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An adaptive importance sampling method with a Kriging metamodel to calculate failure probability †

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Abstract

A Markov chain simulation was performed to extract points in a failure region. A Kriging metamodel was constructed to approximate a limit state based on the points extracted by the Markov chain simulation. A kernel sampling density was constructed to approximate the optimal importance sampling density. The points extracted in the failure region by the Markov chain simulation were assumed as a mean of each kernel. An importance sampling method was applied to calculate the failure probability. In the importance sampling method, points are extracted from the kernel in the vicinity of a limit state. Considering the statistical distance as well as the learning function, additional experimental points were selected for the kriging metamodel. A stable numerical calculation method was applied to find the parameters of the kernel sampling density. The completeness of the Kriging metamodel was evaluated on the basis of possible changes in failure probability.

Keywords: Importance sampling; Kernel density; Kriging metamodel; Markov chain simulation

1. Introduction

Many studies have been made to calculate the probability that a product or a system fails to perform its intended function, with the assumption that the uncertainties in the design are random variables. A performance function is defined as the state of design. The performance function (g) has a positive value in a safe state and a negative value in a failure state. Also, when the performance function is zero, it is known as a limit state.

The failure probability (P_f) is given by Eq. (1) when the uncertainties are defined as a random variable vector $\tilde{\theta}=\left[\theta_1,\theta_2,\cdots,\theta_n\right]^T$ and the joint probability density of $\tilde{\theta}$ is $p\left(\tilde{\theta}\right)$. $I_g\left(\tilde{\theta}\right)$ is an indicator function that is 1 when the performance function is negative. Otherwise, the indicator function is zero. Θ is the domain of possible values of $\tilde{\theta}$. In most cases, it is difficult to calculate Eq. (1) analytically, and an approximated value is calculated by numerical calculations.

$$P_f=\int_{g<0} p\left(\tilde{\theta}\right)d\tilde{\theta}=\int_{\Theta} I_g\left(\tilde{\theta}\right)p\left(\tilde{\theta}\right)d\tilde{\theta}\,. \tag{1}$$

One of the ways to approximate Eq. (1) is to use a Monte Carlo simulation. The Monte Carlo simulation is robust for nonlinear limit states, but requires considerable computational cost [1]. There are two approaches to reduce the computational cost.

The first is to reduce the number of performance function calculations. These include the importance sampling method and subset simulations. The efficiency of the importance sampling method is determined by the importance sampling density. The points in the failure region have been extracted more easily by Markov chain simulation [2]. With these samples, we can construct the importance sampling density in the form of a kernel sampling density.

The second is to reduce the computation time required for one calculation of a performance function. It takes a long computational time when a large numerical analysis is necessary such as finite element analysis. In addition, the Monte Carlo simulation requires a tremendous number of evaluations of a performance function to calculate a meaningful failure probability.

At first, to reduce the computation time, a polynomial-based response surface was applied to approximate the performance function [3, 4]. In the past, the Kriging metamodel was applied to approximate the performance function [5, 6]. In the early stages, many studies constructed a Kriging metamodel precisely for the entire range of the performance function.

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Recently, studies are being conducted to construct a Kriging metamodel precisely in the vicinity of the limit state where it is important to determine whether the value of a performance function is negative or positive. The preciseness of the predicted value of a performance function is no longer important.

In this paper, an improvement of the adaptive importance sampling method is proposed. The method is based on the Markov chain simulation, kernel sampling density, and the Kriging metamodel. A kernel sampling density construction method is proposed to extract more points in the vicinity of a limit state. A stable numerical method is also proposed to find the parameters of the kernel sampling density. A method to evaluate the completeness of the Kriging metamodel is proposed by calculating the possibility that the calculated failure probability changes due to the uncertainty of the Kriging metamodel.

2. Fundamental theory

2.1 Importance sampling method

The importance sampling method has been applied to reduce the number of calculations from a Monte Carlo simulation. In a Monte Carlo simulation, points are extracted according to the probability density function of random variables. But, in the importance sampling method, points are extracted according to the importance sampling density.

The importance sampling density is assumed to extract points more easily in a failure region. In Eq. (2), $p\left(\tilde{\theta}\right)$ is the joint probability density function of a random variable vector and $h\left(\tilde{\theta}\right)$ is the assumed importance sampling density. Eq. (2) is equivalent to Eq. (1)

Because the variance of a predicted failure probability is zero, the $h_{opt}\left(\tilde{\theta}\right)$ in Eq. (3) is the optimal importance sampling density. But, $h_{opt}\left(\tilde{\theta}\right)$ is not practical because it includes a constant P_f. Therefore, the importance sampling density is assumed to be similar to the optimal importance sampling density as much as possible.

$$P_f = \int_{\Theta} I_g\left(\tilde{\theta}\right)\frac{p\left(\tilde{\theta}\right)}{h\left(\tilde{\theta}\right)}h\left(\tilde{\theta}\right)d\tilde{\theta}, \tag{2}$$

$$h_{opt}\left(\tilde{\theta}\right) = \frac{I_g\left(\tilde{\theta}\right)p\left(\tilde{\theta}\right)}{P_f}. \tag{3}$$

2.2 Adaptive importance sampling method

We improved the adaptive importance sampling method proposed by Au [2] by modifying a sampling scheme from a kernel density. The modified sampling scheme is introduced in Sec. 2.3 in this paper. Au performed a Markov chain simulation using a Metropolis algorithm to easily extract points from a failure region. Also, the kernel sampling density is constructed with these points as the mean of each kernel. This kernel sampling density is assumed to be the importance sampling density of Eq. (2). The Metropolis algorithm is performed in the following procedure:

Metropolis algorithm

(1) An arbitrary point is selected in the failure region as the starting point. This point can be chosen by the researcher's intuition and is known to have little effect on the results.

(2) If the current point is $\tilde{\mathrm{A}}$, then one point $\tilde{\mathrm{B}}$ is extracted from a uniform distribution centered on $\tilde{\mathrm{A}}$. The side length of the hyper-rectangle of a uniform distribution is determined by applying Eq. (4) [2]. In Eq. (4), σ_i is a standard deviation of each random variable. *M* is the number of sampling points of Markov chain simulation. *n* is the number of random variables.

$$l_i = 6\sigma_i M^{-1/(n+4)}. \tag{4}$$

(3) $\eta = I_g\left(\tilde{B}\right)p\left(\tilde{B}\right) / I_g\left(\tilde{A}\right)p\left(\tilde{A}\right)$ is calculated.

(4) If $\eta \geq 1$, $\tilde{\mathrm{B}}$ is selected as the next point for a Markov chain simulation.

(5) If $\eta < 1$, $\tilde{\mathrm{B}}$ is selected as the next point for a Markov chain simulation with a probability of η.

In the above procedure, the point selected as the next point of a Markov chain simulation is called an "Accepted point" in this paper. *M* points are extracted in this study. The point that was not selected as the next point is called the "Rejected point" in this paper and the number is M_R.

2.3 Kernel sampling density

In this study, the kernel sampling density of Eq. (5) was constructed by a linear combination of kernels that are a normal distribution of Eq. (6). The covariance matrix of *M* points extracted by the Markov chain simulation is applied to $\boldsymbol{S}$ in Eq. (6). In Eq. (6), *n* is the number of random variables.

Au proposed to uniformly extract points from all kernels [2]. However, in this study, the points were extracted only in the kernel near a limit state. To do this, we calculated the mean (μ) and standard deviation (σ) of the distances from the origin to the points generated by a Markov chain algorithm in the standard normal space. We also only extracted points from kernels that are centered on the points within $\mu + \sigma$. In this study, the number of kernels from which the points are extracted is marked as M_k.

$$k\left(\tilde{\theta}\right) = \frac{1}{M_k}\sum_{i=1}^{M_k} K_i\left(\tilde{\theta}\right), \tag{5}$$

$$K_i\left(\tilde{\theta}\right) = \frac{1}{\sqrt{(2\pi)^n\left|\boldsymbol{\Sigma}_i\right|}} exp\left[-\frac{1}{2}\left(\tilde{\theta}-\tilde{\mu}_i\right)^T \boldsymbol{\Sigma}^{-1}\left(\tilde{\theta}-\tilde{\mu}_i\right)\right] \tag{6}$$

where $\boldsymbol{\Sigma}_i = \boldsymbol{S}\cdot w^2\cdot\lambda_i^2$.

In Eq. (6), *w* is the value for adjusting the covariance of all

kernels, and λ_i is the value for adjusting the covariance of each kernel. In this study, the optimum *w* was found by applying the method suggested by Au [2]. Also, the calculation method of λ_i proposed by Au was improved for numerical calculations.

$$\lambda_i = \left\{ \left[\prod_{j=1}^{M} p\left(\tilde{\theta}^{(j)}\right) \right]^{1/M} \Big/ p\left(\tilde{\theta}^{(i)}\right) \right\}^{\alpha}. \tag{7}$$

Au calculated the λ_i of Eq. (6) through Eq. (7) [2]. When the value of $p\left(\tilde{\theta}\right)$ is small and the value of *M* is large, the numerical value of Eq. (7) becomes zero. To prevent such a numerical error, λ_i is calculated through Eq. (8). The value of α is 0.5, as suggested by Au [2].

$$\begin{aligned} \lambda_i^* &= \log\left(\lambda_i\right) \\ &= \alpha\left[\frac{1}{M} \sum_{j=1}^{M} log\left\{ p\left(\tilde{\theta}^{(j)}\right) \right\} - log\left\{ p\left(\tilde{\theta}^{(i)}\right) \right\} \right]. \\ \lambda_i &= 10^{\lambda_i^*} \end{aligned} \tag{8}$$

Eq. (7) includes $\prod_{j=1}^{M} p\left(\tilde{\theta}^{(j)}\right)$. Because $p\left(\tilde{\theta}^{(j)}\right)$ is always less than one, when *M* is large, $\prod_{j=1}^{M} p\left(\tilde{\theta}^{(j)}\right)$ becomes zero numerically. $\prod_{j=1}^{M} p\left(\tilde{\theta}^{(j)}\right)$ of Eq. (7) is converted to $\frac{1}{M}\sum_{j=1}^{M} log\left\{ p\left(\tilde{\theta}^{(j)}\right) \right\}$ in Eq. (8). $log\left\{ p\left(\tilde{\theta}^{(j)}\right) \right\}$ of Eq. (8) can express a number much smaller than $p\left(\tilde{\theta}^{(j)}\right)$ of Eq. (7) numerically. When $\prod_{j=1}^{M} p\left(\tilde{\theta}^{(j)}\right)$ in Eq. (7) becomes zero numerically, $\frac{1}{M}\sum_{j=1}^{M} log\left\{ p\left(\tilde{\theta}^{(j)}\right) \right\}$ of Eq. (8) does not become zero numerically. When 100 points were extracted from the failure region of Fig. 1, the value of Eq. (7) was zero numerically, whereas the value of Eq. (8) was not zero numerically.

2.4 Kriging metamodel

To reduce the computation time of the performance function, the Kriging metamodel has been applied by many researchers [6].

The Kriging metamodel of Eq. (9) is obtained by adding a stochastic process to a polynomial approximation model [5, 7]. The **F** in Eq. (9) is expressed by Eqs. (10) and (11). $\tilde{f}$ is a vector of *q* functions, and **F** is a $s \times q$ matrix when there are numerous experimental points as *s*. The experimental points are points where both the input and output values of a performance function are known. $\tilde{\beta}$ is equal to the coefficient resulting from the least square method. $\tilde{Z}$ represents the

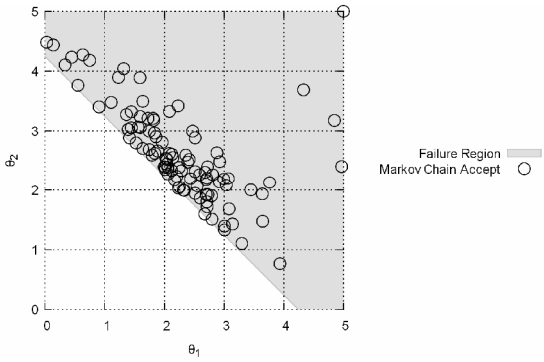


Fig. 1. One hundred points which are extracted from failure region.

Gaussian process. The mean of $\tilde{Z}$ is zero and the variance of $\tilde{Z}$ is σ_Z^2. The covariance of $\tilde{Z}$ is given by Eq. (12).

$$\tilde{Y} = \mathbf{F}\tilde{\beta} + \tilde{Z}, \tag{9}$$

$$\tilde{f} = \left[\zeta_1(\tilde{x}), \zeta_2(\tilde{x}), \cdots, \zeta_q(\tilde{x}) \right]^T, \tag{10}$$

$$\mathbf{F} = \begin{bmatrix} \tilde{f}^T(\tilde{x}_1) \\ \tilde{f}^T(\tilde{x}_2) \\ \vdots \\ \tilde{f}^T(\tilde{x}_s) \end{bmatrix}, \tag{11}$$

$$Cov\left(\tilde{Z}(\tilde{x}_i), \tilde{Z}(\tilde{x}_j)\right) = \sigma_Z^2 \mathbf{C}. \tag{12}$$

Here, **C** is a matrix that has a correlation function $R\left(\tilde{x}_i, \tilde{x}_j\right)$ of Eq. (13) as the *i*, *j*-th element. If the number of experiment points is *s*, the size of **C** is $s \times s$. The correlation function affects the smoothness of the Kriging metamodel, and the Gaussian correlation function of Eq. (13) is applied to the *d*-dimensional random variables.

$$R\left(\tilde{x}_i, \tilde{x}_j\right) = exp\left(-\sum_{\gamma=1}^{d} \rho_\gamma \left| x_\gamma^i - x_\gamma^j \right|^2 \right). \tag{13}$$

x_γ^i is the γ-th element of the *i*-th experiment point. ρ_γ is the scale parameter. $\hat{\tilde{\beta}}$ and $\hat{\sigma}_Z^2$, which are estimates of $\tilde{\beta}$ and σ_Z^2, are given by Eqs. (14) and (15).

$$\hat{\tilde{\beta}} = \left(\mathbf{F}^T \mathbf{C}^{-1} \mathbf{F} \right)^{-1} \mathbf{F}^T \mathbf{C}^{-1} \tilde{y}, \tag{14}$$

$$\hat{\sigma}_Z^2 = \frac{1}{s}\left(\tilde{y} - \mathbf{F}\hat{\tilde{\beta}} \right)^T \mathbf{C}^{-1} \left(\tilde{y} - \mathbf{F}\hat{\tilde{\beta}} \right). \tag{15}$$

$\tilde{y} = \left[y_1, y_2, \cdots, y_s \right]^T$ is a vector that has the output of a performance function at the experimental point as an element. $\hat{\tilde{\beta}}$ and $\hat{\sigma}_Z^2$ can be determined after the ρ_γ of Eq. (13) is determined. The estimate of ρ_γ is determined by the maximum likelihood estimation method and is shown in Eq. (16).

$$\text{Max.} \quad L(\rho) = -\frac{1}{2}\left[s \cdot ln\left(\sigma_Z^2\right) + ln\left\{det\left(\mathbf{C}\right)\right\}\right]$$
$$\text{Subject to} \quad \rho_\gamma \geq 0 \quad \text{for} \quad \gamma = 1, \cdots, d. \tag{16}$$

The estimate of a performance function for any input value is given by Eq. (17). $\tilde{r}_x$ is a correlation vector between an arbitrary input value and an experimental point, and is expressed by Eq. (18).

$$\hat{y}(x) = \tilde{f}_x^T \hat{\beta} + \tilde{r}_x^T \mathbf{C}^{-1}\left(\tilde{y} - \mathbf{F}\hat{\beta}\right), \tag{17}$$

$$\tilde{r}_x = \left[R(\tilde{x}_1, \tilde{x}), R(\tilde{x}_2, \tilde{x}), \cdots, R(\tilde{x}_s, \tilde{x})\right]^T. \tag{18}$$

The Kriging metamodel can calculate the Mean square error (MSE) of the output of Eq. (17) through Eq. (19).

$$\widehat{\sigma_y^2} = \sigma_Z^2\left(1 - \begin{bmatrix} \tilde{f}_x^T & \tilde{r}_x^T \end{bmatrix}\begin{bmatrix} 0 & \mathbf{F}^T \\ \mathbf{F} & \mathbf{C} \end{bmatrix}^{-1}\begin{bmatrix} \tilde{f}_x \\ \tilde{r}_x \end{bmatrix}\right). \tag{19}$$

2.5 Kriging metamodel refinement

Echard defined the learning function of Eq. (20) and proposed a method to refine the Kriging metamodel by adding the point with the smallest learning function value as an experimental point [8]. $\hat{g}(\tilde{\theta})$ is the estimate of the Kriging metamodel at point $\tilde{\theta}$. $\sigma_{\hat{g}}(\tilde{\theta})$ is the uncertainty of the Kriging metamodel estimate at point $\tilde{\theta}$, and is the square root of the value of Eq. (19).

In this study, an experimental point of the Kriging metamodel was added by considering the statistical distance as well as the learning function value. To do this, we calculated the mean (μ) and standard deviation (σ) of the distances from the origin to the accepted points of the Markov chain algorithm in standard normal space. Points whose statistical distances from the origin are greater than $\mu - 2\sigma$ and less than $\mu + \sigma$ were added as experimental points to the Kriging metamodel.

$$U(\tilde{\theta}) = \left|\hat{g}(\tilde{\theta})\right| \Big/ \sigma_{\hat{g}}(\tilde{\theta}). \tag{20}$$

Because the mean distance (μ) is located beyond of a limit state as shown in Fig. 2, the minimum distance of $\mu - 2\sigma$ was considered to include points in the vicinity of the limit state.

When considering the minimum distance of $\mu - \sigma$, the points (Red triangle in Fig. 2) in the vicinity of a limit state can be missed. The points located beyond $\mu + \sigma$ have little effect on the failure probability and need not be considered.

Echard suggested that the Kriging metamodel is refined until the minimum of the learning function values of all the points estimated by the Kriging metamodel becomes 2 or more [8]. In this study, the possibility of changing the predicted failure probability was calculated and used as a criterion for finishing the refinement of the Kriging metamodel.

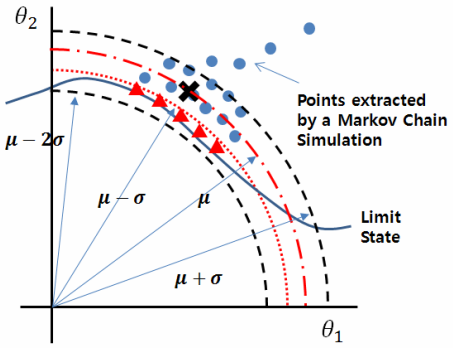


Fig. 2. Example of statistical distance. μ is mean distance and σ is standard deviation of the distance.

The Kriging metamodel is a Gaussian process. The Gaussian process assumes that all the estimated values are a normal distribution. The mean of the estimated value is the predicted value of the Kriging metamodel [7]. The learning function value of Eq. (20) is a statistical distance from zero. This can be seen as an uncertainty in the ability of the Kriging metamodel to predict the negatives/positives of a performance function. Eq. (21) is an average value of the probability that the sign of an estimated value will change at all the points estimated by the Kriging metamodel. In this study, the value of Eq. (21) was applied as a criterion for finishing the Kriging metamodel refinement.

$$T = 1/N_T \sum_{i=1}^{N_T} 1 - \Phi\left(u_{\theta_i}\right), \tag{21}$$

where

$\Phi(\cdot)$: Cumulative distribution function of normal distribution

u_{θ_i} : Value of learning function at point $\tilde{\theta}_i$

N_T : Number of total points of a Kriging metamodel prediction.

The smaller the T value, the higher the accuracy of the Kriging metamodel, but it needs a higher numerical cost. So, it is necessary to find an appropriate compromise between accuracy and numerical cost.

2.6 Procedure and failure probability calculation

In this study, the failure probability was calculated by the procedure below:

Step 1) M points are extracted based on the Metropolis algorithm presented in Sec. 2.2.

Step 2) A kernel sampling density is constructed and points are extracted as presented in Sec. 2.3. N points are extracted in each kernel. Total $M_k \times N$ points are extracted.

Step 3) The Kriging metamodel is constructed, as presented in Sec. 2.4. The experimental points of the Kriging metamodel are the M accepted points and M_R rejected points generated in step 1.

Step 4) The value of T in Eq. (21) is calculated about total $M_k \times N$ points. In Eq. (21), N_T is $M_k \times N$. If the value of T is larger than the pre-defined value, the Kriging metamodel is refined by adding a point that has the smallest value from the learning function of Eq. (20) as an experimental point. This process is repeated until the value of T satisfies the pre-defined value.

Step 5) The failure probability is calculated according to Eqs. (22) and (23), and the variance is calculated according to Eqs. (24) and (25). $\tilde{\theta}_j^i$ is the j-th point extracted from the i-th kernel. If the coefficient of variation (C.O.V, Coefficient of variance) of Eq. (26) is greater than the pre-defined value, the failure probability is calculated again by increasing N from step 2.

$$\widehat{P}_f = \frac{1}{M_k}\sum_{i=1}^{M_k}\hat{P}_{k,i}, \tag{22}$$

$$\hat{P}_{k,i} = \frac{1}{N}\sum_{j=1}^{N} I_g\left(\tilde{\theta}_j^i\right)\frac{p\left(\tilde{\theta}_j^i\right)}{K_i\left(\tilde{\theta}_j^i\right)}. \tag{23}$$

The variance of each kernel and the whole variance are obtained. According to the central limit theorem and formula for the variance of random variable linear summation, the variance of a failure probability is given by Eqs. (24) and (25). The covariance between each extracted points is ignored.

$$Var\left(\widehat{P}_f\right) = \frac{1}{M_k^2}\sum_{i=1}^{M_k} Var\left(\hat{P}_{k,i}\right), \tag{24}$$

$$Var\left(\hat{P}_{k,i}\right) = \frac{1}{N}\frac{1}{N-1}\sum_{j=1}^{N}\left[I_g\left(\tilde{\theta}_j^i\right)\frac{p\left(\tilde{\theta}_j^i\right)}{K_i\left(\tilde{\theta}_j^i\right)} - \hat{P}_{k,i}\right]^2, \tag{25}$$

$$\text{C.O.V} = \sqrt{Var\left(\hat{P}_f\right)} / \hat{P}_f. \tag{26}$$

The denominator of Eq. (26) converges to a failure probability as N increases according to the strong law of large numbers. The numerator of Eq. (26) decreases with an increasing N. Therefore, C.O.V decreases when N increases.

3. Examples

3.1 Example 1

Eq. (27) is an example given in Au's paper [2]. Random variables are an independent standard normal distribution. The Most probable failure point (MPFP) is the nearest point to the origin in the standard normal space, and it is known that the MPFP peripheral region has a large effect on the failure probability. There are three MPFPs in Eq. (27), but Au and Zhao calculated a failure probability only for two MPFPs [2, 9]. To compare with these previous studies, we also calculated a failure probability for only two MPFPs.

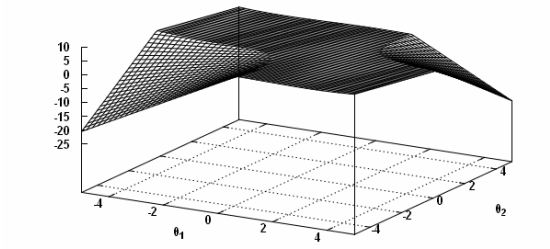
$$\begin{aligned} g &= min\left(g_1, g_2\right), \\ g_1 &= 2 - \theta_2 + exp\left(-\theta_1^2/10\right) + \left(\theta_1/5\right)^4, \\ g_2 &= 4.5 - \theta_1\theta_2. \end{aligned} \tag{27}$$

It is difficult to find two or more MPFPs based on the First order reliability method (FORM). In this study, we could not find MPFPs exactly, but we could find many points in the vicinity of MPFPs more easily by using a Markov chain simulation. With these points, we could approximate the failure probability.

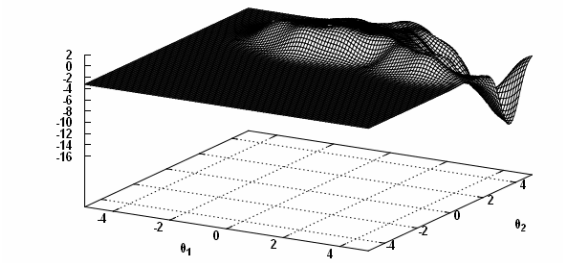
In the Kriging metamodel, **F** in Eq. (9) was assumed as a constant. The criterion of the T value was 0.005. The result is given in Table 1. We can confirm that the proposed method can calculate the failure probability with fewer performance

Table 1. Result of example 1.

	Reference		Proposed method
	Au (1999)	Zhao (2015)	
N_{LSE}	500	163	109
P_f	2.68×10^{-3}	2.69×10^{-3}	2.70×10^{-3}
C.O.V	7 %	5 %	< 5 %



(a) Exact performance function



(b) Approximated performance function by the Kriging metamodel

Fig. 3. Comparison of approximated performance function with the exact performance function of example 1. An approximated performance function of the Kriging metamodel is very different from the exact performance function.

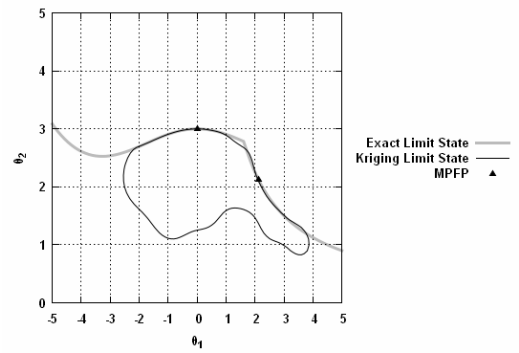


Fig. 4. Comparison of approximated limit state with the exact limit state of example 1. An approximated limit state is similar to the exact limit state in the vicinity of MPFPs.

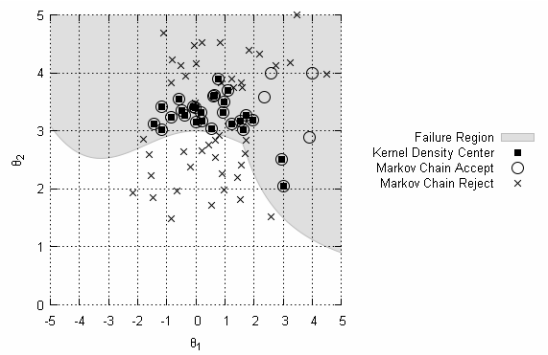


Fig. 5. Markov simulation result and kernel center of example 1. Among the accepted points (Circle) of the Markov simulation, the distant points from MPFP were not selected as the kernel's center (Solid square).

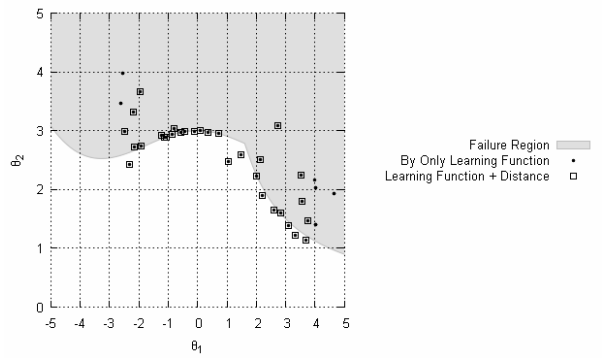


Fig. 6. Added experiment points for the Kriging metamodel of example 1. When considering the statistical distance as well as learning function, the distant points from the MPFP were not selected as the additional experimental points for the Kriging metamodel.

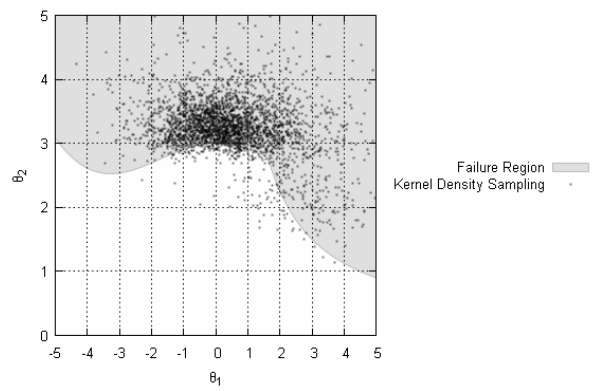


Fig. 7. Sampling points of the kernel density of example 1. Most of the points are extracted from the failure region. This mimics the optimal importance sampling density.

function computations than the previous research results. In Fig. 3, we can compare the exact performance function with the performance function approximated by the Kriging metamodel.

The performance function approximated by the Kriging metamodel is very different from the exact performance function. The overall shape is very different. However, as shown in Fig. 4, the limit state near the MPFPs, which is important in calculating the failure probability, is very similar. In Fig. 4, the limit state of the Kriging metamodel is different from the exact limit state. However, near the MPFPs which have a large effect on failure probability, the limit state of the Kriging metamodel is almost the same as the exact limit state. To reduce numerical cost, it is better to mainly consider the regions that have a large influence on the failure probability. The limit states closer to the MPFPs have more of an influence on the failure probability.

The extracted sampling points are shown in Figs. 5-7. The failure region is painted gray. In Fig. 5, the accepted points (Circle) of a Markov simulation are selected as the center of a kernel (Solid square), except for some points relatively far from MPFPs. This has the effect of reducing the numerical cost by considering mainly the points near the MPFPs that have a large effect on the failure probability. The kernel centers are well distributed around the MPFPs. This makes it possible to approximate the failure probability more accurately.

Among the Markov chain simulation results, both the accepted points and rejected points (Cross) were used as the experimental points for the Kriging metamodel. The added experimental points of the Kriging metamodel are shown in Fig. 6. It can be confirmed that the points added as the experiment points for the Kriging metamodel are selected mainly in the vicinity of the limit state having a large influence on the failure probability.

Echard proposed that additional experimental points for the Kriging metamodel are selected by using the learning functions of Eq. (20) [8]. In this paper, we considered the statistical distance as well as the learning function to select additional experimental points for the Kriging metamodel. Points far from the MPFPs are less influential on the failure probability, so it is better not to consider them to reduce the numerical cost. As shown in Fig. 6, the points far from MPFPs are excluded

Table 2. Results of example 2.

	Reference		Proposed method
	Monte Carlo	MetaAK-IS^2	
N_{LSE}	2.5×10^4	480	451
P_f	7.43×10^{-2}	7.35×10^{-2}	7.53×10^{-2}
C.O.V	2.23 %	2.50 %	3.00 %

from the additional experiments of the Kriging metamodel. This lowers the numerical cost.

In Fig. 7, most of the points extracted from the kernel sampling density are distributed in the failure region. Since Eq. (3) includes an indicator function, only the points in the failure region are extracted in the case of the optimal importance sampling density. From this point of view, the kernel sampling density in Fig. 7 simulates the optimal importance sampling density well.

3.2 Example 2

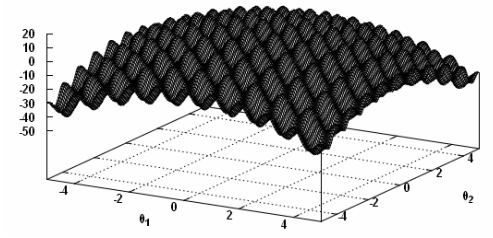
Eq. (28) is an example given in Cadini's paper [10]. Cadini proposed the "MetaAK-IS^2" method, which combines the "metaIS" [11] method and adaptive importance sampling. Random variables are in an independent standard normal distribution. In the Kriging metamodel, **F** in Eq. (9) was assumed as a constant.

The criterion of a T value was 0.1. The results are shown in Table 2. We can confirm that the proposed method calculates the failure probability with fewer performance function computations than the previous research results. In Fig. 8, we compare the performance function approximated by the Kriging metamodel with an exact performance function.

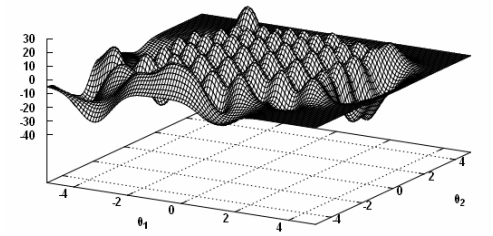
$$g = 10 - \sum_{i=1}^{2}\theta_i^2 - 5\cos\left(2\pi\theta_i\right). \tag{28}$$

The performance function approximated by the Kriging metamodel is very different from the exact performance function. In the approximated performance function, the cosine of the Eq. (28) is reflected, but the second-order polynomial term of Eq. (28) is not reflected. The cosine term is also not well reflected in areas far from the origin. However, as shown in Fig. 9, the limit state near MPFP, which is important in calculating the failure probability, is very similar. In Fig. 9, the limit state of the Kriging metamodel is different from the exact limit state. However, near the MPFPs that have a large effect on the failure probability, the limit state of the Kriging metamodel is almost the same as the exact limit state.

The extracted sampling points are shown in Figs. 10-12. Failure regions are painted gray. The failure regions are very complex and composed of disconnected regions. With such a failure region, it takes many numerical calculations to find all regions that include MPFP. In this example, there are four MPFPs, but we found only one region in the vicinity of MPFP



(a) Exact performance function



(b) Approximated performance function by the Kriging metamodel

Fig. 8. Comparison of approximated performance function with exact performance function of example 2. An approximated performance function of the Kriging metamodel is very different from the exact performance function.

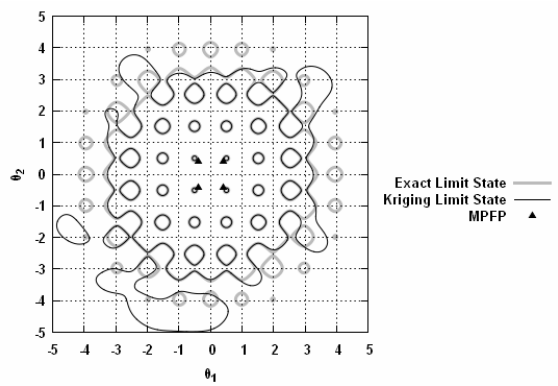


Fig. 9. Comparison of the approximated limit state with the exact limit state of example 2. An approximated limit state is similar to the exact limit state in the vicinity of MPFPs.

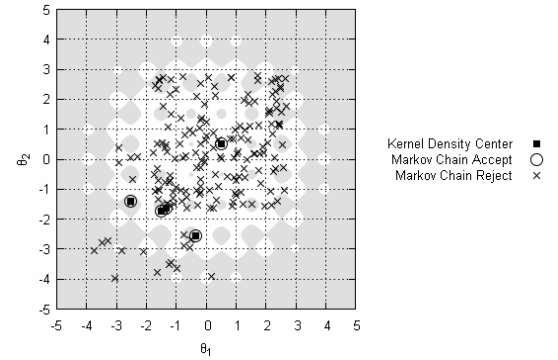


Fig. 10. Markov simulation result and kernel center of example 2. Only one region in the vicinity of MPFP was found.

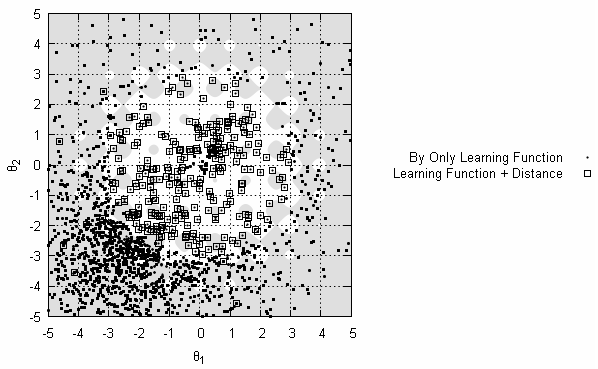


Fig. 11. Added experiment point of the Kriging metamodel of example 2. When considering statistical distance as well as learning function, distant points from MPFP were not selected as additional experimental points for the Kriging metamodel.

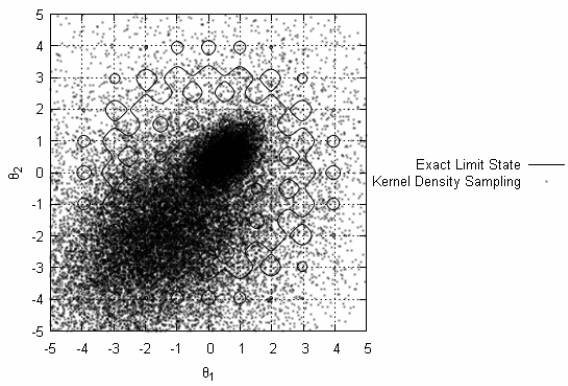


Fig. 12. Sampling points of the kernel density of example 2. Because of the adjusted standard deviation, the kernel density sampling points were distributed in other MPFPs.

and located in the kernel density center at that region.

Although only one kernel density center was generated in the vicinity of a MPFP, the failure probability was calculated correctly within a C.O.V. This is due to the adjustment of the kernel density's standard deviation in Eq. (6). Because of the adjusted standard deviation, the kernel density sampling points were distributed in other MPFPs, as shown in Fig. 12. Additional experimental points of the Kriging metamodel are shown in Fig. 11.

As in the previous example, we selected additional experiment points of the Kriging metamodel by using a learning function and statistical distance. As shown in Fig. 11, points far from the MPFPs are excluded from the additional experiment points of the Kriging metamodel.

3.3 Example 3

The fatigue life of the material has large variation. The equations for the fatigue life predict only an average value of the fatigue life, and there is no information on the variation. To express the variation in the equation, there is a way to assume the parameters included in the fatigue life equation as random variables. In this example, the parameters of the strain-life equation are assumed to be random variables and the fatigue failure probability is calculated.

Table 3. Random variables of example 3.

Random variable	Distribution	Mean	Standard deviation
$\sigma_f^{'}$	Lognormal	709 MPa	Mean × 0.05
b	Lognormal	0.056	
$\varepsilon_f^{'}$	Lognormal	0.12	
c	Lognormal	0.75	
E	Normal	71.7 GPa	
$\Delta\sigma$	Deterministic	820 MPa	0
K'	Deterministic	787 MPa	0
n'	Deterministic	0.07	0

Table 4. Result of example 3.

	Monte Carlo	Proposed method
N_{LSE}	5×10^5	649
P_f	1.00×10^{-3}	1.04×10^{-3}
C.O.V	< 5 %	< 5 %

When the strain is given in Eq. (29), fatigue life can be calculated. To consider the variation of fatigue life, Cho et al. assumed the material constants of Eq. (29) as random variables and calculated the probability of not meeting the predetermined fatigue life [12]. Meggiolaro reported that the fatigue material constants of Eq. (29) are the most similar to the log-logistic distribution for metal materials [13]. In this study, the failure probability for the performance function of Eq. (31) was calculated assuming that the fatigue constants are in a lognormal distribution known to be similar to the log-logistic distribution. The strain's magnitude was calculated by assuming the magnitude of the stress as a constant through Eq. (30).

$$\frac{\Delta\varepsilon}{2}=\frac{\sigma_f^{'}}{E}\left(2N_f\right)^{-b}+\varepsilon_f^{'}\left(2N_f\right),^{-c} \tag{29}$$

$$\Delta\varepsilon=\frac{\Delta\sigma}{E}+2\left(\frac{\Delta\sigma}{2K'}\right)^{\frac{1}{n'}}, \tag{30}$$

$$g=N_f-1000. \tag{31}$$

The results are shown in Table 4. In this study, the performance function was calculated 649 times and the failure probability was 0.104 %. In the Kriging metamodel, **F** in Eq. (9) was assumed to be a constant. The criterion of the T value was 0.01. From considering the Monte Carlo simulation results and C.O.V, we can confirm that the failure probability obtained from the proposed method is a reasonable result.

4. Conclusions

Recently, a failure probability calculation method that applies a Markov chain simulation and a Kriging metamodel to an importance sampling method has been studied by several researchers. The Markov chain simulation is a means to find points in the vicinity of multiple MPFPs. The importance sampling method and Kriging metamodel are methods that can reduce the numerical cost. In this study, we confirmed that the previous evaluation method can be improved by considering the following:

(1) Based on the statistical distance of the points extracted by the Markov chain simulation, only the points closer to the limit state were considered in the failure probability calculation.

(2) After considering the statistical distance as well as learning function, the experimental points of the Kriging metamodel were added.

(3) Log was applied for the stable numerical calculations of the parameter λ_i of a kernel density function needed to construct an importance sampling density.

(4) The completeness of the Kriging metamodel was evaluated on the basis of the possibility of changes in failure probability.

Compared with previous research results that combine the importance sampling method and the Kriging metamodel for the same numerical example, it is confirmed that the number of performance function calculations can be reduced.

From the points extracted from the two-dimensional problem, the kernel sampling density that has the parameter λ_i calculated through an improved numerical method well simulates the ideal importance sampling density.

Nomenclature

g : Limit state function
$\hat{g}$: Predicted value of the limit state function by the Kriging metamodel
h : Importance sampling density
h_{opt} : Optimal importance sampling density
I_g : Indicate function
K_i : Kernel
k : Kernel sampling density
M : Number of accepted points extracted by the Markov chain simulation
M_k : Number of points which used as the kernel center
M_R : Number of rejected points extracted by the Markov chain simulation
N_T : Number of total points from a Kriging metamodel prediction
n : Size of random variable vector
P_f : Failure probability
$\hat{P}_f$: Calculated failure probability
p : Joint probability density of random variable vector
$\mathbf{S}$: Parameter of a kernel sampling density
T : Possibility of a predicted failure probability change
U : Learning function
u_{θ_i} : Value of learning function at point $\tilde{\theta}_i$
w : Parameter of a kernel sampling density
Θ : Domain of possible values of $\tilde{\theta}$
$\tilde{\theta}$: Random variable vector
λ_i : Parameter of a kernel sampling density
$\mathbf{\Sigma_i}$: Covariance matrix of each kernel
Φ : Cumulative distribution function of normal distribution

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