Springer

KSME

Journal of Mechanical Science and Technology 33 (12) (2019) 5963~5977
www.springerlink.com/content/1738-494x(Print)/1976-3824(Online)
DOI 10.1007/s12206-019-1141-0

Reliability-based design optimization of time-dependent systems with stochastic degradation †

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(Manuscript Received May 27, 2019; Revised October 3, 2019; Accepted October 16, 2019)

Abstract

A major hurdle in the application of reliability-based design optimization (RBDO) to time-dependent systems is the continual interplay between calculating time-variant reliability (to ensure reliability policies are met) and moving the design point to optimize some objective function, such as cost, weight, size and so forth. In most cases the reliability can be obtained readily using so-called fast integration methods. However, this option is not available when certain stochastic processes are invoked to model gradual damage or deterioration. In this case, sampling methods must be used. This paper provides a novel way to obviate this inefficiency. First, a meta-model is built to relate time-variant system reliability to the entire design space (and noise space if required). A design of experiments paradigm and Monte Carlo simulation using the mechanistic model determines the corresponding system reliability accurately. A moving least-squares meta-model relates the data. Then, the optimization process to find the best design point, accesses the meta-model to quickly evaluate objectives and reliability constraints. Case-studies include a parallel Daniel's system and a series servo control system. The meta-model approach is simple, accurate and very fast, suggesting an attractive means for RBDO of time-dependent systems.

Keywords: Meta-model; Monetary costs; Optimization; Stochastic process; System reliability

1. Introduction

In many engineering situations, degradation and stochastic loads lead to system deterioration, and this in turn presents a serious problem for the designer. Indeed, in mechanical and structural systems, wear is one of the most critical sources of failure since it effects the life span of bearings, hinges and coupling mechanisms. Examples include vehicle clutches, multi-bar linkages and servo systems that lose their ability to perform to specifications. In electrical systems, the parameters drift from their initial values over time through usage and environmental conditions. For example, in filters, the band frequencies become altered and the attenuation effectiveness degrades.

The analysis of degradation started with Meeker and Escobar [1] over two decades ago when they introduced a convenient statistical framework that in turn spawned various physics-based models. And now, the degradation models [2-4] include a) random variable models, b) marginal distribution models and c) cumulative damage models. The random variable (RV) models (also called degradation path models) randomize the parameters associated with some empirical deterioration law to reflect the sampling variability observed in a sample of the degradation data. The simplest form of a degradation path model of, for example, resistance R is $R = R_0 \pm Ct$ where t is time, R_0 is the initial resistance and C is the random degradation rate. Often a deterministic rate (e.g. c) is used in place of the random rate.

The marginal distribution (MD) models (also referred to as degradation distribution models) provide a new distribution at any time t; however, there are no correlations between different time distributions. A simple MD model has the form $R = R_0 \pm C(\mathbf{p}(t))$ where $\mathbf{p}$ are distribution parameters and $C(\mathbf{p}(t))$ represents a particular distribution at time t. In both RV and MD models, the degradation functions depend on random variables combined with deterministic functions of time leading to non-ergodic processes. The cumulative damage (CD) models (also called shock models) assume that the degradation is caused by shocks or jumps and that damage accumulates additively [4]. These models are used when the temporal uncertainty associated with the deterioration cannot be ignored. In this model $R = R_0 \pm C(t)$ where $C(t)$ is a stochastic process. Examples of CD models include the Weiner process, Gamma processes and inverse Gaussian process. When deterioration is uncertain and non-decreasing,

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the Gamma process is a suitable model [5]. It is apparent then, that system deterioration leads to time-dependent reliability issues, that may be mitigated by reliability-based design optimization (RBDO).

The RBDO problem has a number of related issues. The first deals with how reliability is to be calculated over time; the second deals with the two sources of uncertainty and these include parametric uncertainty in components and excitation uncertainty in loads. Finally, the third issue is the dynamical nature of the system; that is, are the performance measures steady-state, with algebraic equations, or transient, with differential equations. These areas are bridged by considering time-variant parametric uncertainties and stochastic processes.

In time-variant reliability, the uncertainties in the responses or performance measures change over time: examples occur in degradation, or if the excitations are stochastic in nature. Therein the reliability requirements are treated as constraints in the optimization algorithm, hence the optimal solution must ensure that the reliability specifications are met over the complete life-time.

Stochastic loads have been the main source of time-variant reliability issues in RBDO. Kuschel and Rackwitz [6] employed the outcrossing rate to find time-variant reliability and solved the optimization problem under two particular loads. These included differentiable processes and rectangular wave renewal processes. Wang and Wang [7] provided a nested extreme-response method to transform the time-variant RBDO problem into time-invariant RBDO problem. Hu and Du [8] devised the equivalent most-likely failure point (MLFP) and extended sequential reliability assessment algorithm (SORA) to solve time-variant RBDO problems that contained stochastic process loads. Therein first-order reliability method (FORM) was invoked and design parameters comprised either deterministic variables or means of distributions. Jiang et al. [9] produced the time-invariant equivalent method (TIEM) to reduce the number of *cdf* (cumulative distribution function) calculations. FORM was used and design parameters were deterministic variables and means of distributions. The approach assumed that the reliability indexes (over time) were invariant to the design point and hence the curves could be adjusted to meet the life-time reliability constraint. In effect the optimization had to be performed only at initial time. The method seemed to work well for parameter design.

A few papers address RBDO and degradation. Savage and Son [10] applied the set-theory method to find efficiently the *cdf* for multiple response systems and applied it to optimize system costs with respect to deterministic component degradation. Rathod et al. [11] treated probabilistic damage accumulation as a measure of degradation in material fatigue and modelled it as a stationary process that in turn became a constraint in the optimized solution. Singh et al. [12] introduced the composite limit-state to convert a time-variant RBDO problem into a time-invariant problem and then invoked a genetic algorithm to search for the MLFP. The case-study included deterministic degradation and used their composite limit-state to find effectively probabilities.

To make the design process more efficient, meta-models (often called surrogate models) have been introduced. They have had a significant impact in the design of engineering systems in the past two decades. They are computationally efficient substitutes for the mechanistic model and overall they are both accurate and very fast. These two features allow for a variety of timely quality and reliability calculations as well as efficiencies in optimization routines. The success of the meta-model depends on several decisions and these include the following: (a) The proper selection of the input variables (i.e. excitations and component parameters), (b) their ranges (e.g. design space), (c) the choice of the underpinnings of the meta-model, (d) the number of samples and the sampling philosophy used to collect data, and finally, (e) the form of the approximating function. In summary, the primary task is to choose the most accurate and fastest meta-model with the least number of training samples. Overviews of various meta-models are contained in Refs. [13, 14]. The popular Kriging methods are detailed in Refs. [15-17], the moving least squares (sometimes called lazy learning) meta-models are described in Refs. [18, 19]. The Bayesian meta-models are illustrated in Ref. [20].

There is some work in using meta-models to provide efficiencies in time-invariant reliability analysis. The most common approach is to replace the failure surface with a meta-model [21, 22]. Further, there is some work using meta-models in RBDO with time-invariant reliability and these include [23-26].

Finally, for time-variant reliability analysis, the use of meta-models has been included in Savage et al. [27] who predicted the reliability of degrading dynamic systems using various meta-models to link the time-squared-error performance index to the design parameters. Singh et al. [12] used a meta-model as a surrogate for the composite limit-state surface and then used it to determine time-invariant failure. However, the composite limit-state requires multiple MLFPs which first must be identified, and then used to provide an approximate probability. Dregnei et al. [28] developed a random process meta-model that linked the left singular vectors of the responses of a system to the left singular vectors of an uncertain excitation matrix and augmented this with uncertain component dimensions. The meta-model was then used to help determine the life-time reliability. Zhang et al. [29] established a meta-model based on response surface for time-variant limit-state function to estimate time-dependent reliability for nondeterministic structures under stochastic loads. Stochastic loads were discretized as static random variables in the model, and FORM was applied to estimate reliability.

This paper proposes a new method for RBDO of time-variant systems containing stochastic degradation. In past cases the reliability could be obtained readily using so-called fast integration methods. However, this option is not available when the Gamma process is invoked to model degradation, and thus sampling methods must be used. A meta-model restores efficiency. The novel approach herein provides two stages:

Stage one: A meta-model is built that explicitly relates time-variant reliability to the design space (and noise space).
Stage two: The optimization process invokes the typical nested approach, but now the time-variant reliability meta-model is used to quickly evaluate objective functions and constraints.

The impact of the methodology is the greatly reduced design time needed to conduct RBDO for time-variant systems.

The rest of the paper proceeds as follows. Sec. 2 outlines the time-variant reliability problem and ways to find the *cdf* for multi-response systems. Sec. 3 reviews Gaussian stochastic processes and introduces the Gamma process for degradation modelling. Sec. 4 develops the meta-model for time-variant reliability using a moving least-squares foundation. Sec. 5 presents two case-studies to compare speed and accuracy and Sec. 6 sums up the paper and provides the impact of the work herein.

2. Time-variant reliability

For time-variant reliability, we denote the vector **V** as the randomness in the problem comprising the random variables V_j, j = 1, … n. In general, **V** is partitioned into a) initial randomness of, for example, material properties and dimensions, and b) random degradation rates. The probability density functions of **V** are assumed to exist and provide distribution parameters **p**. A conversion to standard normal (i.e. **U**-space) is usually possible through an iso-probabilistic transformation [30], denoted as **V**(**p**, **U**). The transformation is not necessary, but it allows one to see the limit-state surfaces moving in **u**-space with respect to time and allows the use of approximating methods such as FORM. Lastly, we let **W**(t) be a vector of stochastic processes for time t. These may include excitations and loads typically modelled by Gaussian processes as well as cumulative damage degradation modelled by the Gamma process. Other stochastic processes, not invoked herein, may include Poisson and Weiner. Overall, these random effects over time come from dynamically changing environmental conditions and the temporal uncertainties of changes in material properties and structural dimensions.

Let us start with general statements of time-variant reliability in terms of the related c*df*. At the component level, let the limit-state function for the i^{th} component be g_i(**V**, **W**(t), t) where a positive value indicates success and a negative value indicates failure. Then, within the life-time span [0, t_L], we write the failure event

$$E_i(0,t_L) = \{\, g_i(\mathbf{V},\mathbf{W}(t),\ t) \le 0,\ \text{for}\ \exists t \in [0,\ t_L]\} \tag{1}$$

and the *true cdf* for the i^{th} component is

$$F^i(t_L) = P\big(E_i(0,t_L)\big)\,. \tag{2}$$

For systems with multiple components, and the need to determine system failure, we require, for series systems, unions of events (i.e. $\bigcup_i E_i(0,t)$) and for parallel systems, intersections of events (i.e. $\bigcap_i E_i(0,t)$).

The evaluation of Eq. (2) is generally intractable; however, discrete time is of help. Consider the planned time t_L with equally spaced time points obtained from a small, fixed, time step Δt (the length to be determined later). For a time index l = 0, 1,… L, where L is the number of time steps to the planned time, then the time at the l^{th} step is $t_l = l\times\Delta t$. We write a set that represents the *instantaneous failure* region of the i^{th} limit-state function at any selected point-in-time t_l with reference to notation in Eq. (1), as

$$E_{l,i} = \left\{ g_i\left(\mathbf{V},\mathbf{W}(t_l),t_l\right) \le 0 \right\}\,. \tag{3}$$

Note, we must find the stochastic processes **W**(t) at discrete times t_l. For the Gaussian processes, the expansion optimal linear estimation (EOLE) model is typically invoked. The Gamma process can be evaluated using the discrete time step and samples from a Gamma distribution.

Now, an approximation to the true *cdf* in Eq. (2), can be written as

$$\hat{F}^i(t_L) \approx F^i(t_L) = P\left(\bigcup_{l=0}^{L} E_{l,i}\right). \tag{4}$$

Of interest is the so-called instantaneous probability of failure at a fixed time, say $t = t_l$. It may be written as

$$F_l^i(t_l) = P\left(E_{l,i}\right)\,. \tag{5}$$

However, the event implied in Eq. (5) is independent of previous like events. In general, this probability evaluation is not the same as Eqs. (2) and (4) since it does not take into account the time-history of the system, in particular the possible failures that may occur before t_l. In the special case when the limit-state function is monotonically decreasing, the instantaneous probability is, in this case, equal to the cumulative probability.

Now, let us consider multiple failure modes and extend the single event $E_{l,j}$ over multiple limit-state functions (and the time skeleton containing t_l). We start with *subsystems* comprising first parallel connected "components" and then series connected "components".

Parallel connections: The *subsystem* instantaneous failure region at time t_l for the j^{th} subsystem comprising N_j components (connected in *parallel*) is defined to be the set

$$L_{l,j} = \bigcap_{i=1}^{N_j} E_{l,i}\,. \tag{6}$$

Series connections: The *subsystem* instantaneous failure region at time t_l for the j^{th} subsystem comprising S_j components (connected in *series*) is defined to be the set

$$M_{l,j} = \bigcup_{i=1}^{S_j} E_{l,i} \ . \tag{7}$$

Let us now to form systems by connecting the subsystems in the following ways: (Other system configurations are possible. See Ref. [39].)

a) Connect parallel subsystems in series and

b) Connect series subsystems in parallel.

First, for n parallel-subsystems connected in series, the *system* instantaneous failure region at time t_l is defined to be the set

$$\mathbf{E}_l = L_{l,1} \cup L_{l,2} \cup \cdots \cup L_{l,n} = \bigcup_{j=1}^{n} L_{l,j} \ . \tag{8}$$

Next, for s series-subsystems connected in parallel, the *system* instantaneous failure region at time t_l is defined to be the set

$$\mathbf{E}_l = M_{l,1} \bigcap M_{l,2} \bigcap \cdots \bigcap M_{l,s} = \bigcap_{j=1}^{s} M_{l,j} \ . \tag{9}$$

Finally, the *system cumulative failure* set A_l is defined as the set that represents the accumulation of all system instantaneous failure regions for all discrete times up to t_l. This set extends Eqs. (8) and (9) and is written as

$$A_l = \mathbf{E}_0 \cup \mathbf{E}_1 \cup \cdots \cup \mathbf{E}_l = \bigcup_{q=0}^{l} \mathbf{E}_q \ . \tag{10}$$

The *system cumulative safe* set is denoted as $\bar{A}_l$ and is simply

$$\bar{A}_l = \bar{\mathbf{E}}_0 \cap \bar{\mathbf{E}}_1 \cap \cdots \cap \bar{\mathbf{E}}_l = \bigcap_{q=0}^{l} \bar{\mathbf{E}}_q \ . \tag{11}$$

Let us define the emergence of the incremental failure region from the system cumulative safe region, from time t_l during time interval Δt, as $\mathbf{B}_l = A_{l+1} \cap \bar{A}_l$.

But $\mathbf{E}_q \cap \left(\bar{\mathbf{E}}_0 \cap \cdots \cap \bar{\mathbf{E}}_q \cdots \cap \bar{\mathbf{E}}_l\right) = \varnothing$, hence we have more simply

$$\mathbf{B}_l = \mathbf{E}_{l+1} \cap \bar{A}_l \ . \tag{12}$$

It follows that we may write the complete failure history in terms of the incremental failure regions as

$$\begin{aligned} A_{l+1} = \ & \mathbf{E}_0 \cup \left(\mathbf{E}_1 \cap \bar{\mathbf{E}}_0\right) \cup \left(\mathbf{E}_2 \cap \overline{\mathbf{E}_1 \cup \mathbf{E}_0}\right) \cup \cdots \cup \left(\mathbf{E}_{l+1} \cap \bar{A}_l\right) \\ = \ & \mathbf{E}_0 \cup \mathbf{B}_0 \cup \mathbf{B}_1 \cup \cdots \cup \mathbf{B}_l \ . \end{aligned} \tag{13}$$

It can be shown that all events in Eq. (13) are mutually exclusive. We write the *cdf* as

$$F(t_L) = P(\mathbf{E}_0) + P(\mathbf{B}_0) + \cdots + P(\mathbf{B}_l) + \cdots + P(\mathbf{B}_{L-1}) \ . \tag{14}$$

The expression for $\boldsymbol{B}_l$ in Eq. (12) requires the time history of the system responses and thus it is logistically difficult to determine $P(\boldsymbol{B}_l)$. In Monte-Carlo simulation (MCS), a sample from the distributions of the design variables is chosen and then the sign of $\boldsymbol{E}_l$ is determined for time index $l = 0, 1, \ldots L$, stopping and recording the time of first failure [31]: all future times are recorded as fail as well. For all MCS samples, a histogram that represents the terms in Eq. (14) is built, and then the *cdf* is found as the summation of all of the terms up to the time of interest. The reliability at time t_L is simply $R(t_L) = 1 - F(t_L)$.

3. Stochastic processes

The simplest way to incorporate stochastic processes in the reliability calculations is to employ discretized time. The approaches to modelling both excitations and degradation (used herein) are outlined next.

3.1 Excitations Y(t)

An excitation (i.e. a source or load), denoted as $Y(t)$, is typically modelled by a nonstationary Gaussian stochastic process. There are many proposed modelling methods including Karhunen-Loeve [32], polynomial chaos expansion [33], proper orthogonal decomposition [34] and EOLE [35]. The EOLE model is easy to write in matrix form (thus simplifying computer programming) and hence is employed in this paper.

For the Gaussian process, with the discretization of time as above, let the mean function be $\mu_Y(t_i)$, the standard deviation function $\sigma_Y(t_i)$ and the autocorrelation function $\rho_Y(t_i, t_j)$. Then using Appendix, the Gaussian process takes the compact matrix form

$$Y(t) = \mu_Y(t) + [\boldsymbol{\Sigma}(t)]^T (\boldsymbol{\Phi}\tilde{\boldsymbol{\Lambda}})\mathbf{U} \tag{15}$$

where $(\boldsymbol{\Phi}\tilde{\boldsymbol{\Lambda}})$ is a matrix of constants, $\boldsymbol{\Sigma}(t)$ is a time-related vector (containing standard deviation and correlation parameters) and $\mathbf{U}$ is vector of standard normal variables $[U_1, U_2, \ldots U_s]^T$. For a simpler notation, let the distribution parameters in Eq. (15) be written as $\mathbf{q}(t) = [\mu(t), \boldsymbol{\Sigma}(t)]^T$ then more informatively

$$Y(t) = Y\left(\mathbf{q}(t), \mathbf{U}\right) . \tag{16}$$

For several stochastic processes, the distribution parameters form a longer vector $\mathbf{q_Y}(t)$ and the normal variable vectors stack up to become $\mathbf{U_Y}$ so now the vector of stochastic processes is succinctly

$$\mathbf{Y}(t) = \mathbf{Y}\left(\mathbf{q}_{\mathbf{Y}}(t), \mathbf{U}_{\mathbf{Y}}\right) \tag{17}$$

3.2 Component degradation with the gamma process C(t)

The Gamma process is suitable for modelling gradual damage or deterioration when it is monotonically accumulating over time: examples include wear, fatigue, corrosion, crack growth, erosion, consumption, creep, etc. [5]. The Gamma process is a continuous-time process with stationary, independent, non-negative Gamma increments, obtained from the Gamma distribution. Let $G(\alpha, \beta)$ denote the distribution and let its density function be

$$f(\gamma) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)}\gamma^{\alpha-1}e^{-\gamma/\beta} \tag{18}$$

where $\Gamma(\alpha)$ is the so-called gamma function and α and β are the shape and scale parameters respectively.

Let the Gamma process be denoted as $C(t|\mu, \sigma^2)$ with mean μ and variance σ^2. Then, for any time increment $\Delta t = t_L/L > 0$ the increments are [36]

$$C\left(t + \Delta t \mid \mu, \sigma^2\right) - C\left(t \mid \mu, \sigma^2\right) \sim G\left(\alpha\Delta t, \beta\right). \tag{19}$$

We note that the distribution of the increments depends on the length of Δt but not on the time t.

Let us find now suitable distribution parameters for the Gamma distribution in Eq. (18). If the mean value of the process is linear, then we may write mean and variance of the process as

$$\begin{aligned} &E[C(t = t_L)] = \mu = (\alpha t_L)\beta, \\ &Var[C(t = t_L)] = \sigma^2 = (\alpha t_L)\beta^2 . \end{aligned} \tag{20}$$

We have the new parameters for the desired Gamma distribution (in terms of mean, standard deviation and coefficient of variation denoted as CV) as

$$\alpha = \frac{1}{t_L}\frac{\mu^2}{\sigma^2} = \frac{1}{t_L(CV)^2} \tag{21}$$

and

$$\beta = \frac{\sigma^2}{\mu} = \mu(CV)^2 . \tag{22}$$

Let the parameters be written compactly as $\mathbf{r} = [\mu, \sigma, \Delta t]^T$, then equation Eq. (19) provides the series of random variables

$$C(\mathbf{r}, t_l) = C(t_0) + \sum_{i=1}^{l} G\left(\frac{1}{t_L}\frac{\mu^2}{\sigma^2}\Delta t, \frac{\sigma^2}{\mu}\right) \quad l = 1, 2, \ldots \mathrm{L} \tag{23}$$

where $C(t_0) = 0$. For several Gamma processes we have the new vector of parameters $\mathbf{r}_c$ that defines the set of Gamma processes $\mathbf{C}(\mathbf{r}_{\mathbf{C}}, t_l)$.

It is a simple manner to generate the Gamma process over discrete time. For the k^{th} manifestation of a Gamma process, say $C^{(k)}(t)$ with incremental samples $\gamma_i^{(k)}$ chosen according to the Gamma distribution in Eq. (23), we have the process values

$$\begin{aligned} &c^{(k)}(t_0) = 0 \\ &c^{(k)}(t_1) = c^{(k)}(t_0) + \gamma_1^{(k)} \\ &c^{(k)}(t_2) = c^{(k)}(t_1) + \gamma_2^{(k)} \\ &\vdots \\ &c^{(k)}(t_{l+1}) = c^{(k)}(t_l) + \gamma_l^{(k)} \qquad k = 1, 2 \ldots \mathrm{N} \\ &\vdots \\ &c^{(k)}(t_L) = c^{(k)}(t_{L-1}) + \gamma_L^{(k)} . \end{aligned} \tag{24}$$

Hence, for simulation purposes, we generate N Gamma process paths to provide N jump values at each discrete time $t_0, t_1, t_2, \cdots, t_{L-1}$. For several Gamma processes, the terms in Eq. (24) become vectors.

3.3 Reliability-based design optimization

For the random variables $\mathbf{V}$ and their conversion to $\mathbf{U}$-space using the design parameters $\mathbf{p}$, for the Gaussian processes modelled through EOLE, and the Gamma processes with no convenient conversion to $\mathbf{U}$-space, the limit-state functions for the i^{th} component, under stochastic processes, using discrete time events, has a form taken from Eq. (3) as

$$E_{l,i} = g_i\left(\mathbf{V}(\mathbf{p}, \mathbf{U}_{\mathbf{V}}), \mathbf{Y}(\mathbf{q}_{\mathbf{Y}}(t_l), \mathbf{U}_{\mathbf{Y}}), \mathbf{C}(\mathbf{r}_{\mathbf{C}}, t_l), t_l\right) \le 0 \tag{25}$$

and the *cdf* follows Sec. 2.

A typical optimization problem is expressed in terms of an objective function, reliability constraints at various times and the design space for the design parameters. For example, we write

$$\begin{aligned} &\text{Minimize } O(\mathbf{p}) \\ &\text{subject to} \\ &R\left(\mathbf{p}, \mathbf{q}_{\mathbf{Y}}(t_0), \mathbf{r}_{\mathbf{C}}, t_0\right) \ge R_0 \\ &\vdots \\ &R\left(\mathbf{p}, \mathbf{q}_{\mathbf{Y}}(t_L), \mathbf{r}_{\mathbf{C}}, t_L\right) \ge R_L \\ &\mathbf{p}_L \le \mathbf{p} \le \mathbf{p}_U \end{aligned} \tag{26}$$

where $O(\mathbf{p})$ is the objective function, and R_0, R_L etc. are reliability constraints.

4. Meta-model development

The traditional RBDO approach uses a nested, or double loop, method that moves the design point via the outer loop or optimization process and then finds in the inner loop the reliability for the present design point. These loops iterate until convergence. The meta–model approach, presented next, provides a "two-stage" approach where first the reliability is found in terms of the design space and then the optimum solution is found by moving the design point. For the meta-model, let us look at the moving least squares formulation and let both means and tolerances form the design parameter vector.

4.1 Design parameters and training data

A judicious selection of the design variables and their operating ranges is important to keep the meta-model manageable but effective. There are three steps that help.

Step 1. An importance analysis that compares changes in the responses (or performance measures) to material properties and dimensions using sensitivity information can trim the number of design variables to a manageable few.

Step 2. To find reasonable nominal values of the design variables, a so-called parameter design is invoked to maximize the reliability at time zero (i.e. $R(t = 0)$). More specifically, tolerances are fixed at their average values, and the means of the design variables are adjusted accordingly.

Step 3. A sense of the magnitude of the variations about nominal values can be determined from sensitivity or worst-case information using $R(t = 0)$.

The final set of design parameters $\mathbf{p}$ used to form the meta-model are, for example, means and tolerances written as

$$\mathbf{p}^T = [\boldsymbol{\mu}^T \mathbf{tol}^T] = [\mu_1\ \mu_2\ \cdots\ \mu_m\ tol_1\ tol_2\ \cdots\ tol_n] \tag{27}$$

with the design space set out as $\mu_i \in [lsl_i, usl_i]$ for $i = 1, 2,...m$ and $tol_i \in [LSL_i, USL_i]$ for $i = 1, 2,...n$. (One could augment, or replace $\mathbf{p}$, with noise parameters $\mathbf{q}$ and $\mathbf{r}$.) Then, for the j^{th} sample of the parameters (i.e. $\mathbf{p}_j$) the corresponding input data vector, based on a selected polynomial fit, becomes

$$\mathbf{d}(\mathbf{p}_j)^T = \begin{bmatrix} 1 & \mathbf{p}_j^T & f(\mathbf{p}_j^T) \end{bmatrix}_{1\times q} \tag{28}$$

where we have allowed for higher-order terms. The vector length q depends on the order of the polynomial and the sizes of m and n. Let us treat the input data vectors as training data (each stored in the data matrix) and use them in turn to generate the training reliability curves (to be stored in the output matrix). We take our lead from design of experiments (DOE) and Latin Hypercube sampling in particular, and simply select δ samples. The resulting input training matrix becomes

$$\mathbf{D} = \begin{bmatrix} \left[\mathbf{d}(\mathbf{p}_1)\right]^T \\ \left[\mathbf{d}(\mathbf{p}_2)\right]^T \\ \vdots \\ \left[\mathbf{d}(\mathbf{p}_\delta)\right]^T \end{bmatrix}_{\delta\times q} . \tag{29}$$

To generate the output matrix, we invoke the mechanistic model along with the random and stochastic information to generate the reliability curves denoted as $R(t, \mathbf{p}_j)$. Then for discrete time, denoted as $\mathbf{t} = [t_1, t_2, \ldots, t_L]$, we store the discrete values in a corresponding vector $\boldsymbol{R}_j$. For all δ experiments, the output matrix has the structure

$$\bar{\mathbf{R}} = \begin{bmatrix} [\mathbf{R}^T(\mathbf{p}_1)] \\ [\mathbf{R}^T(\mathbf{p}_2)] \\ \vdots \\ [\mathbf{R}^T(\mathbf{p}_\delta)] \end{bmatrix}_{\delta\times L} . \tag{30}$$

4.2 A moving least-squares meta-model

The ubiquitous Kriging meta-model can be found in a variety of places [13-17]; however, the moving least squares meta-model [18, 19] is less well known and is thus outlined next. The idea is to relate the two matrices $\mathbf{D}$ and $\bar{\mathbf{R}}$. Consider the arbitrary input set of parameters $\tilde{\mathbf{p}}$, then a weight matrix $\mathbf{W}(\tilde{\mathbf{p}})$ is required that effectively selects the so-called nearby data sets in $\mathbf{D}$ and $\bar{\mathbf{R}}$. One format of the matrix is simply [18, 19, 22].

$$\mathbf{W}(\tilde{\mathbf{p}}) = diag\begin{bmatrix} \sqrt{w(\tilde{\mathbf{p}}-\mathbf{p}_1)}\ \sqrt{w(\tilde{\mathbf{p}}-\mathbf{p}_2)} \\ \ldots\sqrt{w(\tilde{\mathbf{p}}-\mathbf{p}_\delta)} \end{bmatrix}_{\delta\times\delta} \tag{31}$$

wherein each term has the regularized formulation

$$w(\tilde{\mathbf{p}}-\mathbf{p}_i) = \frac{\hat{w}(\tilde{\mathbf{p}}-\mathbf{p}_i)}{\sum_{j=1}^{n}\hat{w}(\tilde{\mathbf{p}}-\mathbf{p}_j)}$$

with

$$\hat{w}(\tilde{\mathbf{p}}-\mathbf{p}_i) = \frac{\left\{\left(\frac{\|\tilde{\mathbf{p}}-\mathbf{p}_i\|_2}{\hat{r}}\right)^2 + \Xi\right\}^{-2} - (1+\Xi)^{-2}}{\Xi^{-2} - (1+\Xi)^{-2}} . \tag{32}$$

Note herein $\hat{r} = \sqrt{\sum_i (p_i^{\max} - p_i^{\min})^2}$ and $\Xi = 10^{-5}$.

A new input matrix $\mathbf{W}(\tilde{\mathbf{p}})\mathbf{D}$ is formed and related to the new output matrix $\mathbf{W}(\tilde{\mathbf{p}})\bar{\mathbf{R}}$. For a least-squares solution, the normal equations [37] become

$$\mathbf{D}^T\bar{\mathbf{W}}(\tilde{\mathbf{p}})\mathbf{D}\boldsymbol{\Theta}(\tilde{\mathbf{p}}) = \mathbf{D}^T\bar{\mathbf{W}}(\tilde{\mathbf{p}})\bar{\mathbf{R}} \tag{33}$$

where

$$\begin{aligned}\bar{\mathbf{W}}(\tilde{\mathbf{p}}) &= \mathbf{W}^T(\tilde{\mathbf{p}})\mathbf{W}(\tilde{\mathbf{p}}) \\ &= diag\left[w(\tilde{\mathbf{p}}-\mathbf{p}_1)\ w(\tilde{\mathbf{p}}-\mathbf{p}_2)\ \ldots w(\tilde{\mathbf{p}}-\mathbf{p}_\delta)\right]_{\delta\times\delta} .\end{aligned} \tag{34}$$

A solution to Eq. (33) produces the matrix

$$\mathbf{\Theta}(\tilde{\mathbf{p}}) = \left[\mathbf{\theta}_1(\tilde{\mathbf{p}}) \quad \mathbf{\theta}_2(\tilde{\mathbf{p}}) \quad \cdots \quad \mathbf{\theta}_L(\tilde{\mathbf{p}})\right]_{q\times L} . \tag{35}$$

Other approaches, including orthogonal methods [38] may be used to solve the least-squares problem. Finally, an approximation of the reliability curve (i.e. vector) for $\tilde{\mathbf{p}}^T$ is

$$\tilde{\mathbf{R}}^T = \mathbf{d}(\tilde{\mathbf{p}})^T \mathbf{\Theta}(\tilde{\mathbf{p}}) . \tag{36}$$

Note that the k^{th} element of the reliability vector requires only the k^{th} column of the weight matrix, hence

$$\tilde{R}_k = \mathbf{d}(\tilde{\mathbf{p}})^T \mathbf{\theta}_k(\tilde{\mathbf{p}}) . \tag{37}$$

4.3 Error analysis

Errors in the meta-models arise from the following sources: The first source is the number of time instances; that is, the size of Δt used in capturing the time histories of the output function. This number can be increased until improvements cease. The second source is the number of training excitation functions (i.e. δ) chosen. There are several ways to determine this number: the simplest is to use the rule-of-thumb that says multiply the number of parameters (or inputs) by a convenient factor (e.g. ten or twenty) and then add a small contingency factor. Also, the *leave-one–out* method is popular [25].

5. Case studies

The meta-model approach is applied to two systems. In the first system, a parallel Daniels structural system is studied to determine the applicability of the meta-model for modelling the reliability with the stochastic noise. In the second case study, a servo system with series failures is designed using a meta-model of the reliability: optimum means and tolerances with respect to the design space are determined.

5.1 Parallel system

A simple Daniels system with a stochastic load $P(t)$ applied to a beam supported by two rods (i.e. components) is shown in Fig. 1. The support rods are denoted as rod 1 and rod 2, each with widths a_1 and a_2 respectively and depths b_1 and b_2 respectively. The dimensions and material properties are comparable with those from Ref. [39].

The random variables and processes are given in Table 1. Note the yield strengths of the two rods, denoted as σ_1 and σ_2, are fixed random variables. The stochastic load $P(t)$ is Gaussian with the autocorrelation $\rho(t_1, t_2) = \exp[-(t_2 - t_1)^2/\lambda^2]$ and the corrosion of the rods, denoted as $C_1(t)$ and $C_2(t)$, follow Gamma processes. The rod yield strengths are of the larger-is-best type and thus the limit-state functions become

Table 1. Parameters and distributions for the Daniels system.

Variable	Distribution	Mean	Variability
a_1 - V_1	Normal	μ_{a1} in.	σ_{a1}
a_2 - V_2	Normal	μ_{a2} in.	σ_{a2}
b_1 - V_3	Normal	μ_{b1} in.	σ_{b1}
b_2 - V_4	Normal	μ_{b2} in.	σ_{b1}
σ_1 - V_5	Normal	36 kpsi	$\sigma_{\sigma1}$ = 0.36 kpsi
σ_2 – V_6	Normal	36 kpsi	$\sigma_{\sigma2}$ = 0.36 kpsi
$P(t)$ - $Y_I(t)$	Gaussian	$E[Y_1(t)]$	$\sigma[Y_1(t)]$
C_1(t)	Gamma	$E[C_1(t)]$	CV_1
C_2(t)	Gamma	$E[C_1(t)]$	CV_2

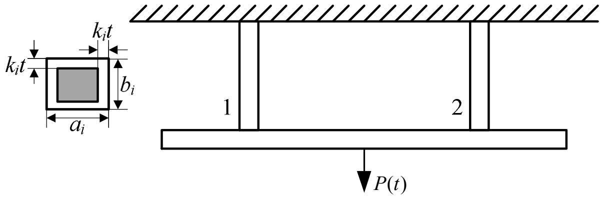

Fig. 1. Daniels system with applied load and two support rods under corrosion.

$$\begin{aligned} g_1(\mathbf{V}, \mathbf{W}(t), t) &= \left(V_1 - 2C_1(t)\right)\left(V_3 - 2C_1(t)\right)V_5 - Y_1(t)/2 \\ g_2(\mathbf{V}, \mathbf{W}(t), t) &= \left(V_2 - 2C_2(t)\right)\left(V_4 - 2C_2(t)\right)V_6 - Y_1(t)/2 . \end{aligned} \tag{38}$$

The life-time for the study is 10 years and the time increments are Δt = 0.1 year.

Study P-A: We let the Gaussian process that models the load have values $E[Y_1(t)]$ = 85 kpsi, $\sigma[Y_1(t)]$ = 8 kpsi and λ = 0.5. The Gaussian process is converted to the EOLE model requiring 33 eigenvalues. A few profiles of sample loads are given in Fig. 2. For the Gamma stochastic degradation processes, the parameters for the Gamma processes are chosen to match-up the initial time and life-time values of a basic linear model. Herein, we let the mean and coefficient of variation of $C_1(t)$ at 10 years be $5\text{x}10^{-3}$ in. and 0.57735, respectively, and the mean and coefficient of variation of $C_2(t)$ at 10 years be $3\text{x}10^{-3}$ in. and 0.57735, respectively. The Gamma distribution parameters are obtained from Eqs. (21) and (22) and provide degradation as shown in Fig. 3.

Let the deign parameters comprise the means of the widths and depths of the two rods so $\mathbf{p}^T = [\mu_{a1}, \mu_{a2}, \mu_{b1}, \mu_{b2}]$. The design space comprises their lower and upper values as given in Table 2. In order to develop the most efficient *reliability* meta-model we need to determine the minimum number of training sets for the four design parameters and the maximum time interval over life-time. As a rule of thumb we chose 20 times the number of design parameters (i.e. 80 here) and keep the time intervals of 0.1 years. The training reliability curves from MCS that correspond to the design space are shown in Fig. 4.

Table 2. Upper and lower specification limits (design space) for meta-model construction.

Design parameters	μ [in.]	
	lsl	*usl*
μ_{a1}	1.17	1.43
μ_{a2}	1.17	1.43
μ_{b1}	1.08	1.32
μ_{b2}	1.08	1.32

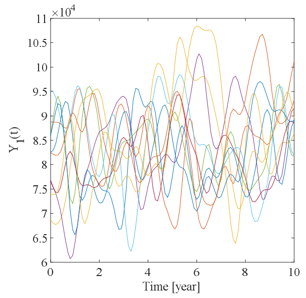

Fig. 2. Ten sampled stochastic load profiles of $Y_1(t)$.

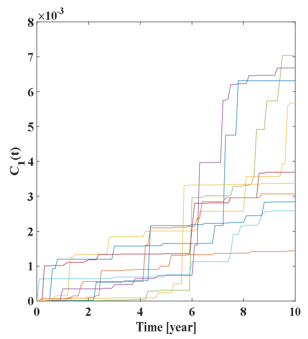

Fig. 3. Ten sampled degradation profiles of $C_1(t)$.

Two different meta-models are built for comparison purposes: these include a quadratic moving least-squares (qMLS) model and a Kriging model. For the least-squares approach, the appropriate form of each input vector $\mathbf{d}(\mathbf{p}_i)^T$ is used to provide the data for the input matrix D. Further, the influence radius in Eq. (32) is 0.5004. Hereinafter, for a Kriging model a linear global function (linear regression model) has been used and along with off-the-shelf software [17]. The accuracies of the two meta-models are checked using the arbitrary set of input parameters $\mathbf{p}_o$ = [1.365, 1.235, 1.14, 1.14]. The system reliability results are compared in Fig. 5. A time-variant reliability index, $\beta(t) = \Phi^{-1}(R(t))$ where $R(t)$ is reliability and $\Phi^{-1}(\,)$ is a standard normal inverse cumulative distribution function is used to analyze the accuracy of the meta-model for reliability [40]. The percentage errors of the two meta-model approaches based on time-variant reliability indices are shown in Fig. 6. Fig. 6 shows mean, minimum, and maximum percentage errors over time for 20 test points which are randomly sampled sets of parameters from design spaces. Both mean and maximum percentage errors for qMLS meta-model are smaller than ones for Kriging meta-model. And the average percentage errors for the qMLS and Kriging meta-models are 2.25 % are 4.68 %, respectively. However, the times to evaluate reliability (over the life-time) are quite different: the qMLS meta-model takes 0.02388 s, the Kriging model takes 9.727 s while the time using Monte Carlo simulation method with N = 100000 is 189.23 sec. Finally, it is noted that Kriging tends to exhibit a more "noisy" reliability profile over the life-time (versus qMLS) and hence this may affect any derivative calculations of the reliability.

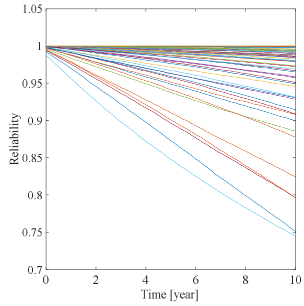

Fig. 4. Reliability (i.e. output) training data (80 samples).

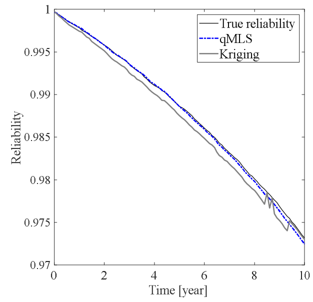

Fig. 5. Comparison of reliability for meta-models.

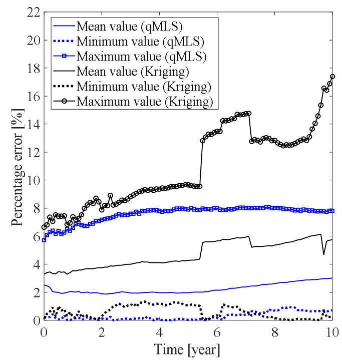

Fig. 6. Comparison of percentage error for 20 test points (study P-A).

Study P-B: In this study, we fix the dimensions and yield strengths of the rods and provide a noise space through some variability of the stochastic process parameters. The means and standard deviations of the rod dimensions are given in Table 3. The noise parameters are taken from both the Gaussian and Gamma processes and more specifically $[\mathbf{q}^T\mathbf{r}^T] = [E[Y_1(t), \sigma[Y_1(t), E[C_1(t), E[C_2(t)]$. The so-called noise

Table 3. Parameters and distributions of the rods for the Daniels system.

Variable	Distribution	Mean	St. Dev.
a_1 - V_1	Normal	1.3 in.	0.01
a_2 - V_2	Normal	1.3 in.	0.05
b_1 - V_3	Normal	1.2 in.	0.01
b_2 - V_4	Normal	1.2 in.	0.05
σ_1 - V_5	Normal	36 kpsi	0.36
σ_2 - V_6	Normal	36 kpsi	0.36

Table 4. Upper and lower limits for meta-model construction.

Parameters for stochastic processes	*lsl*	*usl*
$E[Y_1(t)]$	76500	93500
$\sigma[Y_1(t)]$	7200	8800
$E[C_1(t = 10)]$	0.0045	0.0055
$E[C_2(t = 10)]$	0.0027	0.0033

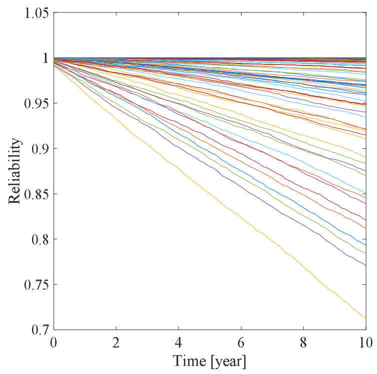

Fig. 7. Reliability (i.e. output) training data (80 samples).

space is given by the variability of these parameters and the four ranges are shown in Table 4. For the four noise parameters we again assemble 80 training sets and use time intervals of 0.1 years. The training reliability curves from MCS are shown in Fig. 7.

The accuracy of our two new meta-models are checked using an arbitrary set of noise parameters, where in particular, $[E[Y_1(t)], \sigma[Y_1(t)], E[C_1(t = 10)], E[C_2(t = 10)]] = [89250, 8400, 0.0053, 0.0032]$. The system reliability results are compared in Fig. 8 and errors are given in Fig. 9. Fig. 9 shows mean, minimum, and maximum percentage errors over time for 20 test points. Again, the accuracy of the qMLS and Kriging meta-models are comparable: the average percentage errors are 4.06 % for qMLS, and 6.55 % for Kriging. And both mean and maximum percentage errors for qMLS meta-model are smaller than ones for Kriging meta-model. However, as in study P-A, the qMLS meta-model evaluates the life-time reliability 400 times faster than Kriging and about 8000 times faster than the MCS procedure. Again, we see the more noisy reliability profile from Kriging.

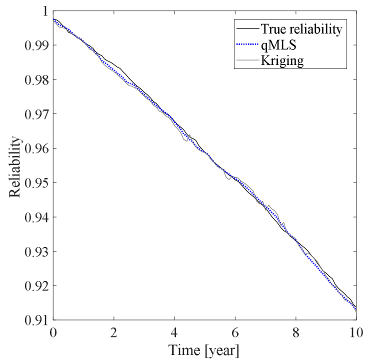

Fig. 8. Comparison of reliability for meta-models.

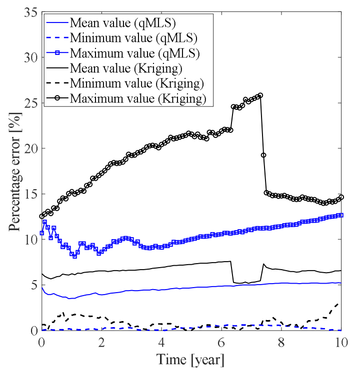

Fig. 9. Comparison of percentage error for 20 test points (study P-B).

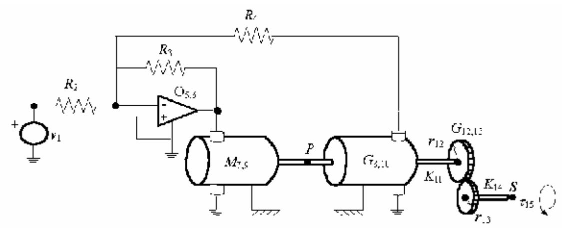

Fig. 10. Electro-mechanical servo system.

5.2 A series system: Servo actuator

The servo system of interest is shown in Fig. 10 and both the component models and interconnection model can be found in more detail in Savage and Carr [41]. The motor and tacho-generator pair are shown as $M_{7,9}$ and $G_{8,10}$, respectfully. Herein the motor and the tacho-generator are identical devices, just interconnected differently to provide the required functions. Other subsystems include the difference amplifier, comprising the three resistances R_2, R_3 and R_4 and an operational amplifier (denoted as $O_{5,6}$) which has an open-loop gain A. The gear train denoted as $G_{12,13}$ has gear ratio $r = r_{12}/r_{13}$. The source and load subsystems are modelled respectfully as follows: A voltage supply v_1 acts as the power source and an applied torque τ_{15} models the load arising from some connected subsystem at the output shaft (Note: The rotational spring constants K_{11} and K_{14} are not factors in the steady-state performance of the servo-system and their models are not

Table 5. Responses, performance metrics and specifications.

Response	Metric	Specifications
Z_1 (t_c)	Smaller-is-best	USL_1 = 0.045 sec
Z_2 (ω_{SS})	Target-is-best	LSL_2 = 551, T = 570 rad/sec, USL_2 = 589
Z_3 (τ_o)	Larger-is-best	LSL_3 = 0.22 N-m

needed herein).

The three performance measures of interest are:

1. The time constant t_c: This term is related directly to the time for the shaft speed, measured at point *S* with respect to a stationary reference frame, to reach steady-state angular velocity,
2. The steady-state shaft speed ω_{SS}, and
3. The initial, or starting torque τ_o supplied to the load at shaft *S*.

The mechanistic models for the three performance measures, in terms of the electro-mechanical parameters with the op-amp gain *A* two-orders of magnitude larger than the resistances, are respectively

$$t_c = \frac{4R_m(R_m + R_4)J}{\kappa^2(2R_m + R_4 + R_3)} \tag{39}$$

$$\omega_{SS} = \frac{rR_3(R_m + R_4)}{\kappa R_2(2R_m + R_4 + R_3)}v_1 - \frac{r^2 R_m(R_m + R_4)}{\kappa^2(2R_m + R_4 + R_3)}\tau_{15} \tag{40}$$

$$\tau_o = \frac{\kappa R_3}{rR_mR_2}v_1 \tag{41}$$

where for the motor and generator, κ denotes the torque constant, *J* is the rotational inertia and R_m is the winding resistance. (The rotational friction and winding inductance are considered to be negligible.)

The design specifications (i.e. performance metrics) for the three performance measures are given in Table 5. Based on Table 5, the four limit-state functions using the three performance measures and four limit specifications are symbolically

$$\begin{aligned} g_1 &= 0.045 - z_1 \\ g_2 &= 589 - z_2 \\ g_3 &= z_2 - 551 \\ g_4 &= z_3 - 0.22 \end{aligned} \tag{42}$$

System uncertainties:

In order to assign, but limit, the randomness in the model we reason as follows. The supply voltage (i.e. v_1 in the schematic) is obtained from a known power supply but may be uncertain owing to manufacturing abilities or the controller requirement. The load torque, (i.e. τ_{15} in the schematic), is known but again may be uncertain owing to the particular end-use. Thus, v_1 is modelled as a random variable with the parameters given in Table 6 and the load torque is modelled as a Gaussian stochastic process using the parameters given in Table 6. The conversion to EOLE requires 12 singular values and provides time-values similar to those in Fig. 2.

Table 6. Specifications for the system variables and stochastic processes.

Variable/process	Distribution	Mean	CV
V_1 (κ)	Normal	μ_1	σ_1/μ_1
V_2 (R_m)	Normal	2.9	0.0067
V_3 (T)	Uniform	335.5 K	[298, 373] K
V_4 (RH)	Uniform	60 %	[30, 90] %
V_5 (v_1)	Normal	12 Volt	0.0033
V_6 (r)	Deterministic	η	-
$C_1(t)$ for $\kappa(t)$	Gamma	0.01 (t = 10)	0.3
$C_2(t)$ for $T(t)$	Gamma	1.75×10^{-5} (t = 10)	0.3
$C_3(t)$ for $RH(t)$	Gamma	2.25×10^{-6} (t = 10)	0.3
$Y(t)$ (τ_{15})	Gaussian	0.01	0.2

To assign variability to any of the eight electrical and mechanical parameters, a sensitivity analysis has been applied with respect to the three performance measures. The results tell us that the most important parameter is the torque constant κ followed by the motor resistance R_m and the gear ratio *r*. The remaining four variables are well down the importance order and thus are fixed at nominal, deterministic, values. More specifically, the rotor inertia *J* (of both the motor and tacho-generator) is set to 1/1000000 $kg\text{-}m^2$. For the difference amplifier, the three resistors R_2, R_3 and R_4 are set at 10k Ω, 40k Ω and 10k Ω respectively and the op-amp gain *A* is set to $5\text{x}10^6$.

Nominal values for κ, R_m and *r*:

The three initial nominal values for κ, R_m and *r* are obtained by minimizing a single, deterministic, system loss function. Target values for the three performance measures and a way to normalize their different units are required. The target for the shaft speed is the natural target value of 570 rad/sec. For the time constant response and the torque response, which have no target value, arbitrary targets are set at values that are 10 % in the favourable direction from the respective limit specification. Each term is normalized using the target-to-limit distance. Then we minimize

$$L(\kappa, R_m, r) = \left(\frac{(t_c - 0.04)}{0.005}\right)^2 + \left(\frac{(\omega_{SS} - 570)}{19}\right)^2 + \left(\frac{(\tau_o - 0.24)}{0.02}\right)^2 \tag{43}$$

where the performance measures, in terms of the design variables, come from Eqs. (39)-(41). We get the initial values as [κ, R_m, *r*] = [7.38/1000 *N-m/A*, 2.9 Ω, 0.455 *m/m*].

Component degradation modelling:

Many DC (direct current) motors use permanent magnets to provide the requisite magnetic flux. This type of motor has good operational characteristics and low cost; however, permanent magnets have a tendency to lose some of their magnetic strength due to both over-use and operating conditions. This reduction in magnetic field strength causes a corresponding reduction in torque output since the so-called torque constant degrades. For the degrading torque constant, the degradation is often written in the random path form, specifically $\kappa(t) = \kappa_0(1 - d \times t)$ where κ_0 is the initial torque-constant value and d is a known degradation rate. For this study we let the degradation be modelled by a stochastic process. Thus, we have

$$X_1(t) = V_1(1 - C_1(t)) \tag{44}$$

where $X_1(t)$ is the torque constant at time t, V_1 is the uncertainty at initial time and $C_1(t)$ is a Gamma process that models the temporal degradation. Herein, we set $C_1(t = 0) = 0$, and choose both $E[C(t = t_L)]$ and $\mathrm{Var}[C(t = t_L)]$ (hence $\mathrm{CV}[C(t = t_L)]$), then use Eqs. (21) and (22) to evaluate the Gamma distribution parameters. The degradation paths are similar in character to those in Fig. 3.

The armature winding resistance R_m increases over time depending on the heat and humidity values of the operating environment [42]. The winding resistance, from Ref. [43], is typically modelled as $R_m(t) = R_0 \exp\left[(\alpha_1 T + \alpha_2 RH)\ t\right]$ where R_0 is the initial resistance value and α_1, α_2 are degradation rates for temperature T and humidity RH. However, with stochastic degradation models, the armature winding resistance, becomes

$$X_2(t) = V_2 \exp\left[(V_3 C_2(t) + V_4 C_3(t)\right] \tag{45}$$

where $X_2(t)$ is the resistance at time t, V_2 is the uncertain resistance at initial time, V_3 is the uncertain temperature in Kelvin and V_4 is the uncertain relative humidity in percentage. $C_2(t)$ and $C_3(t)$ are Gamma degradation processes. The parameters for the random variables and stochastic processes in the torque constant and armature resistance models are given in Table 6. The autocorrelation for $Y(t)$ is $\rho(t_1, t_2) = \exp[-(t_2 - t_1)^2/\lambda^2]$ with $\lambda = 1$ year.

The meta-model:

For design purposes, the torque constant and gear ratio are selected as the design variables and then the mean and tolerance of the torque constant and the nominal gear ratio become the design parameters so $\mathbf{p}^T = [\mu_1, tol_1, \eta]$ where tol_1 is the statistical tolerance $3\sigma_1$. A sensitivity analysis using initial reliability $R(t = 0)$ and entries in $\mathbf{p}$ shows a very sensitivity system and thus to ensure a realistic minimum $R(t = 0)$ the design space in Table 7 is allotted. The meta-model is built using 60 training sets from the design space, evaluated at time instances obtained from a life-time of 10 years and time increments of $\Delta t = 0.1$ year. Corresponding system reliability curves, generated by MCS with N = 100000 samples, are shown in Fig. 11. The quite broad range of curves shows how sensitivity the reliability is to the design parameters.

As a test of the efficacy of the meta-model, the system reliability is obtained for arbitrary values $[\mu_1, tol_1, \eta] = [0.0074, 1.5, 0.45]$. The results from the meta-models are compared and shown in Fig. 12 and the various percentage errors for 20

Table 7. Upper and lower specification limits for design parameters.

Design parameter	*lsl*	*usl*
μ_1	7.3200×10^{-3}	7.4600×10^{-3}
tol_1	$1\ \%\mu_1$	$1.6\ \%\mu_1$
η	0.448	0.462

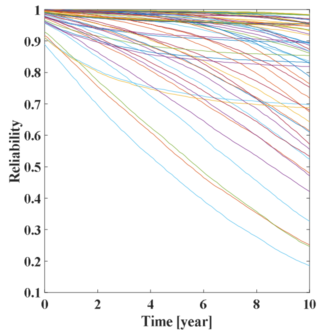

Fig. 11. Reliability curves (i.e. output) from training data (60 samples).

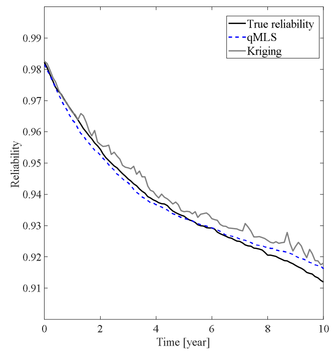

Fig. 12. Comparison of reliability for meta-models.

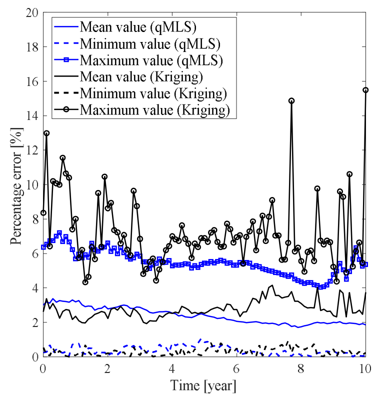

Fig. 13. Comparison of percentage error for 20 test points (servo actuator).

Table 8. Upper and lower specification limits for design parameters.

Parameters and cost [\$]	Initial design	Design for life-time
μ_1	7.3200×10^{-3}	7.4493×10^{-3}
tol_1	1.5000	1.2892
η	0.4500	0.4568
C_P	4.0000	4.0817
C_{LQ}^E	3.0723	0.2725
C_T	7.0723	4.3542

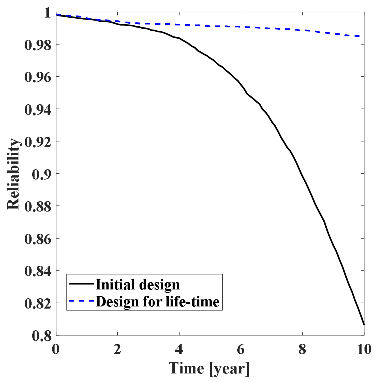

Fig. 14. Reliability curves for initial and optimum designs (design for life-time).

test points are shown in Fig. 13. At most of times, both mean and maximum percentage errors for qMLS meta-model are smaller than ones for Kriging meta-model. And overall the average errors are 2.40 % for the qMLS meta-model and 2.28 % for Kriging. Also, in this example, we see the more noisy reliability profile from Kriging.

Design applications:

The meta-model is now used for design purposes. In order to show the flexibility of the methodology presented herein, three optimum designs are found using two different optimization algorithms [44, 45].

Design for life-time

This problem is expressed in terms of the total cost objective (which is a function of the reliability) comprising production cost, scrap cost and loss of quality cost. In this design there are no specific reliability constraints. We write the *Design for Life-Time* problem as

$$\begin{aligned} &\text{Minimize } C_p(\mathbf{p}) + c_S(1 - R(\mathbf{p}, t_0)) + C_{LQ}^E(\mathbf{p}, \theta, c_F) \\ &\quad\text{subject to} \\ &\quad\mathbf{p}_L \le \mathbf{p} \le \mathbf{p}_U \end{aligned} \tag{46}$$

where the production cost $C_p(\mathbf{p}) = 3.5 + 0.75/tol_1$, and

$$C_{LQ}^E(\mathbf{p}, \theta, c_F) = c_F \sum_{l=1}^{L} \left(\left(R(\mathbf{p}, t_{l-1}) - R(\mathbf{p}, t_l) \right) e^{-\theta t} \right)$$

For the second term (i.e. scrap cost) we set c_S = \$20, and for the third term (i.e. loss of quality cost or opportunity costs [46]) we set c_F = \$20 and θ = 3 % in Eq. (46). The optimum results are shown in Table 8 along with the reliability curves in Fig. 14. The reduction in total cost and increase in overall system reliability is substantial. The optimization design is reached in 133 iterations with the meta-models and 390 iterations using traditional MCS.

Table 9. Design results for design for dependability.

Parameters and cost [\$]	Initial design	Design for dependability	
		Case (a)	Case (b)
μ_1	7.3200×10^{-3}	7.3907×10^{-3}	7.4179×10^{-3}
tol_1	1.500	1.4638	1.3395
η	0.4500	0.454	0.4547
C_P	4.000	4.0124	4.0599
C_{LQ}^E	1.6131	0.2563	0.1553
C_T	5.6131	4.2687	4.2152

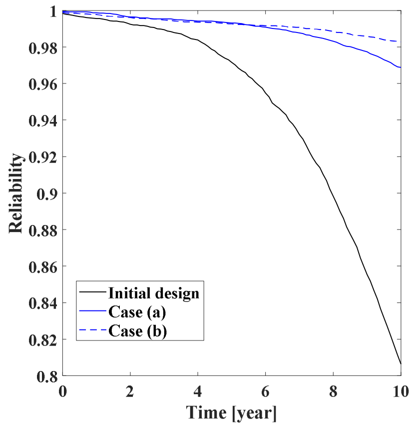

Fig. 15. Reliability curves for initial and optimum designs (design for dependability).

Design for dependability

This problem also uses the total cost as the objective but now specific reliability constraints are introduced at initial time and some later time. We write the *Design for Dependability problem* as

$$\begin{aligned} &\text{Minimize } C_p(\mathbf{p}) + c_S(1 - R(\mathbf{p}, t_0)) + C_{LQ}^E(\mathbf{p}, \theta, c_F) \\ &\quad\text{subject to} \\ &\quad R(\mathbf{p}, t_0) \ge R_0, R(\mathbf{p}, t_M) \ge R_M \\ &\quad\mathbf{p}_L \le \mathbf{p} \le \mathbf{p}_U \end{aligned} \tag{47}$$

For the scrap and loss of loss of quality costs, we set c_s = \$20, c_F = \$10 and θ = 3 % in Eq. (47). For the reliability constraints we consider two cases:

Case (a): R_0 = 0.999, and R_M = 0.95 at t_L = 10 year,

Case (b): R_0 = 0.999, and R_M = 0.99 at t_M = 7 year.

The optimum results are shown in Table 9 and the accompanying reliability curves are shown in Fig. 15. The optimum design for case (a) has been reached in 164 iterations using the meta-model (to provide reliability) and 4205 iterations with the traditional MCS. For case (b), the optimum design has been reached in 105 iterations using the meta-model; however, no solution was reached after 7000 iterations with the traditional MCS.

Discussion:

We note that the single reliability meta-model has been used for three design scenarios. In essence the investment of 60 MCS to train the reliability meta-model has obviated the need to perform thousands of lengthy MCS in the traditional two-loop optimization algorithms. Further, the more consistent gradients in the meta-model provide for a more assured convergent to an optimal solution.

6. Conclusions

In this paper we have presented an efficient, two-stage, methodology for RBDO of engineering systems with time-dependent system reliability. The time-dependence is caused by stochastic loads and stochastic degradations of structural and material characteristics. The traditional time-variant reliability calculation is essentially uncoupled from the optimization process via a meta-model that gives reliability in terms of the selected design parameters and their design space. Then, integrated design to minimize the objective function while meeting reliability constraints becomes trivial. The robustness of the methodology is shown by augmenting the usual design space with a noise space comprising uncertainty in the parameters and stochastic processes modelling both loads and degradation.

To form the meta-model, training samples of the design parameters (and noise parameters if required) are selected and stored in an input matrix. Then, they are used, along with the mechanistic model, to generate corresponding vectors from time-sampled reliability to form up the output matrix. The Gaussian process (used for excitations) and the Gamma process (used for the degradation models) are straightforward to program via MCS. The two matrices have been linked by two meta-model approaches: Moving least-squares and Kriging. The errors and their control are based primarily on the number of training sets and the length of the time increment.

The moving least squares meta-model has been found to be much faster and as accurate as the ubiquitous Kriging model. The accuracy of the moving least squares meta model arises from the consistent shape of the family of reliability curves and the use of a regularized formula for choosing the weights that determine the so-called nearby samples

The case-studies have pointed out the efficacy of the meta-model approach. Herein, the meta-model produces the reliability curve in a fraction of the time needed by the traditional MCS: this has led to the shortening of the optimization time by several orders of magnitude while retaining reliability accuracy in the order of a fraction of a percentage. For more complex and/or implicit problems, it is expected that the methodology would show increased computational leverage.

Acknowledgments

The research was supported by a grant from 2018 Subsidy for overseas dispatch research of Andong National University.

Nomenclature

p	: Vector of design parameters
q	: Vector of Gaussian noise parameters
r	: Vector of Gamma noise parameters
V	: Vector of random variables
U	: Standard normal space variables
t	: Time
t_L	: Life-time
$Y(t)$	: Gaussian process
$C(t)$	: Gamma process
$W(t)$	: Stochastic process
$F(t)$	: Cumulative distribution function (*cdf*)
$R(t)$	: Reliability function
R_M	: Reliability policies at time M
C	: Cost
$g_i(\mathbf{V}, \mathbf{W}(t_l), t_l)$	: i^{th} time-variant limit-state function
A_l	: System failure region
$\mathbf{B}_l$	: Incremental system failure region
RBDO	: Reliability-based design optimization
FORM	: First-order reliability method
MLFP	: Most-likely failure point
MCS	: Monte-Carlo simulation

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Appendix

Let the Gaussian process, with the discretization of time given in Sec. 2, have the mean function $\mu_Y(t_i)$, the standard deviation function $\sigma_Y(t_i)$ and the autocorrelation function $\rho_Y(t_i,t_j)$. The symmetric Covariance matrix (denoted as $\boldsymbol{\Sigma}$), has elements of the form $Cov(t_i,t_j)=\sigma_Y(t_i)\sigma_Y(t_j)\rho_Y(t_i,t_j)$, where i = 0, 1,… L and j = 0, 1,… L. An eigenvalue solution of $\boldsymbol{\Sigma}_{N\times N}$ provides its eigenvectors and eigenvalues; however, for s retained significant eigenvalues, we have the reduced matrix of orthonormal vectors $\boldsymbol{\Phi} = [\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2,\ldots, \boldsymbol{\varphi}_s]_{N\times s}$ and the reduced matrix of eigenvalues $\boldsymbol{\Lambda} = [\lambda_1, \lambda_2,\ldots \lambda_s]_{s\times s}$. Then the approximating symmetric covariance matrix is $\Sigma \approx \boldsymbol{\Phi}\mathbf{I}\boldsymbol{\Phi}^T$. There are various ways to determine the number of retained eigenvalues and eigenvectors (i.e. s) for sufficient accuracy [47].

Of importance, Li [36] defines the time-related vector (containing standard deviation and correlation parameters)

$$\boldsymbol{\Sigma}(t)=[\sigma_Y(t)\rho_Y(t,t_0)\sigma_Y(t_0)\cdots\sigma_Y(t)\rho_Y(t,t_L)\sigma_Y(t_L)]^T \qquad (A.1)$$

and provides a companion vector of standard normal variables $\mathbf{U} = [U_1, U_2, \ldots U_s]^T$. Finally, the expansion of the process is written as

$$Y(t)=\mu_Y(t)+[\boldsymbol{\Sigma}(t)]^T\boldsymbol{\Sigma}^{-1}\sum_{i=1}^{r}\sqrt{I_i}U_i\boldsymbol{\varphi}_i \qquad (A.2)$$

If we define the scalar function $\varphi_i(t)=[\boldsymbol{\Sigma}(t)]^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\varphi}_i$, provide a new matrix, $\tilde{\boldsymbol{\Lambda}}=diag[1/\sqrt{\lambda_1}\ 1/\sqrt{\lambda_2}\cdots1/\sqrt{\lambda_r}]_{r\times r}$, and use the eigenvalue-eigenvector relation $\Sigma\boldsymbol{\Phi} = \boldsymbol{\Phi}\mathbf{I}$, then Eq. (A.2) takes the compact matrix form

$$Y(t)=\mu_Y(t)+[\boldsymbol{\Sigma}(t)]^T(\boldsymbol{\Phi}\tilde{\boldsymbol{\Lambda}})\mathbf{U} \qquad (A.3)$$

where $(\boldsymbol{\Phi}\tilde{\boldsymbol{\Lambda}})$ is a matrix of constants.

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