parameter β, the bending stiffness of the structure under lateral loading increases. It is observed that for the same values of nonlocal parameter β, the deflection of the free end (ξ_{end}) of the nano-beam and its associated dimensionless final length ($\bar{L}_{final}$) increase by increasing the loading coefficient α.

In Table 4, the obtained results of numerical solution for a cantilever nano-beam are presented using the linear curvature assumption for the load coefficients of α = 1, 3 and different values of the nonlocal parameter β with $k_2 = k_s = 0$ and different linear stiffness coefficient of the foundation k_1. According to Table 4, for any specified values of loading coefficient α and nonlocal parameter β, the free end deflection of the deformed cantilever nano-beam (ξ_{end}) and its associated dimensionless final length ($\bar{L}_{final}$) are reduced with increasing the linear elastic stiffness of the foundation k_1. It is also observed that for any specified values of loading coefficient α and nonlocal parameter β, the value of η_{end} (i.e., the position at the free end of the deflected nano-beam in which the maximum deflection (ξ_{end}) occurs) is reduced with increasing the linear stiffness coefficient of the foundation. The reason is that the augmentation of the structure stiffness occurs in the presence of linear stiffness of the foundation.

In Table 5, the results of the numerical solution for a cantilever nano-beam are presented using the linear curvature assumption for $k_1 = k_s = 0$ and different values of α, β, and k_2. According to Table 5, for the cantilever nano-beam at the same values of the loading coefficient α and nonlocal parameter β, the deflection of the free end of nano-beam (ξ_{end}) and its associated dimensionless final length ($\bar{L}_{final}$) is reduced by increasing the nonlinear elastic stiffness of the foundation k_2. Also, it is observed that by increasing the nonlinear elastic stiffness of foundation k_2, the value of η_{end} is reduced. The reason is that the augmentation of the structure stiffness occurs in the presence of nonlinear elastic stiffness of the foundation.

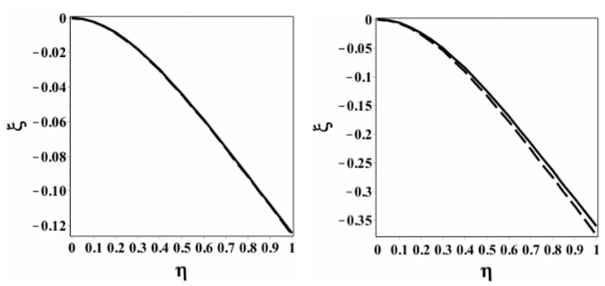
In Figs. 3(a) and (b), the variation of the bending angle φ of elastic curve of the classical cantilever Euler-Bernoulli beam (i.e., β = 0) without foundation ($k_1 = k_2 = k_s = 0$) is indicated versus the dimensionless longitudinal distance η by considering the nonlinear and linear curvatures in deformation of the beam for α = 1 and α = 3, respectively. It is observed that by considering the nonlinear curvature condition, the predicted slope (bending slope angle φ) value is less than the one obtained by considering the linear curvature condition. It is also observed that by increasing the loading coefficient α, difference between the obtained results by linear and nonlinear analysis for the bending angle increases. Moreover, the bending slope angle φ increases by increasing the loading coefficient α, no matter what type of linear or nonlinear curvature assumption is employed.

In Figs. 4(a) and (b), the variation of the large deflection ξ of the classical cantilever Euler-Bernoulli beam versus the dimensionless longitudinal distance η for $\beta = 0$ and $k_1 = k_2 = k_s = 0$ is shown by considering the nonlinear and linear curvatures in deformation of the beam for $\alpha = 1$ and $\alpha = 3$, respectively. It can be seen that by considering the nonlinear curvature, the predicted deflection values are less than the values obtained by considering linear curvature. It is also observed that the vertical deflection of the beam increases by increasing the loading coefficient α, regardless of type of linear or nonlinear curvature assumption is used.

In Figs. 5(a) and (b), the variation of bending slope angle φ of the Euler-Bernoulli cantilever nano-beam versus the dimensionless longitudinal distance η with $\alpha = 3$ and $k_1 = k_2 = k_s = 0$ is depicted, considering nonlinear and linear curvature for β = 0.05 and $\beta = 0.1$, respectively. It is observed that by considering the nonlinear curvature, the predicted value for bending slope angle φ for $\beta = 0.05$ is almost less than the value obtained by considering linear curvature, and a reverse trend prevails for $\beta = 0.1$. From the comparison of the Figs. 5(a) and (b), it is also observed that the bending angle φ decreases by increasing the nonlocal parameter β at any specified value of η, regardless of the type of linear or nonlinear curvature is used for deformation in the numerical solution of the deflection analysis of the nano-beam.

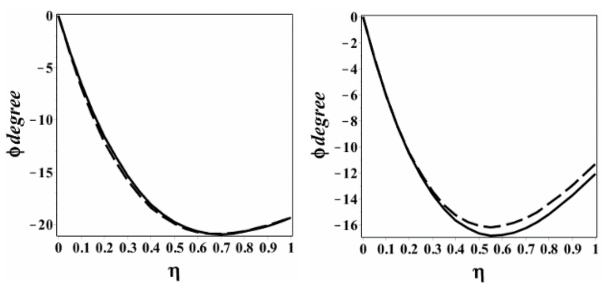
In Figs. 6(a) and (b), the variation of the deflection ξ and bending angle φ of the Euler-Bernoulli cantilever nano-beam versus the dimensionless longitudinal distance η is shown, respectively, for $\alpha = 1$ and $k_1 = k_2 = k_s = 0$ and different values of nonlocal parameter β considering the nonlinear curvature. From Fig. 6(a), it is observed that by increasing the value of nonlocal parameter β, the absolute value of deflection ξ of the nano-beam decreases. Also, the reduction trend of the beam deflection is increasing in the high values of nonlocal parameter. From Fig. 6(b), it is observed that the absolute value of the bending angle φ along the length of the nano-beam η is incremental for $\beta = 0$ and 0.01. For the subsequent values of β (β = 0.02, 0.05, 0.1, 0.15), the absolute value of bending angle φ has an increasing trend up to longitudinal distances of $\eta = 0.8$, $\eta = 0.65$, $\eta = 0.55$ and $\eta = 0.45$, respectively, and then the absolute value of bending angle decreases afterwards.

In Figs. 7(a) and (b), the variation of the deflection ξ and bending angle φ of the Euler-Bernoulli cantilever nano-beam versus the dimensionless longitudinal distance η is illustrated, respectively, for $\alpha = 3$ and $k_1 = k_2 = k_s = 0$ and different values of the nonlocal parameter β (β = 0, 0.01, 0.05, 0.1, 0.15), considering the nonlinear curvature in the large deflection analysis of the nano-beam. From Fig. 7(a), it is observed that by increasing the nonlocal parameter β, the absolute value of the vertical deflection ξ of the nano-beam decreases. From Fig. 7(b), for β = 0, 0.01 the absolute value of the bending angle φ increases with increasing the dimensionless longitudinal distance η. But,



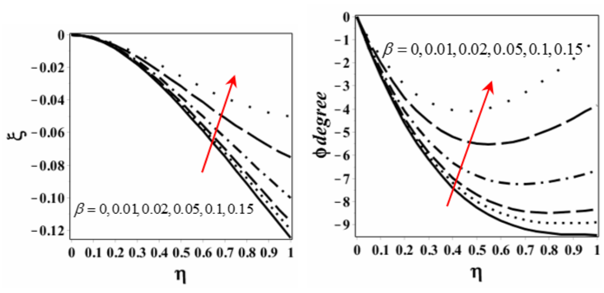
(a) (b)

Fig. 4. Variation of the vertical deflection ξ of the classical cantilever Euler-Bernoulli versus the dimensionless longitudinal distance η using linear and nonlinear curvature for $\beta = 0$ and $k_1 = k_2 = k_s = 0$: (a) $\alpha = 1$; (b) $\alpha = 3$, ------ linear curvature (dashed line), — nonlinear curvature (solid line).



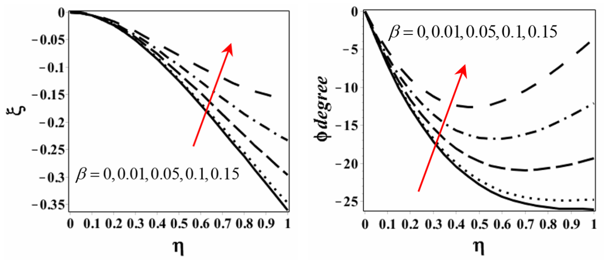
(a) (b)

Fig. 5. Variation of the bending angle φ of the cantilever nano-beam versus the dimensionless longitudinal distance η based on nonlocal elasticity theory with $\alpha = 3$ and $k_1 = k_2 = k_s = 0$: (a) $\beta = 0.05$; (b) $\beta = 0.1$, ------ linear curvature (dashed line), — nonlinear curvature (solid line).



(a) (b)

Fig. 6. (a) Variation of the vertical deflection ξ of the cantilever nano-beam vs. the dimensionless longitudinal distance η; (b) variation of the bending angle φ vs. the longitudinal distance η, based on the nonlocal elasticity theory considering the nonlinear curvature for $\alpha = 1$ and $k_1 = k_2 = k_s = 0$ and different values of the nonlocal parameter β = 0, 0.01, 0.02, 0.05, 0.1 and 0.15.



(a) (b)

Fig. 7. (a) Variation of the vertical deflection of the nano-beam versus the dimensionless longitudinal distance; (b) variation of the bending angle versus the longitudinal distance based on the nonlocal elasticity theory considering the nonlinear curvature for α = 3 and $k_1 = k_2 = k_s$ = 0 and different values of the nonlocal parameter β = 0, 0.01, 0.05, 0.1 and 0.15.

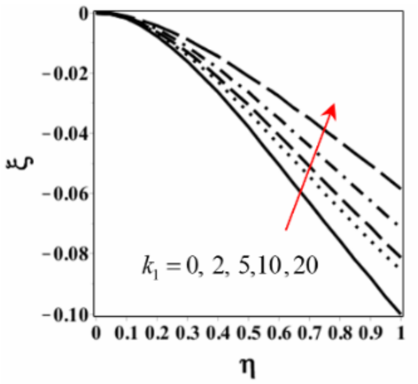
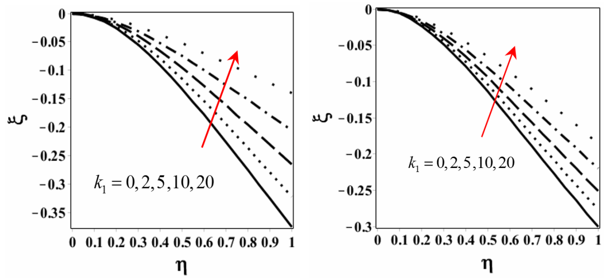


Fig. 8. Variation of the vertical deflection ξ of the nao-beam versus the dimensionless longitudinal distance η based on the nonlocal elasticity theory considering the linear curvature for α = 1, β = 0.05, $k_2 = k_s$ = 0 and different values of the linear stiffness coefficient of the foundation k_1 = 0, 2, 5, 10 and 20.

for β = 0.05, 0.1, 0.15, the absolute value of bending angle φ has an increasing trend up to longitudinal distances of η = 0.65, η = 0.55 and η = 0.45, respectively, and then the bending slope angle φ decreases afterwards.

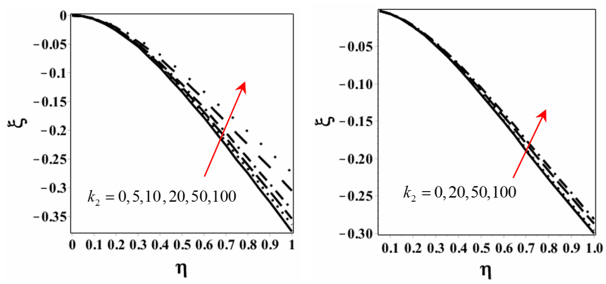
In Fig. 8, the variation of deflection ξ of the cantilever Euler-Bernoulli nano-beam is plotted versus the dimensionless longitudinal distance η based on the nonlocal elasticity theory for α = 1, β = 0.05, $k_2 = k_s$ = 0 and different values of the linear stiffness coefficient k_1 (k_1 = 0, 2, 5, 10, 20) of the foundation considering linear curvature in the large deflection analysis of the nano-beam. It can be seen that the absolute value of deflection ξ decreases by increasing the value of linear stiffness coefficient of the foundation.

In Figs. 9(a) and (b), the variation of deflection ξ of the Euler-Bernoulli cantilever nano-beam versus the dimensionless longitudinal distance η is illustrated for α = 3, $k_2 = k_s$ = 0 and different values of linear stiffness coefficient of the foundation k_1 = 0, 2, 5, 10, 20 based on the classical theory (β = 0) and nonlocal elasticity theory (β = 0.05), respectively, considering the linear curvature in the large deflection analysis of the nano-beam. It can be seen that the absolute value of deflection ξ decreases by increasing the linear stiffness coefficient of the foundation. Also,



(a) (b)

Fig. 9. Variation of the vertical deflection ξ of the cantilever nano-beam versus the dimensionless longitudinal distance η based on classical and nonlocal elasticity theories considering the linear curvature for α = 3, $k_2 = k_s$ = 0 and different values of the linear stiffness coefficient of the foundation k_1 = 0, 2, 5, 10, 20: (a) β = 0; (b) β = 0.05.



(a) (b)

Fig. 10. Variation of the vertical deflection ξ of the cantilever nano-beam versus dimensionless longitudinal distance η based on the classical and nonlocal elasticity theory considering the linear curvature for α = 3, $k_1 = k_s$ = 0 and different values of the nonlinear stiffness coefficient of the foundation k_2 = 0, 5, 10, 20, 50, 100: (a) β = 0; (b) β = 0.05.

from this figure it can be observed that by increasing the nonlocal parameter β, in the same values of linear stiffness coefficient of the foundation, lower values of deflection are obtained. Moreover, from comparison of Figs. 8 and 9(b), for the same values of linear stiffness coefficient of the foundation k_1, the deflection increases by increasing the loading coefficient α.

In Figs. 10(a) and (b), the variation of deflection ξ of the Euler-Bernoulli cantilever nano-beam is shown versus the dimensionless longitudinal distance η based on the classical theory (β = 0) and nonlocal elasticity theory (β = 0.05), respectively, with α = 3, $k_1 = k_s$ = 0 and different values of the nonlinear stiffness coefficient of the foundation k_2 = 0, 5, 10, 20, 50, 100, considering linear curvature in the large deflection analysis of the nano-beam. From this figure it can be seen that the absolute value of deflection ξ decreases by increasing the nonlinear stiffness coefficient of the foundation. Moreover, by comparing Figs. 10(a) and (b) it is observed that by increasing the nonlocal parameter β, for the same values of nonlinear stiffness coefficient of the foundation, lower absolute values of deflection ξ are obtained.

In Fig. 11, the variation of the vertical deflection ξ of the cantilever Euler-Bernoulli nano-beam is shown versus the dimensionless longitudinal distance η based on the nonlocal elasticity

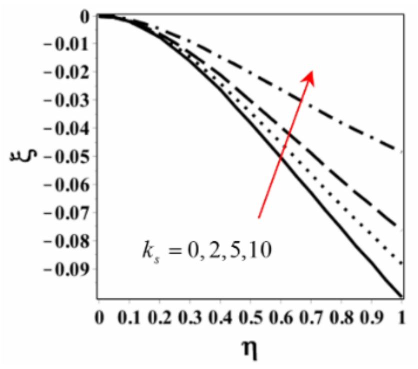


Fig. 11. Variation of the vertical deflection ξ of the cantilever nano-beam versus the dimensionless longitudinal distance η based on the nonlocal elasticity theory considering the nonlinear curvature for α = 1, β = 0.05, $k_1 = k_2$ = 0 and different values of the shear stiffness coefficient of the foundation k_s = 0, 2, 5 and 10.

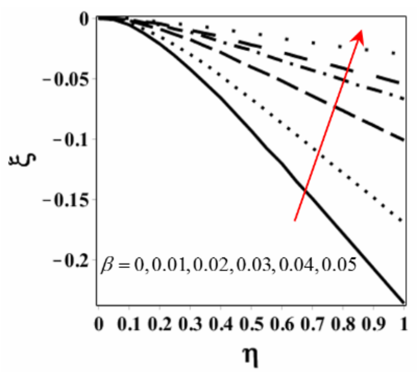


Fig. 12. Variation of the vertical deflection ξ of the cantilever nano-beam versus the dimensionless longitudinal distance η based on the nonlocal elasticity theory considering the nonlinear curvature at α = 1, k_1 = 100, k_2 = 50, k_s = 20 and different values of the nonlocal parameter β = 0, 0.01, 0.02, 0.03, 0.04 and 0.05.

theory for α = 1, β = 0.05, $k_1 = k_2$ = 0 and different values of the shear stiffness coefficient of the foundation k_s = 0, 2, 5, 10 by considering the nonlinear curvature in the large deflection analysis of the nano-beam. From this figure it can be seen that the absolute value of deflection decreases by increasing the shear stiffness coefficient of the foundation.

In Fig. 12, the variation of the deflection ξ of the Euler-Bernoulli cantilever nano-beam versus the dimensionless longitudinal distance η is shown based on the nonlocal elasticity theory for α = 1, k_1 = 100, k_2 = 50, k_s = 20 and different values of the nonlocal parameter β = 0, 0.01, 0.02, 0.03, 0.04, 0.05 by considering the nonlinear curvature in the large deflection analysis of the nano- beam. It can be seen that the absolute value of deflection decreases by increasing the nonlocal parameter.

In Fig. 13, the variation of the deflection ξ of the cantilever Euler-Bernoulli nano-beam versus the dimensionless longitudinal distance η is shown based on the nonlocal elasticity theory for α = 1, k_1 = 100, k_2 = 50, k_s = 20 and different values of the nonlocal parameter β = 0, 0.01, 0.02 by considering the linear

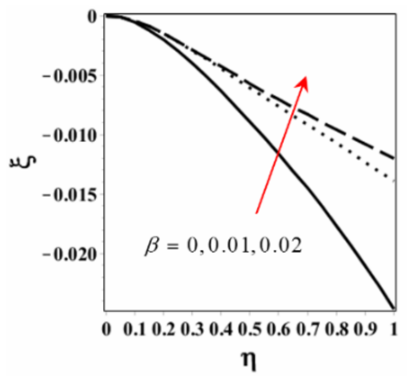


Fig. 13. Variation of the vertical deflection ξ of the cantilever nano-beam in terms of the dimensionless longitudinal distance η based on the nonlocal elasticity theory considering the linear curvature for α = 1, k_1 = 100, k_2 = 50, k_s = 20 and different values of nonlocal parameter β = 0, 0.01 and 0.02.

curvature. It can be seen that the absolute value of deflection decreases by increasing the nonlocal parameter.

4. Conclusions

The differential equation and boundary conditions governing the large bending deflection of a cantilever nano-beam under a uniformly distributed external load (always perpendicular to the longitudinal axis of undeformed beam) and on the nonlinear Winkler-Pasternak elastic foundation were derived using Eringen's nonlocal elasticity and considering the linear and nonlinear curvatures relations. The governing differential equations were numerically solved by the finite difference algorithm. The accuracy of the results was confirmed by comparing with the results reported in the literature. The effects of the nonlocal parameter, the loading coefficient and linear/nonlinear and shear stiffnesses of the foundation were investigated on the deflection, bending slope angle and the final length change of the nano-beam. A summary of the obtained results is as follows:

1) In a constant value of the loading coefficient, the bending angle of the free end of the cantilever nano-beam, the vertical deflection and the final length of the nano-beam decrease with increasing the nonlocal parameter.

2) The vertical deflection values and the bending slope angle of the nano-beam increase by increasing the loading parameter.

3) The effect of nonlocal parameter on the bending behavior (nonlinear large deflection) of the nano-beam is more significant at higher values of the applied uniformly distributed lateral load.

4) In a constant value of the loading parameter and nonlocal parameter, by increasing the linear and nonlinear elastic stiffness, and the shear stiffness coefficients of the foundation, the deflection and final length of the nano-beam are reduced. But the reduction rate of the deflection considering the nonlinear stiffness of the foundation is lower than the ones obtained considering the linear elastic and shear stiffness coefficients of the foundation.

Conflict of interest

The author of this paper clearly declares that he has no conflict of interest.

Nomenclature

k_1 : Linear stiffness coefficient of the foundation
k_2 : Nonlinear stiffness coefficient of the foundation
k_s : Shear stiffness coefficient of the foundation
β : Nonlocal parameter of cantilever nano-beam
α : Dimensionless load parameter
φ : Bending slope angle of cantilever nano-beam
η : Dimensionless longitudinal distance of cantilever nano-beam
ξ : Dimensionless nonlinear deflection of cantilever nano-beam

References

[1] H. D. Conway, XCIV. The large deflection of simply supported beams, *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 38 (287) (1947) 905-911.
[2] T. Belendez, C. Neipp and A. Belendez, Large and small deflections of a cantilever beam, *European J. of Physics*, 23 (3) (2002) 371-379.
[3] J. Peddieson, G. R. Buchanan and R. P. McNitt, Application of nonlocal continuum models to nanotechnology, *International J. of Engineering Science*, 41 (2003) 305-312.
[4] Q. Wang and K. M. Liew, Application of nonlocal continuum mechanics to static analysis of micro- and nano-structures, *Physics. Letters A*, 363 (2007) 236-242.
[5] J. N. Reddy, Nonlocal theories for bending, buckling and vibration of beams, *International J. of Engineering Science*, 45 (2007) 288-307.
[6] A. Banerjee, B. Bhattacharya and A. K. Mallik, Large deflection of cantilever beams with geometric non-linearity: analytical and numerical approaches, *International J. of Non-Linear Mechanics*, 43 (2008) 366-376.
[7] J. Wang, J.-K. Chen and S. Liao, An explicit solution of the large deformation of a cantilever beam under point load at the free tip, *J. of Computational and Applied Mathematics*, 212 (2008) 320-330.
[8] J. L. Herder and N. Tolou, A semianalytical approach to large deflections in compliant beams under point load, *Mathematical Problems in Engineering*, 2009 (2009) 1-14.
[9] D. Zeng and Q. Zheng, Large deflection theory of nanobeams, *Acta Mechanica Solida Sinica*, 23 (5) (2010) 394-399.
[10] L. Chen, An integral approach for large deflection cantilever beams, *International J. of Non-Linear Mechanics*, 45 (3) (2010) 301-305.
[11] M. Mutyalarao, D. Bharathi and B. Nageswara Rao, On the uniqueness of large deflections of a uniform cantilever beam under a tip-concentrated rotational load, *International J. of Non-Linear Mechanics*, 45 (4) (2010) 433-441.
[12] M. Mutyalarao, D. Bharathi and B. N. Rao, Large deflections of a cantilever beam under an inclined end load, *Applied Mathematica and Computation*, 217 (7) (2010) 3607-3613.
[13] O. Civalek and C. Demir, Bending analysis of microtubules using nonlocal Euler-Bernoulli beam theory, *Applied Mathematical Modelling*, 35 (2011) 2053-2067.
[14] H. T. Thai, A nonlocal beam theory for bending, buckling, and vibration of nanobeams, *International J. of Engineering Science*, 52 (2012) 56-64.
[15] H. Tari, On the parametric large deflection study of Euler-Bernoulli cantilever beams subjected to combined tip point loading, *International J. of Non-Linear Mechanics*, 49 (2013) 90-99.
[16] M. Batista, Large deflections of a beam subject to three-point bending, *International J. of Non-Linear Mechanics*, 69 (2015) 84-92.
[17] J. W. Yan, L. H. Tong, C. Li, Y. Zhu and Z. W. Wang, Exact solutions of bending deflections for nano-beams and nano-plates based on nonlocal elasticity theory, *Composite Structures*, 125 (2015) 304-313.
[18] T. S. Jang, A new semi-analytical approach to large deflections of Bernoulli-Euler-v. Karman beams on a linear elastic foundation: nonlinear analysis of infinite beams, *International J. of Mechanical Sciences*, 66 (2013) 22-32.
[19] C. Li, L. Yao, W. Chen and S. Li, Comments on nonlocal effects in nano-cantilever beams, *International J. of Engineering Science*, 87 (2015) 47-57.
[20] A. Moheyydin and S. R. Jafarizadeh, Study of large deflection in nano-beams using the nonlocal elasticity theory, *Iranian J. of Science and Technology,* 41 (2019) 222-233.
[21] G. I. Giannopoulos and S. K. Georgantzinos, Establishing detection maps for carbon nanotube mass sensors: molecular versus continuum mechanics, *Acta Mechanica*, 228 (2017) 2377-2390.
[22] A. C. Ugural, *Stresses in Plates and Shells*, 2nd Edition, McGraw-Hill, New York (1999).
[23] J. S. Rao, *Dynamics of Plates*, Narosa Publishing House, New Delhi (1999).
[24] P. Lu, Dynamic analysis of axially prestressed micro/nano-beam structures based on nonlocal beam theory, *J. of Applied Physics*, 101 (2007) 073504.
[25] S. Narendar and S. Gopalakrishnan, Nonlocal wave propagation in rotating nanotube, *Results in Physics*, 1 (2011) 17-25.

Ahmad Mamandi is an Associate Professor of Aerospace Engineering, Parand Branch, Islamic Azad University, Parand, Iran. He received his Ph.D. in Aerospace Engineering from Science and Research Branch, Islamic Azad University, Tehran, Iran in January 2010. His research interests include nonlinear vibration, recent theories of beams, plates and shells, FSI phenomena, MEMS/NEMS, aeroelasticity, fracture mechanics and fatigue analysis of structures.