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Nanofluid thin film flow and heat transfer over an unsteady stretching elastic sheet by LSM †

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Abstract

This study is carried out on the unsteady flow and heat transfer of a nanofluid in a stretching flat plate. Least square method is implemented for solving the governing equations. It also attempts to demonstrate the accuracy of the aforementioned method compared with a numerical one, Runge-Kutta fourth order. Furthermore, the impact of some physical parameters like unsteadiness parameter (S), Prandtl number (Pr) and the nanoparticles volume fraction (ϕ) on the temperature and velocity profiles is scrutinized carefully. Accordingly, the results obtained from this study reveal that the temperature enhances by means of augmenting the nanoparticles volume fraction. At $\eta \in \{0, 0.5\}$, the velocity decreases as a result of a rise in nanoparticles volume fraction and at $\eta \in \{0.5, 1\}$, an opposite treatment takes place. Moreover, velocity distribution augments by raising the S value, however an inverse trend is observed in temperature values. Moreover, the local skin friction coefficient indicated a notable rise by increasing the S parameter as well as a steady decrease by rising ϕ . Finally, water-Alumina nanofluid demonstrated better heat transfer enhancement compared to other types of nanofluids.

Keywords: Least square method (LSM); Nanofluid volume fraction; Thin film nanofluid; Unsteady stretching elastic sheet

1. Introduction

Enhancement of heat transfer by the addition of nanosized particles to a base fluid is a fast-growing field of interest and its application in various industries from microchannel cooling and floor heating to heat recovery systems has thrived during the recent years. The nano-scaled particles have evoked specific attention as a consequence of barriers in pressure drop or making the mixture homogenous for all particle dimensions. Since these particles are approximately in the same size of the base fluid molecules, they highly perceive stable suspensions during a long period of time. Convective heat transfer characteristic of nanofluids depends on flow pattern, nanofluid volume fraction and geometrical configuration of the particles [1].

Initially, the term “nanofluid” was presented by Choi [2] who described the fluid consisted of nanoparticles dispersing in a fluid. Thereupon, Masuda et al. [3] pursued the concept of this analysis and eventually a substantial number of researches were remarkably implemented in this field of study. During recent years, nanofluids have been brought into service by many industries and its analytical and numerical methods have inspired researchers in various fields of study [4-8].

For instance, Sheikholeslami et al. [9] looked over the performance of nanofluids between two rotating plates. They demonstrated that during injection and suction process, the rate of heat transfer could be improved by enhancing the volume fraction of nanoparticles. Furthermore, Hatami and Ganji [8] used the analytical Least square method (LSM) to study the role of Cu-water nanofluid flow in Microchannel heat sink (MCHS) cooling. Their proposed analytical method was then exerted in other heat transfer problems as well [10-13], by means of its simplicity and high accuracy. Consequently, Hatami and Ganji [13] probed the impact of variable magnetic field on the nanofluid flow behavior between two parallel disks. Besides, Domairry and Hatami [14] analyzed the impact of squeezing film of Cu-water nanofluid between parallel plates. In another investigation by Akbar and Nadeem [15], Homotopy Perturbation Method was applied to realize the motion of a MHD Jeffrey nanofluid in a channel. They [16] also took the initiative to employ the mentioned method to simulate the Phan-Thien-Tanner (PTT) nanofluid in a cylindrical coordinate system. Then, Nadeem et al. [17] reported a boundary layer formation when the velocity of viscid free-

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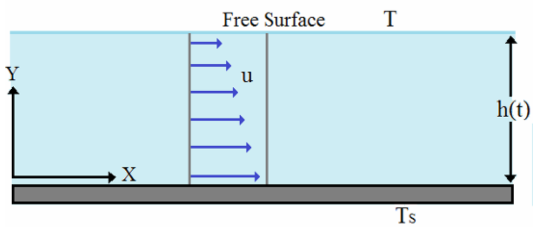

Fig. 1. The schematic diagram of the physical model.

stream is higher than stretching velocity.

Further studies are presented by Akbar et al. [18-22]. They added more complexity into the current nvestigations by considering the external magnetic field on nanofluids passing through a vertical stretching sheet. They studied different aspects including the effect of nanoparticle aggregation [18], exponential temperature-dependent viscosity [19] thermal radiation [20], double-diffusive natural convective boundary-layer flow [21] and permeable wall [22] in their models.

During the recent years, the wide application of Weighted residuals methods (WRMs) [23] including collocation, Galerkin and Least square method (LSM) attracted the researchers' attention looking for more reliable methods for solving the nonlinear differential equations. Stern and Rasmussen [24] solved the third order differential equation via the collocation method. Vaferi et al. [25] conducted an investigation into the practicability of resolving the diffusivity phenomenon in the transient flow passing the radial coordinates with the the Orthogonal collocation method. Latterly, Hatami et al. [26] made use of LSM for the porous fin problems with transverse magnetic field. Hatami and Ganji [27] detected that LSM is the most suitable analytical approach for nonlinear equations.

In the present study, the unsteady flow and the heat transfer of a nanofluid moving on a flat plate are studied and the nonlinear differential equations governing the presented system are solved by Least square method. Besides the impact of physical factors such as unsteadiness parameter (S), Prandtl number (Pr) and the volume fraction of nanofluids (ϕ) on the temperature and velocity distributions is explored thoroughly.

2. Problem description

In this research, it is considered that a nano-liquid film is flowing through a thin elastic sheet. The related physical model is outlined in Fig. 1.

It is assumed that the surface at y = 0 is stretched with the following velocity profile as:

$$U_w = \frac{bx}{1-\alpha t}. \tag{1}$$

In the above equation, b and α are constants. Finally, the profile of temperature on the sheet is formulated as follows:

$$T_s = T_0 - T_r\left(\frac{bx^2}{2\upsilon_f}\right)(1-\alpha t)^{-3/2}. \tag{2}$$

In Eq. (2), T_0 , T_r and ν_f are the slit temperature, the reference temperature and kinematic viscosity of the base fluid. It is presumed that the gravitational effect is negligible and the film is stable and uniform. Therefore, according to the proposed mathematical model by Tiwari and Das [28], the differential equations governing are expressed as below:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \frac{\mu_{nf}}{\rho_{nf}}\frac{\partial^2 u}{\partial y^2}. \tag{4}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_{nf}\frac{\partial^2 T}{\partial y^2}. \tag{5}$$

where the velocity components in x and y directions are shown by u and v, respectively. α_{nf}, ρ_{nf} and μ_{nf} are the thermal diffusivity, density and viscosity of nanofluid defined as:

$$\alpha_{nf} = \frac{k_{nf}}{(\rho C_p)_{nf}}, \rho_{nf} = (1-\phi)\rho_f + \phi\rho_s, \mu_{nf} = \frac{\mu_{nf}}{(1-\phi)^{2.5}}. \tag{6}$$

Afterwards:

$$\left(\rho C_p\right)_{nf} = (1-\phi)\left(\rho C_p\right)_f + \phi\left(\rho C_p\right)_s,$$
$$\frac{k_{nf}}{k_f} = \frac{\left(k_s + 2k_f\right) - 2\phi\left(k_f - k_s\right)}{\left(k_s + 2k_f\right) + \phi\left(k_f - k_s\right)}. \tag{7}$$

where ϕ , k_{nf} and $(\rho C_p)_{nf}$ are the volume fraction, thermal conductivity and the heat capacitance of the nanofluid, respectively. Then, the related boundary conditions for the differential equation governing the mentioned system are defined as:

$$\text{when} \quad y = 0 \rightarrow u = U_w, v = 0, T = T_s. \tag{8}$$

Subsequently:

$$\text{when} \quad y = h(t) \rightarrow \frac{\partial u}{\partial y} = 0, \frac{\partial T}{\partial y} = 0, v = \frac{\partial T}{\partial y}. \tag{9}$$

In the above equation, the thickness of the film is shown by $h(t)$. Based on the Schlichting and Gersten [29], the thickness of boundary-layer δ(x) is proportional to $(x\nu_f/U_w)^{0.5}$, hence the variable η is defined as follows:

$$\eta = y\sqrt{\frac{U_w}{x\nu_f}} = y\sqrt{\frac{b}{(1-\alpha t)\nu_f}}. \tag{10}$$

It is necessary to mention that the above equation can be achieved by the expressions presented by Görtler [29]. In accordance to this explanation and substituting U(x) = U_w, it is possible to define ξ, η and ψ(x, y), which is brought completely in Ref. [30]. Regarding the above explanations, the following new variables are introduced as [30]:

$$\psi = \beta\left[\frac{\nu_f b}{1-\alpha t}\right]^{1/2} x f(\eta). \tag{11}$$

$$T = T_0 - T_r\left(\frac{bx^2}{2\nu_f}\right)(1-\alpha t)^{3/2}\theta(\eta). \tag{12}$$

$$\eta = \frac{1}{\beta}\left[\frac{b}{\nu_f(1-\alpha t)}\right]^{1/2} y. \tag{13}$$

where ψ(x, y) is the stream function explained in the usual way, for example, $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$ and $\beta > 0$ is the non-dimensional film thickness parameter given by $\beta = (hb/v_f)(1-\alpha t)^{-1/2}$. For the limiting cases $\beta = 0$ and $\beta = \infty$, this transformation is no longer effective and particular approaches are needed to give solutions as shown in Wang [31]. As a result, the velocity components u and v can be defined as:

$$u = \frac{\partial\psi}{\partial y} = \left(\frac{bx}{1-\alpha t}\right) f'(\eta). \tag{14}$$

$$v = -\frac{\partial\psi}{\partial x} = -\beta\left(\frac{\nu_f b}{1-\alpha t}\right)^{1/2} f(\eta). \tag{15}$$

Inserting the similarity variables into the momentum and energy equations results in:

$$\varepsilon_1 f''' + \beta^2\left[ff'' - (f')^2 - S\left(f' + \frac{\eta}{2}f''\right)\right] = 0. \tag{16}$$

$$\frac{\varepsilon_2}{2}\theta'' - \beta^2\left[\frac{S}{2}(3\theta + \eta\theta') + 2\theta f' - f\theta'\right] = 0. \tag{17}$$

where the relevant boundary conditions can be rewritten as:

$$\text{at } \eta = 0 \rightarrow \; f(0) = 0,\; f'(0) = 1,\; \theta(0) = 1. \tag{18}$$

$$\text{at } \eta = 1 \rightarrow \; f(1) = \frac{S}{2},\; \theta'(1) = 0. \tag{19}$$

In the afore-mentioned equations, $Pr = (v_f/\alpha_f)$ and $S = (\alpha / b)$ are the Prandtl number and unsteadiness parameter, respectively. Finally, ε_1 and ε_2 are two constants explained in the following form:

$$\varepsilon_1 = \frac{1}{(1-\phi)^{2.5}\left[(1-\phi) + \phi\rho_s/\rho_f\right]}. \tag{20}$$

$$\varepsilon_2 = \frac{(k_{nf}/k_f)}{\left[(1-\phi) + \phi(\rho C_p)_s/(\rho C_p)_f\right]}. \tag{21}$$

The physical quantities for this problem are the Nusselt number Nu and skin friction coefficient C_f as follows:

$$C_f = \frac{\tau_w}{\rho_f (U_w)^2},\; Nu = \frac{q_w x}{k_f (T_s - T_0)}. \tag{22}$$

In the above equation, the heat flux and skin friction at the surface are defined in the following form:

$$\tau_w = \mu_{nf}\left(\frac{\partial u}{\partial y}\right)_{y=0},\; q_w = -k_{nf}\left(\frac{\partial T}{\partial y}\right)_{y=0}. \tag{23}$$

Therefore, after substituting Eq. (23) into Eq. (22), we will have the final form of the physical quantities as follows:

$$C_{fx}\,\mathrm{Re}_x^{-1/2} = \frac{1}{\beta(1-\phi)^{2.5}} f''(0). \tag{24}$$

$$Nu_x\,\mathrm{Re}_x^{-1/2} = -\frac{k_{nf}}{k_f\beta}\theta'(0). \tag{25}$$

It is notable that in Eqs. (24) and (25) $Re_x = U_{wx}/v_f$ is the local Reynolds number. Wang [31] calculated that the solutions are available in the range of $0 \le S \le 2$ for a liquid film problem occurred by an unsteady stretching surface.

3. Analytical and numerical methods

3.1 Least square method (LSM)

Sheikholeslami, Hatami and Ganji [32] expressed that LSM is one of the WRMs constructed for minimizing the residuals of the trial function. It is important to mention that the trial function must satisfy the boundary conditions and equations, hence, the trial function can be considered as:

$$f(\eta) = \eta + C_1(200\eta^2 + \eta) + C_2(300\eta^3 + \eta) + C_3(400\eta^4 + \eta) + C_4(500\eta^5 + \eta) + C_5(600\eta^6 + \eta). \tag{26}$$

$$\theta(\eta) = 1 + C_6(\eta - 1/2\eta^2) + C_7(\eta - 1/3\eta^3) + C_8(\eta - 1/4\eta^4) + C_9(\eta - 1/5\eta^5) + C_{10}(\eta - 1/6\eta^6). \tag{27}$$

By introducing this equation into Eqs. (16) and (17), the residual function can be extracted and by substituting them into Eqs. (26) and (27), a set of ten equations and ten unknown coefficients C1-C10 will be determined.

3.2 Numerical method (NUM)

The above system of non-linear ordinary differential equations along with the boundary conditions is solved numerically using the algebra package Maple 16.0 with a boundary value (B-V) problem procedure. This algorithm is based on ourth-fifth order Runge-Kutta-Fehlberg procedure which improves the Euler method by adding a midpoint in the step which increases the accuracy by one order. Thus, the midpoint method is used as a suitable numerical technique.

4. Results and discussion

In this research, LSM method was adopted to achieve an

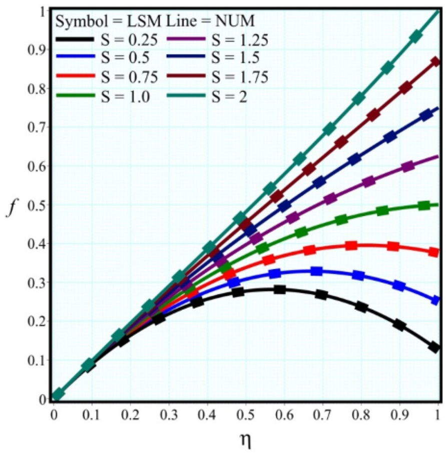

Fig. 2. A comparison between the obtained results by LSM and Numerical solution in terms of varying f(η) for different values of unsteadiness parameter (S) (Water - Copper).

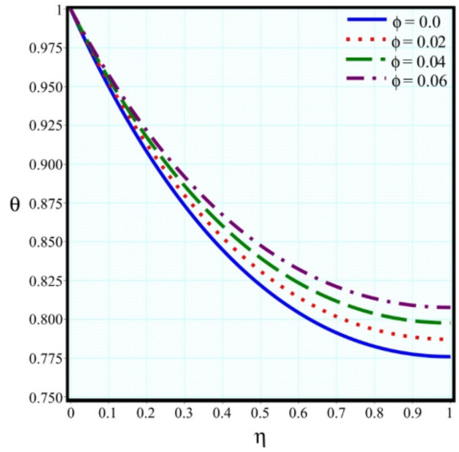

Fig. 5. Temperature distributions for four different values of volume fraction of the nanofluid (ϕ) (Water - Copper).

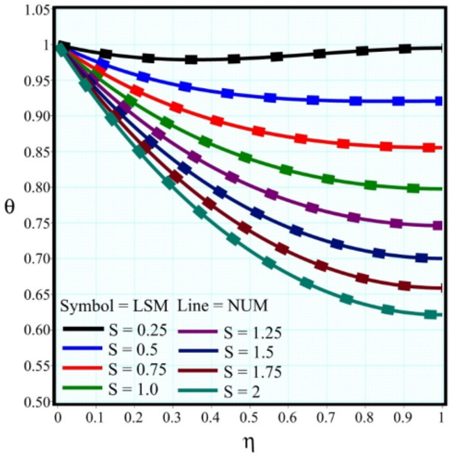

Fig. 3. A comparison between the obtained results by LSM and Numerical solution in terms of varying θ(η) for different values of unsteadiness parameter (S) (Water - Copper).

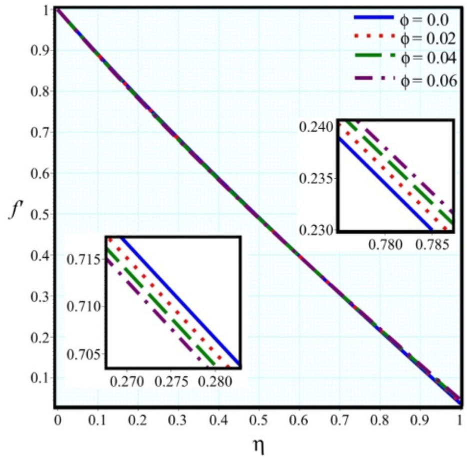

Fig. 6. Variation of velocity for four different values of volume fraction of the nanofluid (ϕ) (Water - Copper).

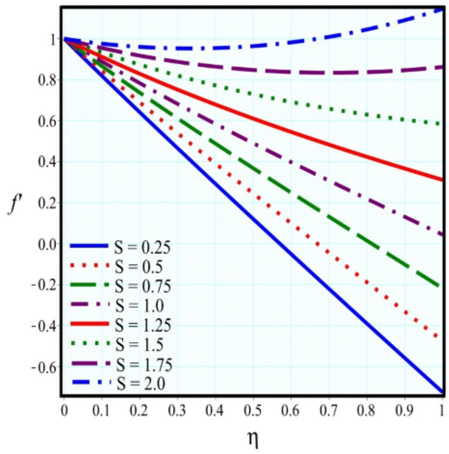

Fig. 4. The variation of velocity profile for some values of unsteadiness parameter (Water - Copper).

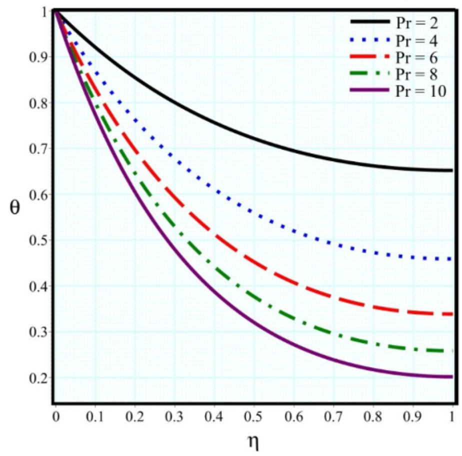

Fig. 7. Variation of θ(η) in terms of various amounts of Prandtl number (Water - Copper).

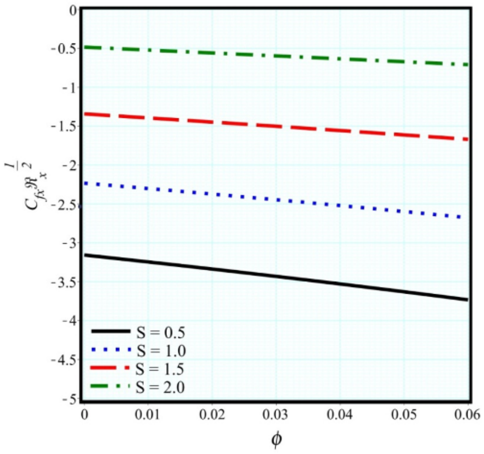

Fig. 8. Variation of local skin friction coefficient in terms of four different amounts of S (Water - Copper).

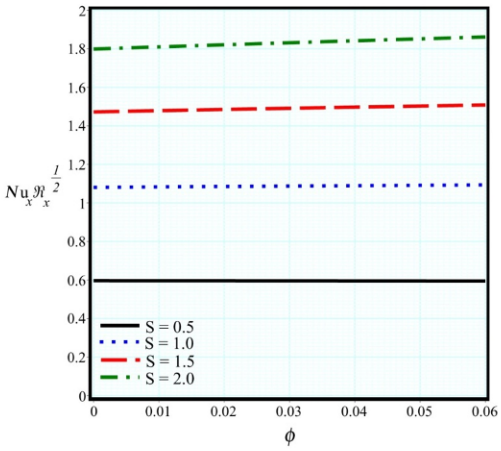

Fig. 9. Effect of (ϕ) on the local Nusselt number in the specified domain (Water - Copper).

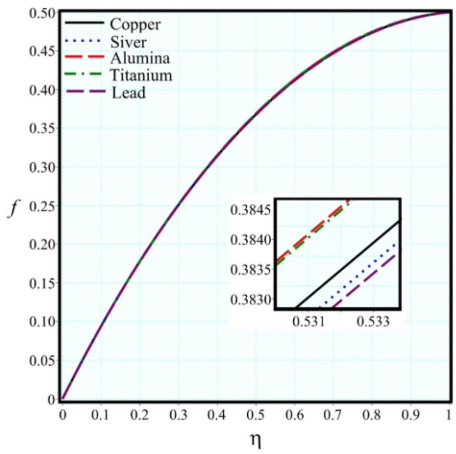

Fig. 10. Effect of different types of nanofluids on f(η).

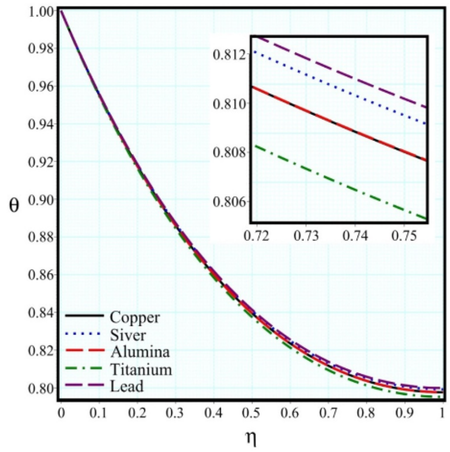

Fig. 11. Effect of different types of nanofluids on θ(η).

explicit analytical solution of the unsteady flow and the related heat transfer of a nanofluid passing through a horizontal plate. (Fig. 1). Copper-water nanofluid passes through the horizontal plate and the impact of some physical parameter (Pr, S and ϕ), as well as various types of nanofluids on the temperature and velocity profiles is studied in this research.

Figs. 2 and 3 indicate that S had an impact on the temperature and f(η) profiles. Accordingly, it is concluded that f(η) increases by a rise in the S parameter. On the contrary, an opposite trend is seen in temperature values, where the value of temperature profile decreases by increasing the parameter S. Moreover, it can be inferred from these figures. that the obtained results from the LSM demonstrate a great compatibility with the data achieved from the numerical model.

Fig. 4 shows that with a rise in η, the velocity will steadily increase. It is worth noting that the rate of the velocity would augment with growth in S parameter. Since the parameter S depends on α and β, it can be deduced that the stretching velocity of wall is a vital factor to specify the velocity profile. The variation of temperature with various volume fractions of nanofluids is shown in Fig. 5. As stated in this figure, increasing the nanoparticles volume leads to the rise of temperature value due to more heat transfer caused by the nanoparticles. This case implies that the nanofluid has a very important and vital impact on the properties of the heat transfer.

Moreover the impact of the nanofluid volume fraction on the velocity is displayed in Fig. 6. An overall observation of Fig. 6 reveals that when $\eta \in \{0, 0.5\}$, the velocity decreases due to the enhancement in nanoparticles volume fraction and when $\eta \in \{0.5, 1\}$, the opposite trend occurs. Furthermore, the effect of Pr number on θ(η) is completely investigated in Fig. 7. As a result, the temperature distribution has a considerable decline due to the increase of Pr number and such a decrease in temperature distribution in free surface (η = 1) will be vanished for larger values of Pr. Thus, it is clear that the surface temperature is equal to ambient temperature.

The variations of local skin friction coefficient and Nusselt number with ϕ are displayed in Figs. 8 and 9. It can be seen that the local skin friction coefficient increases significantly and decreases steadily by increasing the S and ϕ parameters, respectively. From this figure, it can be inferred that the local Nusselt number faces a steady and considerable enhancement by increasing the ϕ and S parameters.

Figs. 10 and 11 show the impact of different structure of nanofluids on f(η) and θ(η). According to the figures, it can be observed that the maximum amount of f(η) and θ(η) occurs when the water-Alumina and water-Lead nanofluids are used. The results indicate that nanoparticles in working fluid mainly cause the increase of the heat transfer rate at any given value of S and the Pr in a thin film nanofluid over a stretching plate.

5. Conclusion

In this study, the LSM is implemented to solve the unsteady flow problem and the related heat transfer of a nanofluid over a horizontal plate. The accuracy of the LSM is checked with the numerical model. The comparison indicates that the LSM has a good compatibility with the numerical data. After validating the model, the impact of physical parameters such as S, Pr and the nanoparticles volume fraction on the temperature and velocity behaviors has been studied in details. It is observed that the values of f(η) increase as the S parameter augments. However, a reverse trend is observed in which an enhancement in the S value leads to reduction in the temperature profile and by increasing the η, the velocity will increase steadily. Regarding the role of nanofluid volume fraction, it is concluded that the overall convective heat transfer coefficient enhances by adding the nanoparticles into the pure working fluid. Among different types of nanofluid, water-Alumina nanofluid showed better improvement in terms of boosting the velocity and heat transfer in the considered geometry. However, the velocity profile experienced a turning point at η = 0.5.

Nomenclature

S	: Unsteadiness parameter
ϕ	: Solid volume fraction of the nanofluid
Pr	: Prandtl number
u, v	: Velocity components in x and y directions
μ_{nf}	: Viscosity of the nanofluid
ρ_{nf}	: Density of the nanofluid
α_{nf}	: Thermal diffusivity of the nanofluid
k_{nf}	: Thermal conductivity of the nanofluid
$(\rho C_p)_{nf}$	: Heat capacitance of the nanofluid
ρ_f	: Density of the base fluid
μ_f	: Dynamic viscosity of the base fluid
$Re_x^{-0.5}C_{fx}$	: Local skin friction coefficient
$Re_x^{-0.5}Nu_x$	: Local Nusselt number
k_f	: Thermal conductivity of the base fluid
$(\rho C_p)_f$	: Heat capacitance of the base fluid
ρ_s	: Density of the nanoparticle
μ_s	: Dynamic viscosity of the nanoparticle
k_s	: Thermal conductivity of the nanoparticle
$(\rho C_p)_s$	: Heat capacitance of the nanoparticle
H	: Similarity variable
B	: Dimensionless film thickness
$h(t)$	: Film thickness
$\delta(x)$	: Boundary layer thickness
$\varepsilon_1, \varepsilon_2$	: Constants
C_f	: Skin friction coefficient

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