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Stress and damping of wide cantilever beams under free vibration †

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Abstract

Research has shown that the damping of a vibrating structure is highly dependent on its stress function. In this study, the bending stress and damping of wide cantilever beams under free vibration were analyzed using the classical plate and beam theory. The damping stress equation for cantilever beams under free vibration was derived based on the empirical function of unit dissipating energy, whereas the plate bending equation was derived using the double finite integral transform method. The bending stress and damping ratio results from the beam and the plate theory were compared with simulation results from finite element analysis (FEA) for different length-to-width ratios. Results show that the plate theory displayed a good agreement with FEA results in terms of estimated value and trending curve shape when a significantly large number of terms were used. Using a small number of terms resulted in large errors at high length-to-width ratios, but provided sufficient estimates when the length-to-width ratio dropped below four. It was found that the beam theory was only valid for beams with very high length-to-width ratios or square plates. Beyond this ratio, the beam theory recorded a higher error estimate than the plate theory. Overall, the most accurate stress and damping estimations come from the use of plate theory with a very high number of terms.

Keywords: Damping; Stress; Wide cantilever beam; Plate theory; Beam theory

1. Introduction

The harvesting of vibration energy from an ambient source to replace conventional batteries in low-powered devices has been carefully studied over the past decade [1-5]. Approaches for converting vibration energy to electrical energy generally involve a clamp-free cantilever beam subjected to base excitation motion, although the transduction method may vary between piezoelectric, electromagnetic, and electrostatic. Most of the current literature focuses on design optimization of the harvester to increase the device's efficiency [6-11]. However, very few studies have attempted to explore the effect of damping on the performance of these devices.

In an attempt to generate more power from a vibration energy harvester, Dayou et al. [12, 13] proposed a 'split-width' method, hypothesizing that the damping of a cantilever beam increases when the width of the beam is enlarged. The hypothesis was validated with experimental proof. It was concluded that using a smaller beam width was more beneficial since a lower damping resulted in a higher power output. Likewise, Hosseini and Hamedi [14] also noted a similar trend in their work. The reason as to why damping increases when the beam width increases was not explained in their work. However, it is expected that the increase in damping is due to the increase in bending stress in wider beams.

In an earlier study, Lazan [15] proposed a theory relating the damping of a structure to its maximum bending stress. He then developed an empirical formula that relates the energy-dissipation unit of a structure to its stress amplitude based on the many metals he tested. Applying this relation, Lazan [16], Kume et al. [17], and Gounaris and Anifantis [18] each developed a refined empirical equation relating the maximum bending stress and the fatigue limit stress of a cantilever beam under forced vibration to its loss factor. The slight differences that appeared in their equations are due to the different approach taken by the authors in deriving the refined equations, although these differences are reportedly very small. Gounaris and Anifantis [18] demonstrated that by using their equation, it was possible to analytically determine the damping capacity of beam-like structures through an iterative finite element analysis (FEA) process. As compared to the approach by Lazan [15] and Kume et al. [17] in which the maximum bending stress was obtained experimentally, the approach by Gounaris and Anifantis [18] does not require any experimental input. Since damping is highly related to the stress of a struc-

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ture, it is important to apply the correct stress theory to obtain a more accurate damping estimation.

In this work, we analyzed the accuracy of the classical beam theory (Euler-Bernoulli) and the classical plate theory (Kirchhoff-Love) in predicting the stress and damping of wide cantilever beams under free vibration. Other forms of beam and plate theories were not considered in this work as the aspect ratios that are generally used in vibration energy harvesting applications allow for thin beam approximations [19]. The damping-stress relation of a cantilever beam under free vibration was first derived based on the empirical formulae provided by Lazan [15]. The maximum bending stress amplitude and damping ratio prediction from the beam theory and the plate theory were compared with the results from finite element analysis (FEA). The validity of both theories was outlined and their implications on damping prediction were discussed.

2. Damping stress equation for free vibration

The damping factor of a vibrating system greatly influences its output performance. Assuming hysteretic damping, the loss factor of a structure can be described as Eq. (1) [20].

$$\gamma = \frac{D}{2\pi G}, \tag{1}$$

where γ is the loss factor of the structure, D is the energy loss per unit volume in the structure and G is the total strain-energy per unit volume of the structure. This form of damping is related to the internal material damping of a structure. Assuming proportional damping, the first mode damping ratio of a structure is equal to half of its loss factor [21]. The energy loss term D can be determined from Eq. (2) and the total strain energy term, G, is defined by Eq. (3).

$$D = \int f(\sigma_a)\,dV, \tag{2}$$

$$G = \int \frac{{\sigma_a}^2}{2E}\,dV, \tag{3}$$

where V is the volume the structure, σ_a is the bending stress amplitude experienced by the vibrating structure, E is Young's modulus of the structure and $f(\sigma_a)$ is the unit dissipating energy as a function of stress given by

$$f(\sigma) = 6895\left(\frac{\sigma_a}{\sigma_f}\right)^{2.3} + 41360\left(\frac{\sigma_a}{\sigma_f}\right)^{8}, \tag{4}$$

where σ_f is the fatigue limit stress of the structure. The fatigue limit stress usually known. Note that Eq. (4) represents an empirical formula derived by Lazan [15] from many tested samples. For cases of forced vibration, σ_a refers to the stress amplitude at resonance. For free vibration, σ_a reflects the initial stress applied on the structure to induce vibration. At the given time, the bending stress of the vibrating structure was obtained experimentally. However, it is possible to analytically derive the bending stress of simple structures such as a rectangular beam. For cases of cantilever beams under free vibration, the vibration of the beam is usually induced by an instantaneous load applied at the free-end of the beam. The undamped motion of a beam under this form of vibration can be modelled by

$$z_{rel}(x,t) = \sum_{n=1}^{\infty} \varphi_n(x) \frac{1}{C^2 L} \int_0^L P(x)\varphi_n(x)\,dx \cos(\omega_n t), \tag{5}$$

where $z_{rel}(x,t)$ is the vertical displacement of the cantilever beam at position x along the length of the beam and time t, L is the length of the beam, C is an arbitrary constant, ω_n is the natural frequency of the beam and $\varphi_n(x)$ and $P(x)$ are defined as [22]

$$\varphi_n(x) = C\left[\cosh\frac{\lambda_n}{L}x - \cos\frac{\lambda_n}{L}x - \tau_n\left(\sinh\frac{\lambda_n}{L}x - \sin\frac{\lambda_n}{L}x\right)\right], \tag{6}$$

$$P(x) = \frac{1}{6EI}Px^2(x-3L), \tag{7}$$

where τ_n is a constant, λ_n is a frequency constant, P is the load applied at the free end of the beam and I is the second moment of area of the beam. The subscript n refers to the mode of vibration. Considering the first six modes of vibration at $t=0$, Eq. (5) simplifies to

$$z_{rel}(x,t) = \frac{1}{6EI}PL^3\delta(x) \tag{8}$$

where

$$\begin{aligned}\delta(x) = \frac{1}{C}\big[&-0.970688\varphi_1(x) + 0.024716\varphi_2(x) - \\ &0.003152\varphi_3(x) + 0.000821\varphi_4(x) - \\ &0.000300\varphi_5(x) + 0.000135\varphi_6(x)\big].\end{aligned} \tag{9}$$

By applying the Euler-Bernoulli theory of bending, the bending stress amplitude of a cantilever beam can be described by

$$\sigma_a(x,z) = Ez\frac{\partial^2 z_{rel}(x)}{\partial x^2}, \tag{10}$$

where σ_a is the bending stress amplitude acting on the vibrating cantilever beam at position (x, z) and z is the position along the thickness of the beam relative to the center. Substituting Eqs. (9) and (10) into Eq. (2) and integrating it with respect to the width, w, length, L, and thickness, t, of the beam results in Eq. (11):

$$D = w\int_0^L\int_0^{\frac{h}{2}}\left[6895\left(\frac{zPL^3}{6I\sigma_f}\frac{d^2\delta(x)}{dx^2}\right)^{2.3} + 41360\left(\frac{zPL^3}{6I\sigma_f}\frac{d^2\delta(x)}{dx^2}\right)^{8}\right]dzdx. \quad (11)$$

Solving the first integral with respect to the z direction yields

$$D = w\left\{2089.39\left(\frac{h}{2}\right)^{3.3}\left(\frac{PL^3}{6I\sigma_f}\right)^{2.3}\int_0^L\left[\frac{d^2\delta(x)}{dx^2}\right]^{2.3}dx + 4595.56\left(\frac{h}{2}\right)^{9}\left(\frac{PL^3}{6I\sigma_f}\right)^{8}\int_0^L\left[\frac{d^2\delta(x)}{dx^2}\right]^{8}dx\right\}. \quad (12)$$

Applying the same steps for Eq. (3), the following equation is obtained:

$$G = \frac{w}{2E}\left\{\frac{1}{3}\left(\frac{h}{2}\right)^{3}\left(\frac{PL^3}{6I\sigma_f}\right)^{2}\int_0^L\left[\frac{d^2\delta(x)}{dx^2}\right]^{2}dx\right\}. \quad (13)$$

Substituting Eqs. (12) and (13) into Eq. (1) defines the damping stress equation for the first mode damping ratio of a cantilever beam:

$$\zeta_1 = \frac{E}{2}\left[1620.62\left(\frac{hPL^3}{6I}\right)^{0.3}\sigma_f^{-2.3}A_1 + 68.57\left(\frac{hPL^3}{6I}\right)^{6}\sigma_f^{-8}A_2\right], \quad (14)$$

where

$$A_1 = \int_0^L\left[\frac{d^2\delta(x)}{dx^2}\right]^{2.3}dx\left\{\int_0^L\left[\frac{d^2\delta(x)}{dx^2}\right]^{-2}dx\right\}, \quad (15)$$

and

$$A_2 = \int_0^L\left[\frac{d^2\delta(x)}{dx^2}\right]^{8}dx\left\{\int_0^L\left[\frac{d^2\delta(x)}{dx^2}\right]^{-6}dx\right\}. \quad (16)$$

It is difficult to analytically solve the integral parts in Eqs. (15) and (16) due to the power exponent of each term. Nevertheless, the terms in Eqs. (15) and (16) can be approximated using the following expressions with an error less than 1.7 % for all values.

$$A_1 \approx 1.1167\left(\frac{3}{L}\right)^{0.6}, \quad (17)$$

$$A_2 \approx 21.052\left(\frac{3}{L}\right)^{0.6}. \quad (18)$$

Substituting Eqs. (15) and (16) into Eq. (12) results in

$$\zeta_1 = \frac{E}{2}\left[1809.75\left(\frac{hPL}{2I}\right)^{0.3}\sigma_f^{-2.3} + 1443.54\left(\frac{hPL}{2I}\right)^{6}\sigma_f^{-8}\right]. \quad (19)$$

Notice that the term in the rounded brackets in Eq. (19) is actually equal to the solution of Eq. (10) for $x = 0$ and $y = h/2$. This position corresponds to the top and bottom surface of the vibrating beam at the clamped end, which is also the location where the beam experiences its maximum bending stress during vibrations. Hence, Eq. (19) can also be defined as

$$\zeta_1 = \frac{E}{2}\left[1809.75\frac{\sigma_m^{0.3}}{\sigma_f^{2.3}} + 1443.54\frac{\sigma_m^{6}}{\sigma_f^{8}}\right], \quad (20)$$

where σ_m is the maximum bending stress located at the clamped end of the cantilever beam. In addition, the designated term also corresponds to the classical beam bending theory. Note that the same results as in Eq. (20) can be obtained for cases of a clamp-free cantilever beam under harmonic base excitation [17].

Since Eq. (20) was derived from the basic beam theory, Eq. (20) would always predict the same stress and damping for cases where the force per unit beam width (P/w) is constant. Generally, this means that the width of the beam does not affect the maximum bending stress and hence the damping of the beam. This conclusion contradicts the observations made by Dayou et al. [12]. Dayou et al. [12] analyzed damping in both free and forced vibration. It is unclear if the ratio of P/w was fixed in their free vibration analysis. However, a similar trend was also recorded in their forced vibration experiment where the ratio of P/w was constant for this case since a constant forced acceleration was used. Both experiments recorded an increase in damping with increasing beam width. This suggests that the classical beam theory is unable to differentiate the stress and damping of wider beams. A solution for this issue would be to consider plate theory instead of beam theory. Since plates are essentially wide beams, it is assumed that Eq. (20) is applicable for plates.

3. Plate bending theory

Since the maximum stress in Eq. (20) relates to the bending of a cantilever beam under an applied concentrated load, this section will consider the bending of a cantilever plate under the same type of loading. Kirchhoff-Love theory of thin plates states that

$$E^*\left[\frac{\partial^2 W(x,y)}{\partial x^2} + 2\frac{\partial^2 W(x,y)}{\partial x\partial y} + \frac{\partial^2 W(x,y)}{\partial y^2}\right] = q, \quad (21)$$

where $W(x, y)$ is the vertical displacement of the plate at position x along the length and y along the width, q is the loading applied on the plate and E^* is the plate's flexural rigidity defined as

$$E^* = \frac{Eh^3}{12\left(1-v^2\right)}, \tag{22}$$

where v is Poisson's ratio of the plate. The boundary conditions for a cantilever plates are

$$W(x,y)_{x=0} = \frac{\partial W(x,y)}{\partial x}_{x=0} = 0, \tag{23}$$

$$\left[\frac{\partial^2 W(x,y)}{\partial x^2} + v\frac{\partial^2 W(x,y)}{\partial y^2}\right]_{x=L} = \left[\frac{\partial^2 W(x,y)}{\partial y^2} + v\frac{\partial^2 W(x,y)}{\partial x^2}\right]_{y=0,w} = 0, \tag{24}$$

$$\left[\frac{\partial^3 W(x,y)}{\partial x^3} + (2-v)\frac{\partial^2 W(x,y)}{\partial y^2 \partial x}\right]_{x=L} = \left[\frac{\partial^3 W(x,y)}{\partial y^3} + (2-v)\frac{\partial^3 W(x,y)}{\partial x^3 \partial y}\right]_{y=0,w} = 0, \tag{25}$$

$$\frac{\partial^2 W(x,y)}{\partial x \partial y}_{x=0,L\ y=0,w} = 0. \tag{26}$$

Applying the method of double finite integral transform, $W(x,y)$ can be defined as [23]

$$W(x,y) = \frac{2}{Lw}\sum_{i}^{\infty} W_{i0}\sin\left(\frac{\alpha_m}{2}x\right) + \frac{4}{Lw}\sum_{i}^{\infty}\sum_{j}^{\infty} W_{ij}\sin\left(\frac{\alpha_m}{2}x\right)\cos\left(\beta_n y\right), \tag{27}$$

where $i = 1,3,5\ldots,\ j=1\ \ldots$ and

$$\alpha_m = \frac{m\pi}{L}, \tag{28}$$

$$\beta_n = \frac{n\pi}{w}. \tag{29}$$

The term W_{ij} can be derived from the boundary condition problems resulting in [23]

$$W_{ij} = M_{ij}\left[\frac{q_{ij}}{D} + (-1)^n N_{ij}P_j - N_{ij}Q_j + (-1)^{0.5(m-1)} O_{ij}R_i - \frac{\alpha_m}{2}S_i\right], \tag{30}$$

where

$$M_{ij} = \left[\left(\frac{\alpha_m}{2}\right)^2 + \beta_n^2\right]^{-2} \tag{31}$$

$$N_{ij} = v\left(\frac{\alpha_m}{2}\right)^2 + \beta_n^2 \tag{32}$$

$$O_{ij} = \left(\frac{\alpha_m}{2}\right)^2 + v\beta_n^2. \tag{33}$$

The term q_{ij} refers to the double finite integral transform of the load function $q(x,y)$ applied on the plate and can be defined as

$$q_{ij} = \int_0^w\int_0^L q(x,y)\sin\left(\frac{\alpha_m}{2}x\right)\cos\left(\beta_n y\right). \tag{34}$$

For cases of concentrated point load, q_{ij} can be solved by considering the integral identities of delta functions [24].

$$q_{ij} = P\sin\left(\frac{\alpha_m}{2}\eta\right)\cos\left(\beta_n\Omega\right), \tag{35}$$

where (η, Ω) marks the x and y location of the applied load. The terms P_j, Q_j, R_i and S_i from Eq. (30) are unknowns which can be determined by solving the boundary condition problems in Eqs. (23)-(26). The accuracy of Eq. (30) depends on the number of unknown terms used. In this paper, the effect of the number of terms used for each of the four unknown variables on the accuracy of the predicted stress and damping is explained in the next section. The maximum bending stress amplitude at the clamped end of the beam can be calculated using Eq. (36).

$$\sigma_m\left(0,y,\frac{h}{2}\right) = \frac{6E^*}{h^2}\left[\frac{1}{w}S_0 + \frac{2}{w}\sum_{i}^{\infty}S_i\cos\left(\beta_n y\right)\right]. \tag{36}$$

4. Comparison with FEA

In this section, the effect of the beam length to beam width ratio (L/w) on the maximum bending stress amplitude and damping ratio of a steel cantilever beam under concentrated loading was analyzed using FEA. Since the classical beam and plate theory is only valid for thin beams, the effect of beam thickness was not investigated. The loading was applied at the center of the beam's free end. FEA simulation was conducted using ANSYS software. The constant properties of the steel beam are in Table 1. The beam was modelled using three-dimensional elements with a mesh size of 0.2 mm.

Table 1. Properties of steel beam.

E (GPa)	200
v	0.3
L (mm)	100
t (mm)	1
σ_f (MPa)	186

The length-to-width ratio of the beam was varied between

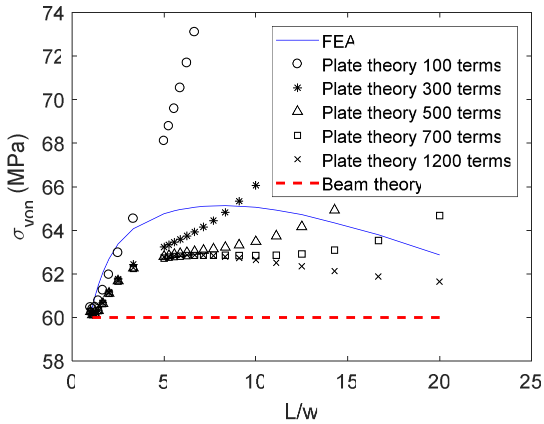

Fig. 1. von Mises stress comparison between beam theory, plate theory and FEA simulation.

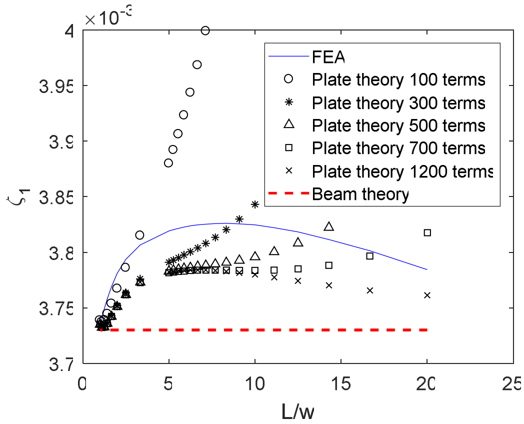

Fig. 2. Damping ratio comparison between beam theory, plate theory and FEA simulation.

$L/w=1$ to $L/w=20$ while maintaining a constant length, thickness and a P/w ratio of 100. The bending stress amplitude for each scenario was recorded at the center of the beam's clamped end. To compare the beam theory and plate theory with FEA, the von Mises stress was considered. Eqs. (37) and (38) describe the von Mises stress of beams and plates:

$$\sigma_{von}^{beam} = \sigma_m, \quad (37)$$

$$\sigma_{von}^{plate} = \sqrt{\frac{\left(\sigma_m - v\sigma_m\right)^2 + \sigma_m{}^2 + \left(v\sigma_m\right)^2}{2}} . \quad (38)$$

The results of the plate theory were computed using 100, 300, 500, 700 and 1200 number of terms for each of the four unknown variables. Fig. 1 illustrates the von Mises stress comparison between the two theories with FEA. Fig. 2 shows the damping ratio comparison of the three methods where the damping ratio was calculated using Eq. (20). Results show the same trend in both stress and damping. This is expected since the recorded stresses in Fig. 1 were much lower than the fatigue limit stress value in Table 1. Hence, Eq. (20) becomes an approximately linear relation. The FEA results in Fig. 2 show that as the width of the beam increases, the damping ratio increases until a specific value and decreases again after that. However, the increase in damping is not very significant. Dayou et al. [12] observed a much higher increase in damping in their experiment than in Fig. 2. The discrepancies that arise here may be due to the different material used and the contribution of other forms of damping in the experiment. Theoretically, the FEA results in Fig. 2 predict that a cantilever beam would experience minimum damping under free vibration motion when $w=L$.

The important matter of this study is to discuss the validity of the classical beam and plate bending for wide beams. It is assumed that the FEA results are accurate. It can be seen that as the ratio of L/w increases, the required number of terms also increases in order to obtain a more accurate stress and damping estimate. Large errors can be observed between the plate theory estimations and FEA results when only 100 terms are used for cases where $L/w>4$. In addition, the trending curves displayed by plate theory when using a lower number of terms are significantly different than the FEA results. This shows that the plate theory becomes more sensitive when larger L/w ratios are used.

Overall, the use of a smaller number of terms in plate theory is only valid when $L/w \le 4$. The plate theory is seen to converge to the beam theory when $w=L$, due to the contribution of the poison ratio effect. In addition, a strong agreement between the beam theory and the FEA results was recorded only when $w=L$. This means that the beam theory can be deemed accurate for square plate estimations. The maximum difference between the beam theory and FEA results in Figs. 1 and 2 is 8.5 % and 2.5 %, whereas the maximum difference for plate theory using 1200 terms is 3.7 % and 1.4 %, respectively. These errors may differ depending on the material properties of the beam itself.

This analysis shows that in determining the bending stress and damping of wide cantilever beams under free vibration, plate theory results in a more accurate prediction. However, the estimations of plate theory highly depend on the number of terms used in Eq. (30), especially when dealing with larger L/w ratios. Generally, the classical beam theory assumes zero shear deformation. Hence, the beam theory estimations would be more accurate when the width of the beam approaches zero. Nevertheless, this study shows that the beam theory is also valid for square plates.

5. Conclusion

This study explored the application of the classical beam and plate theory in predicting the stress and damping of wide cantilever beams. The damping stress equation for cantilever beams under free vibrations was derived from the empirical unit dissipating energy function given by Lazan [15]. The classical plate bending theory was derived using the method of double integral transform. The maximum bending stress and damping ratio results from the beam and plate theory were then compared to FEA results.

Results show that the accuracy of the classical plate theory is highly dependent on the length-to-width ratio of the beam. Since the solution to the derived plate bending equation is a summation of a specified number of four unknown terms, this number plays a major role in the accuracy of the plate theory. It was found that the number of terms required for a sufficient stress and damping estimation increases when the L/w ratio increases. In addition, the trending curve of stress against L/w displayed by the FEA results can only be achieved when a significantly high number of terms are used. If a small number of terms were used, then the results from the plate theory would only be valid for when $L/w \leq 4$. The maximum difference between the plate theory and FEA results in terms of bending stress and damping was recorded to be 3.7 % and 1.4 % when 1200 terms were used for each of the four unknowns.

The results from the classical beam theory suggest that the beam theory is valid for square plates. If the assumptions made in the development of the beam theory were considered, this means that the beam theory is only valid when the ratio of L/w is significantly high or equal to one (square plate). Beyond these ratios, comparison between the beam theory and the FEA results recorded a maximum difference of 8.5 % and 2.5 % in terms of bending stress and damping. Overall, plate theory can be concluded to be more accurate in estimating the bending stress and damping of wide beams, provided that a high number of terms are used. Nevertheless, results show that the width of a cantilever beam has a minor effect on its material damping ratio and that the observations made by Dayou et al. [12] may be due to the contributions of other form of damping.

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