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An efficient method to model the time-domain behavior of viscoelastically damped systems based on an improved fractional derivative model

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Abstract Viscoelastic materials are widely used as an efficient passive control technique to mitigate undesirable vibration and noise in many engineering fields. However, the damping properties of these materials are strongly influenced by operational and environmental factors, such as excitation frequency and operating temperature, leading to some difficulty during the finite element modelling of engineering systems containing these materials, especially for transient analysis. Thus, it is proposed herein an improved fractional derivative model to be combined with the finite element method in a straightforward way to describe the frequency- and temperature-dependent behavior of viscoelastic materials. As an academic application, the influence of the temperature on the transient responses of a viscoelastic sandwich beam subjected to some types of excitations is addressed. Moreover, the parameters of the viscoelastic model are obtained by curve-fitting for each operating temperature and the resulting equations of motion of the viscoelastic system in the time-domain are solved by using the Newmark integration scheme. Through this improved fractional derivative model, the time-domain responses of the viscoelastic sandwich beam are evaluated for several operating temperatures and compared with the corresponding obtained by the open literature to demonstrate the main features and capabilities of the proposed method.

1. Introduction

Viscoelastic materials are those whose deformations exhibit elastic and viscous behaviors. These materials are widely used for vibration control in many engineering fields, such as aeronautics, automobile, robotics and civil structures [1-5]. However, the damping properties of these materials are highly dependent on operational and environmental factors, mainly on the excitation frequency and operating temperature [6]. Thus, to model correctly the transient behavior of structures with viscoelastic materials, it is essential to consider the influence of such factors and their fading memory arising from fluency and relaxation, which makes the development of efficient and accurate viscoelastic material models a challenge.

In this context, many researchers have proposed mathematical models to represent the frequency- and temperature-dependent behavior of linear viscoelastic materials [7]. The first ones, the so-called classical viscoelastic models, are based on associations of viscous dampers and linear springs, which have some limitations in terms of the correct representation of the fluency and relaxation effects of the viscoelastic materials, having a reduced applicability. Also, there are the generalized or modern models, based on the association of the classical models, which supplied some deficiencies of the previous ones. However, it results in complex differential equations with higher order terms to be identified by curve-fitting methods [6].

In the frequency-domain, dynamic analyses are simpler, where a complex modulus function, similar to Hooke's law for elasticity, is able to adequately represent the behavior of viscoelastic materials, especially when it is used in conjunction with the fractional calculus. However, the use of this approach in the time-domain by performing the inverse Fourier transform has some limitations, as discussed by Nashif et al. [6].

From the second half of the twentieth century, several works were developed by Bagley and Torvik [8, 9] regarding the use of the fractional calculus to represent the frequency- and temperature-dependent behavior of viscoelastic materials. They were the pioneers in the development of the so-called fractional derivative models (FDM). Fractional calculus is applied in several areas, being a powerful tool to model phenomena with memory effects, as it is the case of the viscoelastic materials. Therefore, according to Zhou et al. [7], FDM proved to be the most powerful model to describe the dynamic behavior of damping materials.

Despite the great progress of the FDM model proposed by Bagley and Torvik, it was limited to one-dimensional problems. Then, Schmidt and Gaul [10] proposed a fractional derivative model based on the concepts of hydrostatic and deviation tensors for dealing with three-dimensional problems with viscoelastic materials for vibration mitigation. However, the model proposed by those authors presents a self-dependence of the stress fields of the viscoelastic part. This self-dependence makes the transient analyses of viscoelastic systems very costly, sometimes unfeasible.

Later, Galucio et al. [11] proposed an improved version of the FDM based on the introduction of additional anelastic displacements into the finite element (FE) model to eliminate the self-dependence of viscoelastic stress fields. However, it is restricted to one-dimensional problems, as discussed in Ref. [12].

Hence, to overcome the limitations of both models and based on the work by Cunha-Filho et al. [12], we propose an improved and efficient FDM model based on a recurrence term to eliminate the self-dependence of the viscoelastic stresses in the FE modeling of viscoelastically damped systems.

To demonstrate the main features and capability of the proposed methodology, an academic example formed by a three-layer viscoelastic sandwich beam is investigated herein and the results are compared with the corresponding available in the open literature.

At this time, it is important to clarify that this new FDM with a recurrence term is a simpler and more efficient constitutive law compared with the self-dependent model proposed by Schmidt and Gaul [10]. Also, the differences between the proposed new FDM model with the corresponding suggested by Ref. [11] are, first, it is based on the time-domain analysis of the viscoelastic stress history to define the recurrence term and to eliminate its self-dependence, while the model proposed by Galucio et al. [11] uses additional non-physical internal variables for the anelastic displacements for the viscoelastic part. As a result, the proposed model is more efficient in terms of the computational cost required to perform transient analyses of viscoelastic systems, since all the FE matrices can be computed outside the integration schemes. Also, the proposed model is general and can be extended to deal with three-dimensional stress state problems, while the fractional derivative model proposed by the authors in Ref. [11] has some difficulties, as discussed by Refs. [12, 13].

2. The FDM with the recurrence

Based on a four parameters fractional derivative model [7, 14], the behavior of the beam structure with a viscoelastic material, accounting for the bending and shear uncoupled equations, is described as follows:

$$\sigma_x(t) + a_E \frac{d^\alpha}{dt^\alpha}\sigma_x(t) = E_0\varepsilon_x + E_\infty \frac{d^\alpha}{dt^\alpha}\varepsilon_x(t) . \tag{1}$$

$$\tau_{xz}(t) + a_G \frac{d^\alpha}{dt^\alpha}\tau_{xz}(t) = 2G_0\varepsilon_{xz}(t) + 2G_\infty \frac{d^\alpha}{dt^\alpha}\varepsilon_{xz}(t) . \tag{2}$$

To perform the fractional derivatives appearing in Eqs. (1) and (2), the Grünwald-Letnikov approximation [11] is used herein. For a function, $f(t)$, their mathematical model is given by Eq. (3), where $\Delta t = t/N$ is the time step, N is the number of discretization points, N_L is the memory size, and the Grünwald terms A_{j+1} are given by the recurrence Eq. (4). It is important to emphasize that the Grünwald coefficients represent the fading memory in viscoelastic materials.

$$\frac{d^\alpha}{dt^\alpha} f(t) \approx \Delta t^{-\alpha} \sum_{j=0}^{N_L} A_{j+1} f(t - j\Delta t) , \tag{3}$$

$$A_{j+1} = \frac{j-\alpha-1}{j} A_j \text{ , with } A_1 = 1 . \tag{4}$$

By using Eq. (3), Eq. (1) becomes:

$$\sigma_x(t) = \frac{E_0\varepsilon_x(t) + E_\infty \Delta t^{-\alpha}\sum_{j=0}^{N_L} A_{j+1}\varepsilon_x(t - j\Delta t)}{1 + a_E\Delta t^{-\alpha}} - \frac{a_E\Delta t^{-\alpha}\sum_{j=1}^{N_L} A_{j+1}\sigma_x(t - j\Delta t)}{1 + a_E\Delta t^{-\alpha}} . \tag{5}$$

For a viscoelastic material, the constitutive law depends not only on the deformation in the current state, but also on the stress and strain histories. This fact makes the transient analysis of viscoelastic systems more complicated and very costly. Thus, we developed a recurrence term to eliminate the self-dependence of the stress history. This is done by expanding Eq. (5) at subsequent time steps to identify the influence of the stress history. For the 1st, 2nd and 3rd steps of time, Eq. (5) can be written as Eqs. (6)-(8), where $\sigma_x(t) = \sigma_t$ and $\varepsilon_x(t) = \varepsilon_t$.

$$\sigma_{\Delta t} = \frac{E_0 + E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} \varepsilon_{\Delta t} , \tag{6}$$

$$\sigma_{2\Delta t} = \frac{E_0 + E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} \varepsilon_{2\Delta t} + \left(\frac{E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_2 - \frac{a_E \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_2 \frac{E_0 + E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} \right) \varepsilon_{\Delta t}, \quad (7)$$

$$\begin{aligned}\sigma_{3\Delta t} = {} & \frac{E_0 + E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} \varepsilon_{3\Delta t} + \left(\frac{E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_2 - \frac{a_E \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_2 \frac{E_0 + E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} \right) \varepsilon_{2\Delta t} \\ & + \left(\frac{E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_3 - \frac{a_E \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_3 \frac{E_0 + E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} \right. \\ & - \frac{a_E \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_2 \left(\frac{E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_2 \right. \\ & \left. \left. - \frac{a_E \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_2 \frac{E_0 + E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} \right) \right) \varepsilon_{\Delta t}. \end{aligned} \quad (8)$$

From the previous equations, it is clear that the time history affects the stress state. This pattern will be repeated for the next time steps and Eq. (5) can be simplified into Eq. (9) for the recurrence terms given by Eqs. (10) and (11).

$$\sigma_t = \sum_{j=0}^{N_t} \beta_{j+1}^E \varepsilon_{t-j\Delta t} , \quad (9)$$

$$\beta_{j+1}^E = \frac{E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_{j+1} - \sum_{i=0}^{j} \frac{a_E \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_{i+1} \beta_{j+1-i}^E , \quad (10)$$

$$\beta_1^E = \frac{E_0 + E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} . \quad (11)$$

By applying the same procedure for the shear stress leads to the following relations:

$$\tau_t = \sum_{j=0}^{N_t} \beta_{j+1}^G \gamma_{t-j\Delta t} , \quad (12)$$

$$\beta_{j+1}^G = \frac{2G_\infty \Delta t^{-\alpha}}{1 + a_G \Delta t^{-\alpha}} A_{j+1} - \sum_{i=0}^{j} \frac{a_G \Delta t^{-\alpha}}{1 + a_G \Delta t^{-\alpha}} A_{i+1} \beta_{j+1-i}^G , \quad (13)$$

$$\beta_1^G = \frac{2G_0 + 2G_\infty \Delta t^{-\alpha}}{1 + a_G \Delta t^{-\alpha}} . \quad (14)$$

Hence, this improved and more efficient fractional derivative model eliminates the self-dependence of the stress field of the viscoelastic part, where the time history is given by the recurrence terms. It makes the transient analysis of viscoelastically-damped systems more efficient compared with the straight use of Eq. (5), as will be demonstrated later in the applications.

For self-dependent models, the matrix associated with the stress history of each element must be integrated at each time step in Newmark procedure. It makes the analysis very costly in terms of the computational time, especially for more complex structures composed by a large number of degrees of freedom (DOFs).

3. Incorporation of the new FDM into the sandwich beam

Now, consider a three-layer sandwich beam with a viscoelastic core and two elastic faces, where the layers are perfectly bonded to present the same transversal displacements and rotations and to be in a plane stress state. For the elastic layers, the Euler-Bernoulli theory is assumed, and for the viscoelastic core, Timoshenko's theory is retained. Also, it is assumed that the deformations are not sufficiently large to exceed the linear regime.

Based on these assumptions and the classical laminate theory (CLT) [13, 15], the displacements fields for a model with four DOFs (u, w, β and θ) are given as follows:

$$u^{(1)}(x,z,t) = u(x,t) - z\frac{\partial w(x,t)}{\partial x} , \quad (15)$$

$$u^{(2)}(x,z,t) = u(x,t) - z\frac{\partial w(x,t)}{\partial x} + \left(z - \frac{h_1}{2}\right)\beta(x,t) , \quad (16)$$

$$u^{(3)}(x,z,t) = u(x,t) - z\frac{\partial w(x,t)}{\partial x} + h_2\beta(x,t) \quad (17)$$

where u and β are interpolated by using a linear function, and a cubic function is used for the transverse displacement, w, with $\theta = \partial w / \partial x$. The displacement fields are approximated in FE method by Eqs. (18) and (19), where $\{q_{(e)}\} = [u_i\, u_{i+1}\, w_i\, \theta_i\, w_{i+1}\, \theta_{i+1}\, \beta_i\, \beta_{i+1}]^T$.

$$w(x,t) = [N_w(x)]\{q_{(e)}\} , \quad (18)$$

$$u^{(k)}(x,z,t) = \left[N_u^{(k)}(x,z)\right]\{q_{(e)}\}, \text{ for k=1, 2, 3} . \quad (19)$$

The shape functions are given as:

$$\begin{aligned}
[N_w] &= [0 \quad 0 \quad N_3 \quad N_4 \quad N_5 \quad N_6 \quad 0 \quad 0], \\
\left[N_u^{(1)}\right] &= [N_1 \quad N_2 \quad -z\cdot N_3' \quad -z\cdot N_4' \quad -z\cdot N_5' \quad -z\cdot N_6' \quad 0 \quad 0], \\
\left[N_u^{(2)}\right] &= [N_1 \quad N_2 \quad -z\cdot N_3' \quad -z\cdot N_4' \quad -z\cdot N_5' \quad -z\cdot N_6' \quad z_2\cdot N_1 \quad z_2\cdot N_2], \\
\left[N_u^{(3)}\right] &= [N_1 \quad N_2 \quad -z\cdot N_3' \quad -z\cdot N_4' \quad -z\cdot N_5' \quad -z\cdot N_6' \quad h_2 N_1 \quad h_2 N_2], \\
[N_\beta] &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad N_1 \quad N_2],
\end{aligned}$$

with,

$$z_2 = z - \frac{h_1}{2};\; z_{01} = -\frac{h_1}{2};\; z_{02} = \frac{h_1}{2};\; z_{03} = \frac{h_1}{2} + h_2;$$

$$N_1 = 1 - \frac{x}{li};\; N_2 = \frac{x}{li};\; N_3 = 2\left(\frac{x}{li}\right)^3 - 3\left(\frac{x}{li}\right)^2 + 1;$$

$$N_4 = x\left(1 - \frac{x}{li}\right)^2;\; N_5 = 3\left(\frac{x}{li}\right)^2 - 2\left(\frac{x}{li}\right)^3;\; N_6 = \frac{x^2}{li}\left(\frac{x}{li} - 1\right).$$

According to elasticity theory [16], strain-displacement relations are given as:

$$\varepsilon_x^{(k)}(x,z,t)=\left[\frac{d}{dx}N_u^{(k)}(x,z)\right]\{q_{(e)}\}, \tag{20}$$

$$\gamma_{xz}^{(2)}(x,t)=\left[N_\beta(x)\right]\{q_{(e)}\}. \tag{21}$$

Now, the kinetic and strain energies can be determined for each layer. For the elastic layers, Eq. (22) describes the strain energy, where the stiffness matrix defined by Eq. (23).

$$V_e^{(e)}=\frac{1}{2}\{q_e\}^T\left[K_e^{(e)}\right]\{q_e\}, \tag{22}$$

$$\left[K_e^{(e)}\right]=b\sum_{k=1,3}E_k\int_{z_{0k}}^{z_{0k}+h_k}\int_0^{li}\left[N'^{(k)}_u\right]^T\left[N'^{(k)}_u\right]dxdz. \tag{23}$$

For a viscoelastic material, the strain energy depends on the previous times, which are associated with the recurrence terms. It makes it impossible to describe the strain energy of the viscoelastic core through a single stiffness matrix. Indeed, Eq. (24) represents the strain energy for the viscoelastic core and there will be a stiffness matrix for each recurrence term expressed by Eq. (25).

$$V_e^{(v)}=\frac{1}{2}\{q_e(t)\}^T\sum_{j=0}^{N_l}\left[K_e^{**(v)}\right]_j\{q_e(t-j\Delta t)\}, \tag{24}$$

$$\begin{aligned}\left[K_e^{**(v)}\right]_j=b\Big(\beta_{j+1}^E\int_{z_{02}}^{z_{02}+h_2}\int_0^{li}\left[N'^{(2)}_u\right]^T\left[N'^{(2)}_u\right]dxdz\\+\beta_{j+1}^G\int_{z_{02}}^{z_{02}+h_2}\int_0^{li}\left[N_\beta^{(2)}\right]^T\left[N_\beta^{(2)}\right]dxdz\Big).\end{aligned} \tag{25}$$

The kinetic energy is described by Eq. (26), where the mass matrix is given by Eq. (27).

$$T_e=\frac{1}{2}\{\dot q_e\}^T[M_e]\{\dot q_e\}, \tag{26}$$

$$\begin{aligned}[M_e]=\sum_{k=1}^{3}\rho_k\Big(b\int_{z_{0k}}^{z_{0k}+h_k}\int_0^{li}\left[N_u^{(k)}\right]^T\left[N_u^{(k)}\right]\mathrm{dxdz}\\+\mathrm{A_k}\int_0^{li}\left[N_w^{(k)}\right]^T\left[N_w^{(k)}\right]\mathrm{dx}\Big).\end{aligned} \tag{27}$$

The global FE matrices are assembled using standard FE procedures accounting for the node connectivity and the elementary matrices. By using Lagrange's equations, it is possible to obtain the following equations of motion of a viscoelastic system:

$$[M]\{\ddot q(t)\}+\left[K^{(e)}\right]\{q(t)\}+\sum_{j=0}^{N_l}\left[K^{**(v)}\right]_j\{q(t-j\Delta t)\}=\{f(t)\}. \tag{28}$$

To simplify Eq. (28), the first term (j = 0) can be evaluated, resulting in the following form:

$$[M]\{\ddot q(t)\}+\left(\left[K^{(e)}\right]+\left[K^{*(v)}\right]\right)\{q(t)\}=\{f(t)\}-\{f_v(t)\} \tag{29}$$

where the viscoelastic damping forces are given as:

$$\{f_v(t)\}=\sum_{j=1}^{N_l}\left[K^{**(v)}\right]_j\{q(t-j\Delta t)\}\text{ and }\left[K^{*(v)}\right]=\left[K^{**(v)}\right]_0.$$

Eq. (29) was solved by using the Newmark constant average acceleration method (β = 0.5 and γ = 0.25), where the main steps of the numerical procedure are summarized as follows:

1) Define the parameters of the FDM: $\alpha, a_G, G_0, G_\infty, N_L$;

2) Define the time interval: $t_i:\Delta t:t_f$;

3) Compute the recurrence terms: β_E, β_G;

4) Compute the mass and stiffness matrices of the elastic layers: $[M], \left[K^{(e)}\right]$;

5) Compute the stiffness matrix of the viscoelastic layer for $j=0$:

$\left[K^{*(v)}\right]=\left[K^{**(v)}\right]_0$;

6) Compute the effective stiffness matrix:

$[K]=[M]a_0+\left[K^{(e)}\right]+\left[K^{*(v)}\right]$;

7) Newmark integration scheme: define $i=2$;

7.1) Define j = 1;

7.2) While $j\le N_L$ and $j<i$:

$\{f_v(t)\}=\sum_{j=1}^{N_l}\left[K^{**(v)}\right]_j\{q(t-j\Delta t)\}$;

7.3) Compute:

$P_i=\{f(t)\}-\{f_v(t)\}+(a_0q_{i-1}+a_2\dot q_{i-1}+a_3\ddot q_{i-1})[M]$,

$q_{i+1}=inv([K])P_i$,

$\ddot q_{i+1}=a_0(q_i-q_{i-1})-a_2\dot q_{i-1}-a_3\ddot q_{i-1}$,

$\dot q_i=\dot q_{i-1}+a_6\ddot q_{i-1}+a_7q_i$;

8) Update $i=i$ +1.

4. Curve fitting procedure

By using the frequency-temperature superposition principle (FTSP) [6], the storage modulus and loss factor at different temperatures can be combined in a single master curve at a reference temperature. It can be done by using a shift factor, α_T, to consider the temperature, according to a reduced frequency, $\omega_r=\alpha_T\cdot\omega$, which represents the frequency at which the properties at the reference temperature are the same as those for the frequency and temperature of interest. Thus, the complex modulus, which is frequency- and temperature-dependent, becomes a function of the reduced frequency.

Soovere and Drake [17] proposed analytical expressions to describe the master curves of several viscoelastic materials. For the 3M ISD-112 used in this study, the complex shear modulus and shift factor are given as:

$$\begin{aligned}G^*(\omega_r)=0.4307+1200\Bigg[1+3.241\left(\frac{i\omega_r}{1543000}\right)^{-0.18}\\+\left(\frac{i\omega_r}{1543000}\right)^{-0.6847}\Bigg]^{-1}MPa,\end{aligned} \tag{30}$$

Table 1. FDM parameters for the ISD112.

T [°C]	G_0 [Pa]	G_∞ [Pa·s$^\alpha$]	a_G [s$^\alpha$]	α [-]
10	392340	151312	8.06×10^{-4}	0.664
20	414446	55756	3.60×10^{-4}	0.673
30	425512	23702	1.83×10^{-4}	0.678
40	428743	11533	1.04×10^{-4}	0.680

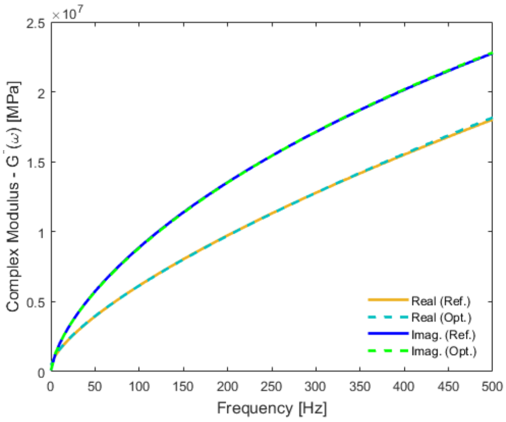

Fig. 1. FDM parameters for 10 °C.

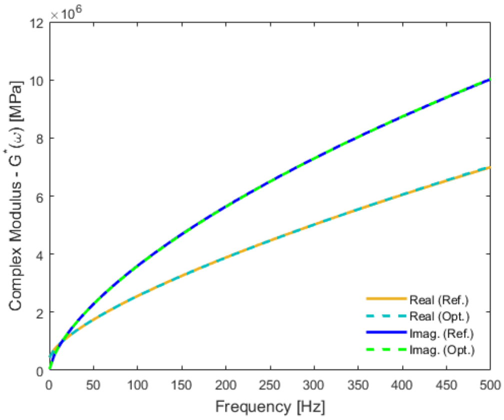

Fig. 2. FDM parameters for 20 °C.

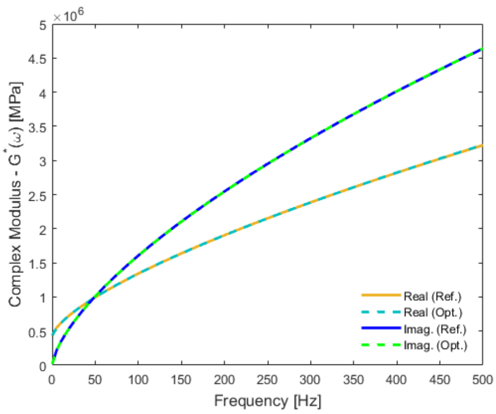

Fig. 3. FDM parameters for 30 °C.

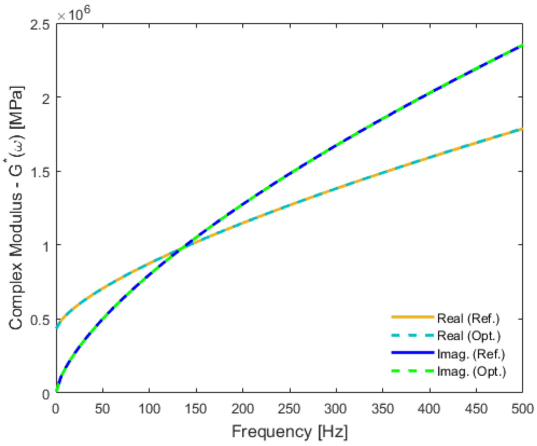

Fig. 4. FDM parameters for 40 °C.

$$\log \alpha_T = -3758.4\left(\frac{1}{T} - 0.00345\right) - 225.06\log(0.00345T) + 0.23273(T-290). \quad (31)$$

To determine the recurrence terms of the FDM model, a curve fitting procedure is performed herein using the heuristic method of differential evolution [18] to minimize the difference between the complex modulus defined by Eq. (30) and the corresponding obtained by applying the Fourier transform of Eq. (2).

Figs. 1-4 compare the adjusted curves for the storage and loss moduli with the corresponding obtained by the analytical expression for various values of temperature. The FDM parameters are given in Table 1 for each temperature.

Based on the elastic and viscoelastic correspondence principle (EVCP) [6], the shear and extensional moduli are related as follows:

$$E_0 = 2G_0(1+\nu), \ \ E_\infty = 2G_\infty(1+\nu), \ \ a_E = a_G. \quad (32)$$

At this time, it is important to clarify that it is common to assume a frequency- and temperature-independent Poisson ratio. Since extensional and shear loss factors are approximately equal, the imaginary part of the Poisson ratio is very small, as discussed in Ref. [6]. Considerable difference in its value only occurs if the viscoelastic changes from rubbery region ($\nu = 0.5$) to glassy region ($\nu = 0.33$), where the viscoelastic material becomes very stiff. This assumption has been adopted by some authors [19, 20], who verified experimentally the variations of the Poisson ratio of PVC specimens.

5. Influence of temperature

In the simulations that follow, it is used a cantilever viscoelastic sandwich beam whose mechanical and geometrical properties are given in Table 2. The sandwich beam is discretized into 25 finite elements and the time interval is divided into 0.1 ms steps with 500 points as memory size. First, the system is sub-

Table 2. Properties of the sandwich beam.

	1st layer	2nd layer	3rd layer
Material	Steel	ISD112	Steel
Length	0.5 m	0.5 m	0.5 m
Width	0.04 m	0.04 m	0.04 m
Thickness	0.015 m	0.005 m	0.01 m
Young's modulus	210 GPa	-	210 GPa
Shear modulus	80 GPa	-	80 GPa
Density	7850 kg/m³	1600 kg/m³	7850 kg/m³

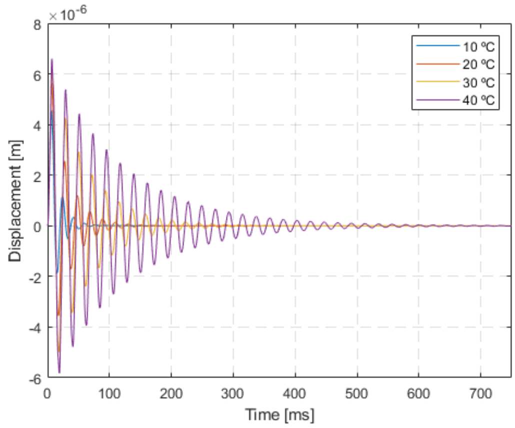

Fig. 5. Transient responses of the sandwich beam for various values of temperature for an impulse forcing.

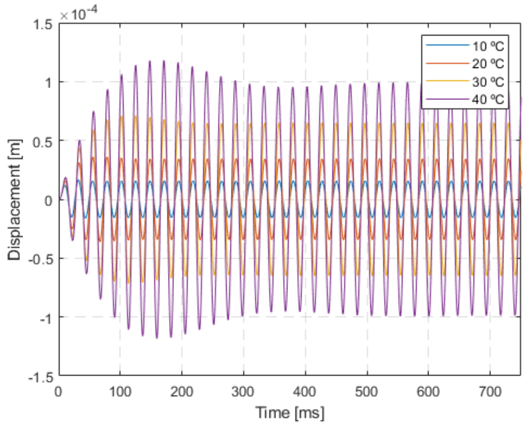

Fig. 6. Transient responses of the sandwich beam for various values of temperature for a harmonic forcing.

jected to a triangular unit impulse loading with a peak at 2 ms, and then to a unit harmonic forcing with a frequency of 43 Hz, which is close to the first natural frequency of the system. Both excitations have been applied on the free end of the beam.

The time-domain responses and the amplitudes of the frequency responses functions (FRFs) for various values of operating temperature of the viscoelastic are shown in Figs. 5-7. By comparing the transient responses, it can be clearly perceived the capacity of the proposed FDM with the recurrence to represent the influence of the operating temperature of the viscoelastic on its damping capacity, as expected.

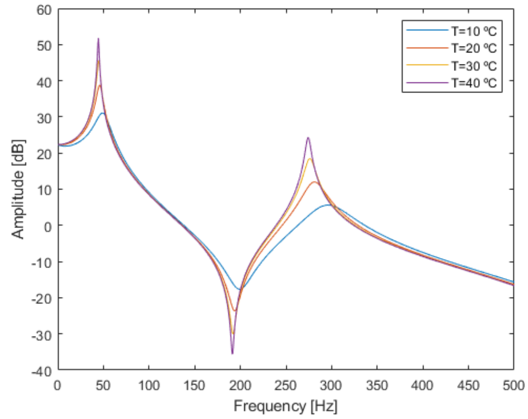

Fig. 7. Amplitudes of the FRFs for various values of temperature.

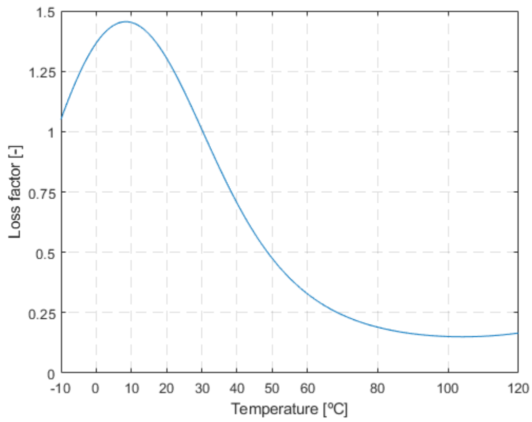

Fig. 8. Influence of the temperature on the loss factor for an excitation frequency of 50 Hz.

By analyzing the amplitudes of the FRFs, it can be also noted the effectiveness of the passive constraining damping layer to mitigate the amplitudes of the resonance peaks and the strong influence of the temperature on its damping performance.

Hence, an increase in the temperature of viscoelastic material by environmental conditions or by its self-heating [20] compromises strongly its efficiency in terms of vibration attenuation.

In this case, the maximum damping capacity of the 3M-ISD112 viscoelastic material occurs at 10 °C, as shown in Fig. 8. Based on this figure, it is possible to see how the operating temperature affects the loss factor, especially for temperatures from 0 °C to 30 °C. From 50 °C to 60 °C, the viscoelastic material changes from transition region to rubber region, where it is observed a small variation of its loss factor with the temperature, but with a reduction on its damping capacity.

6. Efficiency and accuracy of the proposed FDM

To demonstrate the efficiency and accuracy of the proposed

Table 3. CPU times for the FDM models.

FDM	Time [s]
Schmidt and Gaul	36000
With recurrence	21

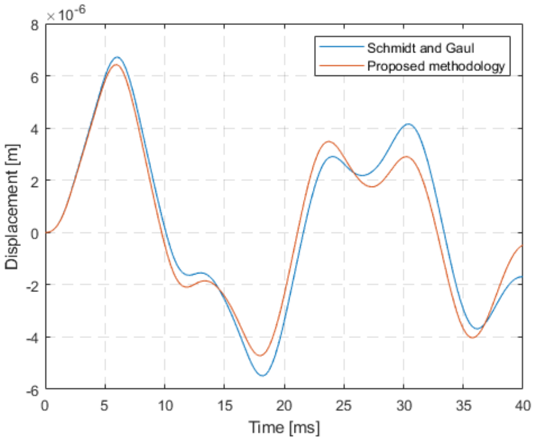

Fig. 9. Comparison between the transient responses obtained by the proposed FDM model with the model by Schmidt and Gaul [10].

FDM, it is compared with the fractional derivative model proposed by Schmidt and Gaul [10]. In this case, it is considered the same sandwich beam described in Table 2. Due to the computational limitations of the FDM proposed by the authors in Ref. [10], the sandwich beam was discretized into 15 finite elements with a time step of 0.25 ms and 300 points as memory size. The beam is subjected to a unit harmonic force with a frequency of 120 Hz. The temperature of operation is 30 °C (see Table 1).

Fig. 9 shows the comparison between the amplitudes of the transient response predicted by the proposed FDM model with the corresponding obtained by the model proposed by Ref. [10]. The correlation between the responses is very satisfactory.

However, the computation time needed for each approach appearing in Table 3 makes clear that the main drawback of the FDM model proposed by Schmidt and Gaul is its computational cost. For this reason, the proposed fractional model with the recurrence term is found to be more efficient and interesting, especially for dealing with more complex structures containing viscoelastic materials for the purposes of vibration attenuation, where simulations can take several days.

The computation times shown in Table 3 are CPU times on an Intel Core i7-6700K CPU @4.00 GHz, 16 GB RAM, 64 Bits.

7. Concluding remarks

This work proposes an improved and efficient fractional derivative model to perform transient analyses of structures incorporating viscoelastic materials for the purposes of vibration mitigation. Also investigated was the influence of the operating temperature on the performance of the viscoelastic sandwich beam. The results clearly demonstrate that, the proposed recurrence term in the classical fractional derivative formulation is a straightforward way to incorporate the viscoelastic damping forces on the transient analyses of systems with viscoelastic materials. In general, for the viscoelastic material used in this study, it was noted that around 10 °C the passive constraining layer is capable to maximize the efficiency of the viscoelastic damping. However, as the operating temperature of the system increases, a significant reduction in the performance of the viscoelastic material was observed in terms of its ability to mitigate the amplitudes of vibrations.

Finally, it is important to emphasize that, the main advantage of the proposed improved FDM is its capacity of modeling simple to more complex structures with viscoelastic materials and the correct representation of their fading memory by using the recurrence formulae. This new approach eliminates the self-dependence of stress field, making the resolution of the dynamic equations in time-domain more efficient and overcoming some limitations of the available FDM models appearing in the open literature.

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Nomenclature

A : Transversal area
A_{j+1} : Grünwald coefficients
a_E : Parameter of the model associated to elongation
a_E : Parameter of the model associated to shear
b : Width
E_0 : Low frequency Young's modulus
E_∞ : High frequency Young's modulus
$\{f\}$: External force vector
$\{f_v\}$: Viscoelastic force vector
G_0 : Low frequency shear modulus
G_∞ : High frequency shear modulus
G^* : Complex shear modulus
h : Thickness
k : Layer identification
$[K]$: Stiffness matrix
li : Length of the finite element
$[M]$: Mass matrix
N : Time step number
N_L : Memory size
$\{q_{(e)}\}$: Degrees of freedom vector
t : Time
T : Temperature

T_e : Elementary kinetic energy
$u^{(k)}$: x axis displacement associated to the k-th layer
V_e : Elementary deformation energy
x : x axis
w : z axis displacement
z : z axis
α : Fractional derivative order
α_T : Shift factor
β : Viscoelastic layer shear deformation
β^E_{j+1} : Recurrence term associated to elongation
β^G_{j+1} : Recurrence term associated to shear
γ : Shear deformation
Δt : Time step
ε : Normal deformation
ν : Poisson's ratio
ρ : Density
σ : Normal stress
τ : Shear stress
ω : Frequency

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