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Kinematics and dynamics analysis of a novel serial-parallel dynamic simulator†

Bo Hu[1,2,*], Liandong Zhang[1,2] and Jingjing Yu[1,2]

[1]*Parallel Robot and Mechatronic System Laboratory of Hebei Province, Yanshan University, Qinhuangdao, Hebei 066004, China*
[2]*Key Laboratory of Advanced Forging & Stamping Technology and Science of Ministry of National Education, Yanshan University, Qinhuangdao, Hebei 066004, China*

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Abstract

A serial-parallel dynamics simulator based on serial-parallel manipulator is proposed. According to the dynamics simulator motion requirement, the proposed serial-parallel dynamics simulator formed by 3-RRS (active revolute joint-revolute joint-spherical joint) and 3-SPR (Spherical joint-active prismatic joint- revolute joint) PMs adopts the outer and inner layout. By integrating the kinematics, constraint and coupling information of the 3-RRS and 3-SPR PMs into the serial-parallel manipulator, the inverse Jacobian matrix, velocity, and acceleration of the serial-parallel dynamics simulator are studied. Based on the principle of virtual work and the kinematics model, the inverse dynamic model is established. Finally, the workspace of the (3-RRS)+(3-SPR) dynamics simulator is constructed.

Keywords: Dynamics simulator; Serial-parallel manipulator; Kinematics; Dynamics

1. Introduction

Dynamics simulators are often used in recreational facilities or devices for simulating the motions of cars and planes. The goal of a dynamics simulator is to give a realistic impression of a driving or flying [1-3]. With the development of mechanism theory, the Stewart platform is used to design a dynamics simulator [4]. The Stewart platform is a good choice for motion platform because it has six Degrees of freedom (DOFs), which can achieve various required motions. In addition, this Parallel manipulator (PM) connects all six legs, forming a closed loop mechanism, which allows the PM to have good accuracy, rigidity and capability of handling a large payload. The idea of a dynamics simulator based on Stewart platform has been demonstrated by very successful applications. Extensive research and application activities have been carried out on the Stewart-like PMs used for dynamics simulators. However, it is quite surprising that little attention has been paid to other novel versions that may be more effective in many practical applications. Motivated by this idea, we present a new concept of series-parallel dynamics simulator, which uses Series-parallel manipulators (S-PMs) as their mechanism body. The proposed concept in this paper uses 3-RRS and 3-SPR PMs and adopts outer and inner layout.

In recent years, the idea of serially connected PMs has been employed to design S-PMs [4-14]. The generated S-PMs have higher stiffness than Serial manipulators (SMs) [4] and a larger workspace than PMs [5-8]. Generally, the PMs included in the S-PMs are selected from some well-known PMs, such as 3-UPU PM [9], 3-RPS PM [10-13], 3-SPR PM [14], Tricept PM [15] and so on, which may lead to some S-PMs [16-20] with good performance. By serially connecting two PMs to form S-PMs, enhanced translational and rotational abilities, high stiffness and huge workspace can be achieved. Based on this concept, a novel (3-RRS)+(3-SPR) S-PM is proposed to design a novel dynamics simulator. Kinematics and dynamics are important issues for dynamics simulator. It is well known that SMs have easy forward kinematics yet difficult inverse kinematics. Inversely, PMs have easy inverse kinematics yet difficult forward kinematics [21, 22]. For the S-PMs, both the forward and the inverse kinematics difficulties are included in S-PMs. In addition, because of their highly nonlinear relations between joint variables and position/orientation of the end effectors for the S-PMs, solving the inverse dynamics of the S-PMs formed by the 3-RRS and 3-SPR PMs is also challenging.

For the above reasons, we aimed at deriving simple and compact formulae for the inverse kinematics velocity, acceleration in compact and explicit form, which is suitable for computer programming, and aims at establishing inverse dynamics for the proposed (3-RRS)+(3-SPR) serial-parallel dynamics simulator. The research provides a theoretical basis for the novel series-parallel dynamics simulator, as well as a feasible approach for establishing the dynamics for other S-PMs.

*Corresponding author. Tel.: +86 13230307516, Fax.: +86 3358057031
E-mail address: hubo@ysu.edu.cn
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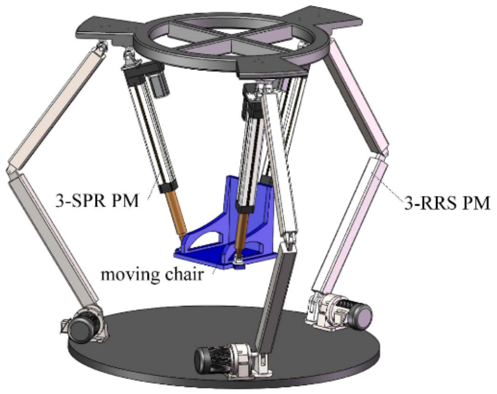


Fig. 1. CAD model of (3-RRS)+(3-SPR) S-PM used for dynamics simulator.

2. Conceptual design of the novel dynamics simulator

The general S-PMs are formed by two PMs connected in serials. The traditional S-PMs adopt the upper and lower layout [5-12]. This layout includes a lower PM and an upper PM connected serially. Different from the traditional layout of S-PMs, the concept of dynamics simulator in this paper adopts an outer and inner layout. Fig. 1 shows a CAD model of the novel (3-RRS)+(3-SPR) serial parallel dynamics simulator, which consists of an outer 3-RRS PM and an inner 3-SPR PM. The motion chair is fixed on the moving platform of the inner PM, which can achieve various required motions such as swinging, lifting, rotation. This device can be used as recreational facility in home theaters, entertainment places and so on.

Compared with traditional dynamics simulators, this concept has these advantages:

(1) This concept has high rotation motion ability because the rotation of the motion chair is the superposition of the outer and inner PMs.

(2) Because the motion chair is located at the inner platform, it has the advantage of compacted structure and small space-occupancy.

3. Displacement analysis of the (3-RRS)+(3-SPR) S-PM

The displacement analysis for the (3-RRS)+(3-SPR) S-PM mechanism includes two parts: The direct displacement analysis and the inverse displacement analysis. The direct displacement analysis is to calculate the pose parameters of the terminal platform relative to the base with the given actuated joint parameters. The inverse position analysis aims to calculate the actuated joint parameters from the given pose of the terminal platform relative to the base. The forward displacement of the (3-RRS)+(3-SPR) S-PM can be easily derived by using superposing method based on the forward displacements of two single PMs. However, the inverse displacement is a difficult work. This section aims at solving the inverse displacement of the (3-RRS)+(3-SPR) S-PM.

Fig. 2 shows the sketch of the (3-RRS)+(3-SPR) S-PM. Let

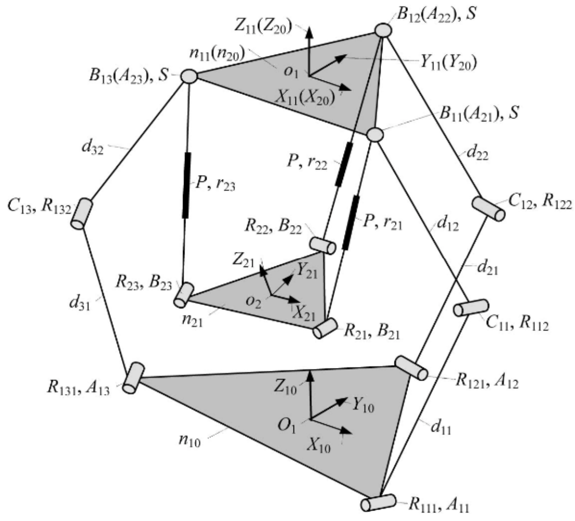


Fig. 2. Sketch of (3-RRS)+(3-SPR) S-PM.

the PM from outer to inner is the i-th PM of the S-PM. Let n_{i0} and n_{i1} (i = 1, 2) be the base and moving platform of PM i, respectively. Then n_{10} and n_{21} denote the base and terminal platform of the whole S-PM, respectively. In structure, n_{11} and n_{20} are fixed with their centers kept coincident (see Fig. 2).

The 3-RRS PM includes a base n_{10}, a moving platform n_{11} and three RRS type driving legs r_{1j} (j = 1, 2, 3). n_{10} and n_{11} are two equilateral triangles. The j-th RRS leg connects n_{10} with n_{11} by a revolute joint R_{1j1} with a rotational actuator at A_{1j}, two serial connected links d_{j1} and d_{j2}, one revolute joint R_{1j2} at points C_{1j} and one spherical joint S at point B_{1j}.The 3-SPR PM includes a base n_{20}, a moving platform n_{21} and three SPR type driving legs r_{2j}(j = 1, 2, 3). n_{20} and n_{21} are two equilateral triangles. The j-th SPR leg connects n_{20} with n_{21} by using a spherical joint S at A_{2j} on n_{20}, a prismatic joint P along r_{2j}, and a revolute joint R_{2j1} at B_{2j} on n_{21}.

Let $\perp$ be the perpendicular constraint and || be the parallel constraint respectively. Establish coordinate frames $\{n_{2j}\}$ (i = 1, 2; j = 0, 1) at the center of n_{ij} with X_{ij}, Y_{ij} and Z_{ij}(i = 1, 2; j = 0, 1) are three orthogonal coordinate axes and some constraints ($X_{ij} \| A_{i1}A_{i3}$, $Y_{ij} \perp A_{i1}A_{i3}$, $Z_{ij} \perp n_{ij}$) are satisfied in i-th PM. The geometrical constraints in the 3-RRS PM can be expressed as follows:

$$\begin{matrix} R_{111} \| R_{112} \| A_{12}A_{13},\ R_{121} \| R_{122} \| A_{11}A_{13}, \\ R_{131} \| R_{132} \| A_{11}A_{12},\ R_{1j1} \perp d_{j1}\ R_{1j1} \perp d_{j2} \end{matrix} \quad (j = 1, 2, 3). \tag{1a}$$

The geometrical constraints in the 3-SPR PM can be expressed as follows:

$$R_{211} \| B_{22}B_{23},\ R_{221} \| B_{21}B_{23},\ R_{231} \| B_{21}B_{22},\ R_{2j1} \perp r_{2j}\ . \tag{1b}$$

For the 3-RRS and the 3-SPR PM, the unit vectors $\boldsymbol{R}_{ij1}$ of R_{ij1} (j = 1, 2, 3) in $\{n_{i0}\}$ can be expressed as follows:

$$
\begin{aligned}
&{}^{n_{i0}}\boldsymbol{R}_{i11} = \frac{1}{2}\begin{bmatrix}1 & q & 0\end{bmatrix}^{\mathrm{T}},\ {}^{n_{i0}}\boldsymbol{R}_{i21} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^{\mathrm{T}},\\
&{}^{n_{i0}}\boldsymbol{R}_{i31} = \frac{1}{2}\begin{bmatrix}-1 & q & 0\end{bmatrix}^{\mathrm{T}},\ q=\sqrt{3}.
\end{aligned} \tag{2}
$$

For the i-th PM, the points $A_{ij}(j = 1, 2, 3)$ in n_{i0} can be expressed as:

$$
\begin{aligned}
&{}^{n_{i0}}\boldsymbol{A}_{i1} = \begin{bmatrix} X_{A_{i1}} \\ Y_{A_{i1}} \\ Z_{A_{i1}} \end{bmatrix} = \frac{E_i}{2}\begin{bmatrix} q \\ -1 \\ 0 \end{bmatrix},\ {}^{n_{i0}}\boldsymbol{A}_{i2} = \begin{bmatrix} X_{A_{i2}} \\ Y_{A_{i2}} \\ Z_{A_{i2}} \end{bmatrix} = \begin{bmatrix} 0 \\ E_i \\ 0 \end{bmatrix},\\
&{}^{n_{i0}}\boldsymbol{A}_{i3} = \begin{bmatrix} X_{A_{i3}} \\ Y_{A_{i3}} \\ Z_{A_{i3}} \end{bmatrix} = -\frac{E_i}{2}\begin{bmatrix} q \\ 1 \\ 0 \end{bmatrix},\ q=\sqrt{3}.
\end{aligned} \tag{3a}
$$

The points $B_{ij}(j = 1, 2, 3)$ in n_{i1} can be expressed as:

$$
\begin{aligned}
&{}^{n_{i1}}\boldsymbol{B}_{i1} = \begin{bmatrix} X_{B_{i1}} \\ Y_{B_{i1}} \\ Z_{B_{i1}} \end{bmatrix} = \frac{e_i}{2}\begin{bmatrix} q \\ -1 \\ 0 \end{bmatrix},\ {}^{n_{i1}}\boldsymbol{B}_{i2} = \begin{bmatrix} X_{B_{i2}} \\ Y_{B_{i2}} \\ Z_{B_{i2}} \end{bmatrix} = \begin{bmatrix} 0 \\ e_i \\ 0 \end{bmatrix},\\
&{}^{n_{i1}}\boldsymbol{B}_{i3} = \begin{bmatrix} X_{B_{i3}} \\ Y_{B_{i3}} \\ Z_{B_{i3}} \end{bmatrix} = -\frac{e_i}{2}\begin{bmatrix} q \\ 1 \\ 0 \end{bmatrix}
\end{aligned} \tag{3b}
$$

where E_i denotes the distance from O_i to A_i, e_i denotes the distance from o_i to B_i.

Let ${}^{n_{i0}}_{n_{i1}}\mathrm{R}$ denote the rotational matrix of n_{i1} relative to n_{i0}. Let ${}^{n_{i0}}_{n_{i1}}\mathrm{R}$ be formed by XYX Euler rotations with α_i, β_i and λ_i are three Euler angles; it leads to

$$
\begin{aligned}
{}^{n_{i0}}_{n_{i1}}\mathbf{R} &= \begin{bmatrix} {}^{n_{i0}}x_{li} & {}^{n_{i0}}y_{li} & {}^{n_{i0}}z_{li} \\ {}^{n_{i0}}x_{mi} & {}^{n_{i0}}y_{mi} & {}^{n_{i0}}z_{mi} \\ {}^{n_{i0}}x_{ni} & {}^{n_{i0}}y_{ni} & {}^{n_{i0}}z_{ni} \end{bmatrix}\\
&= \begin{bmatrix} c_{\beta_i} & s_{\beta_i}s_{\lambda_i} & s_{\beta_i}c_{\lambda_i} \\ s_{\alpha_i}s_{\beta_i} & c_{\alpha_i}c_{\lambda_i} - s_{\alpha_i}c_{\beta_i}s_{\lambda_i} & -c_{\alpha_i}s_{\lambda_i} - s_{\alpha_i}c_{\beta_i}c_{\lambda_i} \\ -c_{\alpha_i}s_{\beta_i} & s_{\alpha_i}c_{\lambda_i} + c_{\alpha_i}c_{\beta_i}s_{\lambda_i} & -s_{\alpha_i}s_{\lambda_i} + c_{\alpha_i}c_{\beta_i}c_{\lambda_i} \end{bmatrix}
\end{aligned} \tag{3c}
$$

where (${}^{n_{i0}}x_{li}$ ${}^{n_{i0}}x_{mi}$ ${}^{n_{i0}}x_{ni}$ ${}^{n_{i0}}y_{li}$ ${}^{n_{i0}}y_{mi}$ ${}^{n_{i0}}y_{ni}$ ${}^{n_{i0}}z_{li}$ ${}^{n_{i0}}z_{mi}$ ${}^{n_{i0}}z_{ni}$) are nine orientation parameters of ${}^{n_{i0}}_{n_{i1}}\mathbf{R}$.

A composite rotational matrix ${}^{n_{10}}_{n_{21}}\mathbf{R}$ from n_{21} relative to n_{10} can be expressed as follows:

$$
{}^{n_{10}}_{n_{21}}\mathbf{R} = {}^{n_{10}}_{n_{11}}\mathbf{R}\,{}^{n_{11}}_{n_{20}}\mathbf{R}\,{}^{n_{20}}_{n_{21}}\mathbf{R} = \begin{bmatrix} {}^{n_{10}}x_{l2} & {}^{n_{10}}y_{l2} & {}^{n_{10}}z_{l2} \\ {}^{n_{10}}x_{m2} & {}^{n_{10}}y_{m2} & {}^{n_{10}}z_{m2} \\ {}^{n_{10}}x_{n2} & {}^{n_{10}}y_{n2} & {}^{n_{10}}z_{n2} \end{bmatrix} \tag{4a}
$$

where

$$
{}^{n_{11}}_{n_{20}}\mathbf{R} = \mathbf{E}_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4b}
$$

The position vectors $\boldsymbol{B}_{ij}$ (j = 1, 2, 3) in $\{n_{i0}\}$ for each PM can be expressed as follows:

$$
{}^{n_{i0}}\boldsymbol{B}_{ij} = \begin{bmatrix} {}^{n_{i0}}X_{B_{ij}} \\ {}^{n_{i0}}Y_{B_{ij}} \\ {}^{n_{i0}}Z_{B_{ij}} \end{bmatrix} = {}^{n_{i0}}_{n_{i1}}\mathbf{R}\ {}^{n_{i1}}\boldsymbol{B}_{ij} + {}^{n_{i0}}\boldsymbol{o}_i,\ {}^{n_{i0}}\boldsymbol{o}_i = \begin{bmatrix} {}^{n_{i0}}X_{o_i} \\ {}^{n_{i0}}Y_{o_i} \\ {}^{n_{i0}}Z_{o_i} \end{bmatrix}. \tag{5a}
$$

The center of o_2 relative to base n_{10} can be expressed as follows:

$$
{}^{n_{10}}\mathbf{o}_2 = {}^{n_{10}}\mathbf{o}_1 + {}^{n_{10}}_{n_{20}}\mathbf{R}\ {}^{n_{20}}\mathbf{o}_2,\quad {}^{n_{10}}_{n_{20}}\mathbf{R} = {}^{n_{10}}_{n_{11}}\mathbf{R}\ {}^{n_{11}}_{n_{20}}\mathbf{R} = {}^{n_{10}}_{n_{11}}\mathbf{R}. \tag{5b}
$$

When ${}^{n_{10}}\mathbf{o}_2$ and ${}^{n_{10}}_{n_{20}}\mathbf{R}$ are given, the position vectors $\boldsymbol{B}_{2j}$(j = 1, 2, 3) of the inner PM in $\{n_{10}\}$ can be expressed as follows:

$$
{}^{n_{10}}\boldsymbol{B}_{2j} = \begin{bmatrix} {}^{n_{10}}X_{B_{1j}} \\ {}^{n_{10}}Y_{B_{1j}} \\ {}^{n_{10}}Z_{B_{1j}} \end{bmatrix} = {}^{n_{10}}_{n_{21}}\mathbf{R}\ {}^{n_{21}}\boldsymbol{B}_{2j} + {}^{n_{10}}\mathbf{o}_2,\ {}^{n_{10}}\mathbf{o}_2 = \begin{bmatrix} {}^{n_{10}}X_{o_2} \\ {}^{n_{10}}Y_{o_2} \\ {}^{n_{10}}Z_{o_2} \end{bmatrix}. \tag{5c}
$$

3.1 The inverse kinematics analysis

From Eqs. (1a) and (1b) it leads to,

$$
{}^{n_{i0}}\boldsymbol{R}_{ij1}\cdot({}^{n_{i0}}\boldsymbol{B}_{ij} - {}^{n_{i0}}\boldsymbol{A}_{ij}) = 0\ (i = 1, 2; j = 1, 2, 3)\,. \tag{6}
$$

When i = 1, from Eqs. (3a), (3b), (5a) and (6),

$$
{}^{n_{10}}X_{B_{11}} = -q\,{}^{n_{10}}Y_{B_{11}},\ {}^{n_{10}}X_{B_{12}} = {}^{n_{10}}X_{A_{12}} = 0,\ {}^{n_{10}}X_{B_{13}} = q\,{}^{n_{10}}Y_{B_{13}}. \tag{7}
$$

When i = 2, from Eqs. (3a), (3b), (5a) and (6),

$$
\begin{aligned}
&{}^{n_{10}}Z_{B_{11}} = s_{11} + s_{12}\,{}^{n_{10}}Y_{B_{11}},\\
&s_{11} = [({}^{n_{20}}x_{l2} + q\,{}^{n_{20}}y_{l2})\,{}^{n_{10}}X_{B_{21}} + ({}^{n_{20}}x_{m2} + q\,{}^{n_{20}}y_{m2})\,{}^{n_{10}}Y_{B_{21}}\\
&\quad + ({}^{n_{20}}x_{n2} + q\,{}^{n_{20}}y_{n2})\,{}^{n_{10}}Z_{B_{21}}]/({}^{n_{20}}x_{n2} + q\,{}^{n_{20}}y_{n2}),\\
&s_{12} = (q\,{}^{n_{20}}x_{l2} + 3\,{}^{n_{20}}y_{l2} - {}^{n_{20}}x_{m2} - q\,{}^{n_{20}}y_{m2})\,/\,({}^{n_{20}}x_{n2} + q\,{}^{n_{20}}y_{n2}),
\end{aligned} \tag{8a}
$$

$$
\begin{aligned}
&{}^{n_{10}}Z_{B_{12}} = s_{21} + s_{22}\,{}^{n_{10}}Y_{B_{12}},\\
&s_{21} = ({}^{n_{20}}x_{l2}\,{}^{n_{10}}X_{B_{22}} + {}^{n_{20}}x_{m2}\,{}^{n_{10}}Y_{B_{22}} + {}^{n_{20}}x_{n2}\,{}^{n_{10}}Z_{B_{22}})\,/\,{}^{n_{20}}x_{n2},\\
&s_{22} = -{}^{n_{20}}x_{m2}\,/\,{}^{n_{20}}x_{n2}
\end{aligned} \tag{8b}
$$

$$
\begin{aligned}
&{}^{n_{10}}Z_{B_{13}} = s_{31} + s_{32}\,{}^{n_{10}}Y_{B_{13}}\\
&s_{31} = [(-{}^{n_{20}}x_{l2} + q\,{}^{n_{20}}y_{l2})\,{}^{n_{10}}X_{B_{23}} - ({}^{n_{20}}x_{m2} - q\,{}^{n_{20}}y_{m2})\,{}^{n_{10}}Y_{B_{23}}\\
&\quad - ({}^{n_{20}}x_{n2} - q\,{}^{n_{20}}y_{n2})\,{}^{n_{10}}Z_{B_{23}}]/(-{}^{n_{20}}x_{n2} + q\,{}^{n_{20}}y_{n2}),\\
&s_{32} = (q\,{}^{n_{20}}x_{l2} - 3\,{}^{n_{20}}y_{l2} + {}^{n_{20}}x_{m2} - q\,{}^{n_{20}}y_{m2})\,/\,(-{}^{n_{20}}x_{n2} + q\,{}^{n_{20}}y_{n2})\,.
\end{aligned} \tag{8c}
$$

When ${}^{n_{10}}\mathbf{o}_2$ and ${}^{n_{10}}_{n_{20}}\mathbf{R}$ are given, ${}^{n_{10}}\boldsymbol{B}_{2j}$ can be easily solved from Eq. (5c) and then $s_{ij}(i = 1,2,3; j = 1,2,3,4,5)$ in Eqs. (8a)-(8c) can be easily obtained.

The points B_{1j} have the dimensional constraints as follows:

$$({}^{n_{10}}\boldsymbol{B}_{11} - {}^{n_{10}}\boldsymbol{B}_{12}) \cdot ({}^{n_{10}}\boldsymbol{B}_{11} - {}^{n_{10}}\boldsymbol{B}_{12}) = 3e_1^2 \tag{9a}$$

$$({}^{n_{10}}\boldsymbol{B}_{13} - {}^{n_{10}}\boldsymbol{B}_{12}) \cdot ({}^{n_{10}}\boldsymbol{B}_{13} - {}^{n_{10}}\boldsymbol{B}_{12}) = 3e_1^2 \tag{9b}$$

$$({}^{n_{10}}\boldsymbol{B}_{11} - {}^{n_{10}}\boldsymbol{B}_{13}) \cdot ({}^{n_{10}}\boldsymbol{B}_{11} - {}^{n_{10}}\boldsymbol{B}_{13}) = 3e_1^2 . \tag{9c}$$

By substituting Eqs. (7), (8a)-(8c) into Eqs. (9a)-(9c),

$$\begin{gathered} u_{12}\,{}^{n_{10}}Y_{B_{11}}^2 + u_{11}\,{}^{n_{10}}Y_{B_{11}} + u_{10} = 0, \\ u_{12} = p_{15},\ u_{11} = p_{12} + p_{13}\,{}^{n_{10}}Y_{B_{12}},\ u_{10} = p_{14}\,{}^{n_{10}}Y_{B_{12}}^2 + p_{11}\,{}^{n_{10}}Y_{B_{12}} + p_{10}, \\ p_{15} = 4 + s_{12}^2,\ p_{14} = 1 + s_{22}^2, \\ p_{13} = -2 - 2s_{12}s_{22},\ p_{12} = -2s_{22}(s_{11} - s_{21}),\ p_{11} = 2s_{12}(s_{11} - s_{21}), \\ p_{10} = (s_{11} - s_{21})^2 - 3e_1^2, \end{gathered} \tag{10a}$$

$$\begin{gathered} u_{22}\,{}^{n_{10}}Y_{B_{13}}^2 + u_{21}\,{}^{n_{10}}Y_{B_{13}} + u_{20} = 0, \\ u_{22} = p_{25},\ u_{21} = p_{23} + p_{22}\,{}^{n_{10}}Y_{B_{12}},\ u_{20} = p_{24}\,{}^{n_{10}}Y_{B_{12}}^2 + p_{21}\,{}^{n_{10}}Y_{B_{12}} + p_{20}, \\ p_{25} = 4 + s_{32}^2,\ p_{24} = 1 + s_{22}^2, \\ p_{23} = -2 - 2s_{32}s_{22},\ p_{22} = 2s_{32}(s_{31} - s_{21}),\ p_{21} = -2s_{22}(s_{31} - s_{21}), \\ p_{20} = (s_{31} - s_{21})^2 - 3e_1^2, \end{gathered} \tag{10b}$$

$$\begin{gathered} u_{32}\,{}^{n_{10}}Y_{B_{13}}^2 + u_{31}\,{}^{n_{10}}Y_{B_{13}} + u_{30} = 0, \\ u_{32} = p_{35},\ u_{31} = p_{32} + p_{33}\,{}^{n_{10}}Y_{B_{11}},\ u_{30} = p_{34}\,{}^{n_{10}}Y_{B_{11}}^2 + p_{31}\,{}^{n_{10}}Y_{B_{11}} + p_{30}, \\ p_{35} = 4 + s_{32}^2,\ p_{34} = 4 + s_{12}^2, \\ p_{33} = 4 - 2s_{32}s_{12}, p_{32} = 2s_{32}(s_{31} - s_{11}),\ p_{31} = -2s_{12}(s_{31} - s_{11}), \\ p_{30} = (s_{31} - s_{11})^2 - 3e_1^2. \end{gathered} \tag{10c}$$

Here, ${}^{n_{10}}Y_{B_{11}}$, ${}^{n_{10}}Y_{B_{12}}$ and ${}^{n_{10}}Y_{B_{13}}$ are three unknowns in Eqs. (10a)-(10c), $u_{1j}(i = 1, 2)$ are the polynomials in ${}^{n_{10}}Y_{B_{12}}$, u_{3j} are the polynomials in ${}^{n_{10}}Y_{B_{11}}$.

From Eqs. (10b) and (10c),

$$t_{24}\,{}^{n_{10}}Y_{B_{11}}^4 + t_{23}\,{}^{n_{10}}Y_{B_{11}}^3 + t_{22}\,{}^{n_{10}}Y_{B_{11}}^2 + t_{21}\,{}^{n_{10}}Y_{B_{11}} + t_{20} = 0 \tag{11}$$

where

$$\begin{aligned} & t_4 = u_{22}^2 p_{34}^2,\ t_3 = u_{22}p_{34}(u_{21}p_{33} + 2u_{22}p_{31}), \\ & t_2 = u_{22}^2(p_{31}^2 + 2p_{30}p_{34}) - p_{34}(2u_{20}u_{22}u_{32} + u_{21}^2u_{32}) \\ & \quad + u_{21}u_{22}(p_{33}p_{31} + p_{32}p_{34}) - u_{20}u_{22}p_{33}^2, \\ & t_1 = 2u_{22}^2 p_{30}p_{31} - 2u_{20}u_{22}(p_{32}p_{33} + p_{31}u_{32}) - u_{21}^2 u_{32}p_{31} \\ & \quad + u_{21}u_{22}(p_{33}p_{30} + p_{32}p_{31}) + u_{20}u_{21}u_{32}p_{33}, \\ & t_0 = (u_{22}p_{30} - u_{20}u_{32})^2 + (u_{22}p_{32} - u_{32}u_{21})(u_{21}p_{30} - u_{20}p_{32}), \end{aligned}$$

here t_j $(j = 0,1,\ldots,4)$ are polynomials in ${}^{n_{10}}Y_{B_{12}}$.

From Eqs. (10a) and (11),

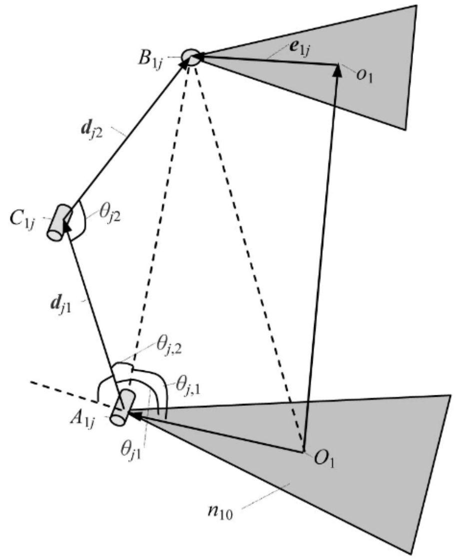


Fig. 3. Schematic representation of the RRS type leg.

$$\mathrm{M}\begin{bmatrix} b_{1y}^5 \\ b_{1y}^4 \\ b_{1y}^3 \\ b_{1y}^2 \\ b_{1y} \\ 1 \end{bmatrix} = 0,\ \mathrm{M} = \begin{bmatrix} 0 & 0 & 0 & u_{12} & u_{11} & u_{10} \\ 0 & 0 & u_{12} & u_{11} & u_{10} & 0 \\ 0 & u_{12} & u_{11} & u_{10} & 0 & 0 \\ u_{12} & u_{11} & u_{10} & 0 & 0 & 0 \\ 0 & t_4 & t_3 & t_2 & t_1 & t_0 \\ t_4 & t_3 & t_2 & t_1 & t_0 & 0 \end{bmatrix}. \tag{12}$$

The necessary condition for Eq. (12) to have nontrivial solutions is

$$\det(\mathrm{M}) = 0 . \tag{13}$$

Eq. (13) is a nonlinear equation with regard to ${}^{n_{10}}Y_{B_{12}}$. By using Matlab software, the unknown in Eq. (13) can be easily solved. Expanding Eq. (13) results in an eighth-degree polynomial in $Y_{B_{12}}$. It follows that there are at most eight solutions for $Y_{B_{12}}$. When ${}^{n_{10}}\boldsymbol{o}_2$ and ${}^{n_{10}}_{n_{20}}\mathbf{R}$ are given, ${}^{n_{10}}Y_{B_{12}}$ can be solved from Eq. (13) and then ${}^{n_{10}}Y_{B_{11}}$ and ${}^{n_{10}}Y_{B_{13}}$ can be solved from Eqs. (11b) and (11c), respectively. Other coordinate parameters of $B_{1j}(j = 1,2,3)$ can be derived from Eqs. (7), (8a)-(8c).

After $B_{1j}(j = 1,2,3)$ are derived, the actuator angles of the 3-RRS can be solved. Let θ_{j1} be the rotational angle of R_{1j1}. Since $d_{j1} \perp R_{1j1}$, $O_1A_{1j} \perp R_{1j1}$, θ_{j1} is the angle between d_{j1} and O_1A_{1j}, θ_{j1} can be expressed as:

$$\theta_{j1} = \theta_{j1,1} + \theta_{j1,2} \tag{14a}$$

where $\theta_{j1,1}$ denotes the angle between $A_{1j}O_1$ and $B_{j1}A_{j1}$, $\theta_{j1,2}$ denotes the angle between $B_{j1}A_{j1}$ and d_{j1}.

From Fig. 3, $\theta_{j1,1}$, $\theta_{j1,2}$, θ_{j1} and θ_{j2} can be expressed as follows:

$$\theta_{j1,1} = \arccos(\frac{E_j^2 + L_j^2 - |O_1B_{1j}|^2}{2E_jL_j}),\ \theta_{j1,2} = \arccos(\frac{d_{j1}^2 + L_j^2 - d_{j2}^2}{2d_{j1}L_j}),$$
$$\theta_{j1} = \theta_{j1,1} + \theta_{j1,2} = \arccos(\frac{E_j^2 + L_j^2 - |O_1B_{1j}|^2}{2E_jL_j}) + \arccos(\frac{d_{j1}^2 + L_j^2 - d_{j2}^2}{2d_{j1}L_j}),$$
$$\theta_{j2} = \arccos(\frac{d_{j1}^2 + d_{j2}^2 - L_j^2}{2d_{j1}d_{j2}}). \tag{14b}$$

For the 3-SPR PM, the length of r_{2j} (j = 1, 2, 3) can be derived as follows:

$$r_{2j} = \left| {}^{n_{20}}\mathbf{B}_{2j} - {}^{n_{20}}\mathbf{A}_{2j} \right|. \tag{15}$$

From Eqs. (14b) and (15), the inverse displacement of the 3-RRS+3SPR S-PM can be solved.

3.2 The pose decoupling equations of the 3-RRS PM

For the 3-RRS PM, from Eqs. (3a), (3b), (5a) and (6) it leads to,

$${}^{n_{10}}X_{o_1} = -e_1{}^{n_{10}}y_{l_1} \tag{16a}$$
$${}^{n_{10}}Y_{o_1} = e_1({}^{n_{10}}y_{m_1} - {}^{n_{10}}x_{l_1})/2 \tag{16b}$$
$${}^{n_{10}}x_{m_1} = {}^{n_{10}}y_{l_1},\ {}^{n_{10}}x_{n_1} = -{}^{n_{10}}z_{l_1},\ {}^{n_{10}}y_{n_1} = -{}^{n_{10}}z_{m_1}. \tag{16c}$$

From Eqs. (3c) and (16c) it leads to,

$$\alpha_1 = \lambda_1. \tag{17a}$$

From Eqs. (3c) and (17a), ${}^{n_{10}}_{n_{11}}\mathbf{R}$ for the 3-RRS PM can be simplified as the following:

$${}^{n_{10}}_{n_{11}}\mathbf{R} = \begin{bmatrix} c_{\beta_1} & s_{\beta_1}s_{\alpha_1} & s_{\beta_1}c_{\alpha_1} \\ s_{\alpha_1}s_{\beta_1} & c_{\alpha_1}^2 - s_{\alpha_1}^2c_{\beta_1} & -c_{\alpha_1}s_{\alpha_1} - s_{\alpha_1}c_{\beta_1}c_{\alpha_1} \\ -c_{\alpha_1}s_{\beta_1} & s_{\alpha_1}c_{\alpha_1} + c_{\alpha_1}c_{\beta_1}s_{\alpha_1} & -s_{\alpha_1}^2 + c_{\alpha_1}^2c_{\beta_1} \end{bmatrix}. \tag{17b}$$

From Eqs. (16a)-(16c) and (17b),

$${}^{n_{10}}X_{o_1} = -e_1s_{\alpha_1}s_{\beta_1} \tag{18a}$$
$${}^{n_{10}}Y_{o_1} = e_1(c_{\alpha_1}^2 - s_{\alpha_1}^2c_{\beta_1} - c_{\beta_1})/2 \tag{18b}$$
$${}^{n_{10}}Z_{o_1} = {}^{n_{10}}Z_{o_1}. \tag{18c}$$

3.3 The pose decoupling equations for the 3-SPR PM

From Eqs. (3a), (3b), (5a) and (6),

$${}^{n_{20}}X_{o_2} = \frac{E_2x_{m_2}(3y_{m_2} - x_{l_2}) + 2{}^{n_{20}}Z_{o_2}z_{l_2}}{2z_{n_q}}$$
$${}^{n_{20}}Y_{o_2} = \frac{E_2x_{l_2}(x_{l_2} - y_{m_2}) - 2E_2y_{l_2}x_{m_2} + 2{}^{n_{20}}Z_{o_2}z_{m_2}}{2z_{n_2}}. \tag{19a}$$
$$x_{m_2} = y_{l_2}$$

From Eqs. (3c) and (19a), ${}^{n_{20}}_{n_{21}}\mathbf{R}$ for the 3-SPR PM can be simplified as follows:

$${}^{n_{20}}_{n_{21}}\mathrm{R} = \begin{bmatrix} c_{\beta_2} & s_{\beta_2}s_{\alpha_2} & s_{\beta_2}c_{\alpha_2} \\ s_{\alpha_2}s_{\beta_2} & c_{\alpha_2}^2 - s_{\alpha_2}^2c_{\beta_2} & -c_{\alpha_2}s_{\alpha_2} - s_{\alpha_2}c_{\beta_2}c_{\alpha_2} \\ -c_{\alpha_2}s_{\beta_2} & s_{\alpha_2}c_{\alpha_2} + c_{\alpha_2}c_{\beta_2}s_{\alpha_2} & -s_{\alpha_2}^2 + c_{\alpha_2}^2c_{\beta_2} \end{bmatrix}. \tag{19b}$$

From Eqs. (19a) and (19b) it leads to,

$${}^{n_{20}}X_{o_2} = \frac{E_2s_{\alpha_2}s_{\beta_2}[3(c_{\alpha_2}^2 - s_{\alpha_2}^2c_{\beta_2}) - c_{\beta_2}] + 2{}^{n_{20}}Z_{o_2}s_{\beta_2}c_{\alpha_2}}{2(-s_{\alpha_2}^2 + c_{\alpha_2}^2c_{\beta_2})}$$
$${}^{n_{20}}Y_{o_2} = \frac{E_2c_{\beta_2}(c_{\beta_2} - c_{\alpha_2}^2 + s_{\alpha_2}^2c_{\beta_2}) - 2Es_{\alpha_2}^2s_{\beta_2}^2 - 2{}^{n_{20}}Z_{o_2}c_{\alpha_2}s_{\alpha_2}(1 + c_{\beta_2})}{2(-s_{\alpha_2}^2 + c_{\alpha_2}^2c_{\beta_2})}. \tag{20}$$

4. Velocity analysis of the (3-RRS)+(3-SPR) S-PM

4.1 Velocity constraint and decoupling analysis of 3-RRS PM

From Eqs. (18a)-(18c), the linear velocity of n_{11} relative to $\{n_{10}\}$ of the 3-RRS PM can be expressed as:

$${}^{n_{10}}\boldsymbol{v}_{o1} = \mathbf{J}_{v1}\begin{bmatrix} \dot{\alpha}_1 \\ \dot{\beta}_1 \\ \dot{Z}_{o1} \end{bmatrix},\ \mathbf{J}_{v1} = \begin{bmatrix} -e_{11}c_{\alpha_1}s_{\beta_1} & -e_{11}s_{\alpha_1}c_{\beta_1} & 0 \\ -e_{11}c_{\alpha_1}s_{\alpha_1}(1 + c_{\beta_1}) & e_{11}s_{\beta_1}(s_{\alpha_1}^2 + 1)/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{21a}$$

From Eqs. (3c) and (17b), the angular velocity of n_{11} relative to $\{n_{10}\}$ of the 3-RRS PM can be expressed as:

$${}^{n_{10}}_{n_{11}}\boldsymbol{\omega} = {}^{n_{10}}\boldsymbol{R}_{\alpha_1}\dot{\alpha}_1 + {}^{n_{10}}\boldsymbol{R}_{\beta_1}\dot{\beta}_1 + {}^{n_{10}}\boldsymbol{R}_{\lambda_1}\dot{\lambda}_1 = \mathbf{J}_{\omega 1}{}^{n_{10}}\boldsymbol{v}_{s_1},\ \mathbf{J}_{\omega 1} = \begin{bmatrix} 1 + c_{\beta_1} & 0 & 0 \\ s_{\alpha_1}s_{\beta_1} & c_{\alpha_1} & 0 \\ -c_{\alpha_1}s_{\beta_1} & s_{\alpha_1} & 0 \end{bmatrix} \tag{21b}$$

here, ${}^{n_{10}}\boldsymbol{R}_{\alpha_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, ${}^{n_{10}}\boldsymbol{R}_{\beta_1} = \begin{bmatrix} 0 \\ c_{\alpha_1} \\ s_{\alpha_1} \end{bmatrix}$, ${}^{n_{10}}\boldsymbol{R}_{\lambda_1} = \begin{bmatrix} c_{\beta_1} \\ s_{\alpha_1}s_{\beta_1} \\ -c_{\alpha_1}s_{\beta_1} \end{bmatrix}$

where ${}^{n_{10}}\boldsymbol{R}_{\alpha_1}$, ${}^{n_{10}}\boldsymbol{R}_{\beta_1}$ and ${}^{n_{10}}\boldsymbol{R}_{\lambda_1}$ are the unit vectors of the axes along α_1, β_1 and λ_1, respectively.

From Eqs. (21a) and (21b) it leads to,

$$\begin{bmatrix} {}^{n_{10}}\boldsymbol{v}_{o_1} \\ {}^{n_{10}}_{n_{11}}\boldsymbol{\omega} \end{bmatrix} = \mathbf{J}_{o_1}\begin{bmatrix} \dot{\alpha}_1 \\ \dot{\beta}_1 \\ {}^{n_{10}}\dot{Z}_{o_1} \end{bmatrix},\ \mathbf{J}_{o_1} = \begin{bmatrix} \mathbf{J}_{v_1} \\ \mathbf{J}_{\omega_1} \end{bmatrix} \tag{21c}$$

Here, $\mathbf{J}o_1$ is a 6×3 form velocity decoupling matrix of the 3-RRS PM.

For the 3-RRS and 3-SPR PMs, the constrained forces/torques which restrain the velocity of the terminal platform of PMs exist. The velocity constraint equations can be obtained by analyzing the constrained forces/torques in the PMs using geometrical approach [14]. From the geometrical approach for determining constrained wrenches, one constrained force which is parallel with R_{1j} and passes through S joint can be determined in each RRS leg. As the constrained forces/torques do no work to the moving platform n_{i1}, the velocity constraint equation for the 3-RRS PM can be determined as the following [14]:

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \mathbf{J}_{\beta_1}\begin{bmatrix}{}^{n_{10}}\boldsymbol{v}_{o_1}\\ {}^{n_{10}}_{n_{11}}\boldsymbol{\omega}\end{bmatrix},\ \mathbf{J}_{\beta_1} = \begin{bmatrix}{}^{n_{10}}\boldsymbol{R}_{111}^{\mathrm{T}} & ({}^{n_{10}}\boldsymbol{d}_{11}\times{}^{n_{10}}\boldsymbol{R}_{111})^{\mathrm{T}}\\ {}^{n_{10}}\boldsymbol{R}_{121}^{\mathrm{T}} & ({}^{n_{10}}\boldsymbol{d}_{12}\times{}^{n_{10}}\boldsymbol{R}_{121})^{\mathrm{T}}\\ {}^{n_{10}}\boldsymbol{R}_{131}^{\mathrm{T}} & ({}^{n_{10}}\boldsymbol{d}_{13}\times{}^{n_{10}}\boldsymbol{R}_{131})^{\mathrm{T}}\end{bmatrix}. \tag{22}$$
$${}^{n_{10}}\boldsymbol{d}_{1j} = {}^{n_{10}}\boldsymbol{B}_{1j} - {}^{n_{10}}\boldsymbol{o}_1$$

Here, $\mathbf{J}_{\beta 1}$ is a 3×6 form velocity constraint matrix for the 3-RRS PM.

4.2 Velocity constraint and decoupling analysis of 3-SPR PM

From Eq. (20), the linear velocity of n_{21} relative to $\{n_{20}\}$ of the 3-SPR PM can be expressed as follows:

$${}^{n_{20}}\boldsymbol{v}_{o_2} = \mathbf{J}_{v_2}\begin{bmatrix}\dot{\alpha}_2\\ \dot{\beta}_2\\ {}^{n_{20}}\dot{Z}_{o_2}\end{bmatrix},\ \mathbf{J}_{v_2} = \begin{bmatrix}\dfrac{\partial^{n_{20}}X_{o_2}}{\partial\alpha_2} & \dfrac{\partial^{n_{20}}X_{o_2}}{\partial\beta_2} & \dfrac{\partial^{n_{20}}X_{o_2}}{\partial^{n_{20}}Z_{o_2}}\\ \dfrac{\partial^{n_{20}}Y_{o_2}}{\partial\alpha_2} & \dfrac{\partial^{n_{20}}Y_{o_2}}{\partial\beta_2} & \dfrac{\partial^{n_{20}}Y_{o_2}}{\partial Z_{o_2}}\\ 0 & 0 & 1\end{bmatrix} \tag{23a}$$

$$\frac{\partial^{n_{20}}X_{o_2}}{\partial\alpha_2} = \frac{Ec_{\alpha_2}s_{\beta_2}(3c_{\alpha_2}^2 - 6s_{\alpha_2}^2 - 9s_{\alpha_2}^2c_{\beta_2} - c_{\beta_2}) - 2^{n_{20}}Z_{o_2}s_{\beta_2}s_{\alpha_2} + 4s_{\alpha_2}c_{\alpha_2}(1+c_{\beta_2})^{n_{20}}X_{o_2}}{2(-s_{\alpha_2}^2 + c_{\alpha_2}^2c_{\beta_2})}$$
$$\frac{\partial^{n_{20}}X_{o_2}}{\partial\beta_2} = \frac{3Es_{\alpha_2}c_{\beta_2}c_{\alpha_2}^2 + Es_{\alpha_2}(3s_{\alpha_2}^2+1)(s_{\beta_2}^2 - c_{\beta_2}^2) + 2^{n_{20}}Z_{o_2}c_{\beta_2}c_{\alpha_2} + 2c_{\alpha_2}^2s_{\beta_2}{}^{n_{20}}X_{o_2}}{2(-s_{\alpha_2}^2 + c_{\alpha_2}^2c_{\beta_2})}$$
$$\frac{\partial^{n_{20}}X_{o_2}}{\partial^{n_{20}}Z_{o_2}} = \frac{s_{\beta_2}c_{\alpha_2}}{(-s_{\alpha_2}^2 + c_{\alpha_2}^2c_{\beta_2})},\ \frac{\partial^{n_{20}}Y_{o_2}}{\partial^{n_{20}}Z_{o_2}} = -\frac{c_{\alpha_2}s_{\alpha_2}(1+c_{\beta_2})}{(-s_{\alpha_2}^2 + c_{\alpha_2}^2c_{\beta_2})}$$
$$\frac{\partial^{n_{20}}Y_{o_2}}{\partial\alpha_2} = \frac{Es_{\alpha_2}c_{\alpha_2}(c_{\beta_2} + c_{\beta_2}^2 - 2s_{\beta_2}^2) - {}^{n_{20}}Z_{o_2}c_{2\alpha_2}(1+c_{\beta_2}) + 2s_{\alpha_2}c_{\alpha_2}(1+c_{\beta_2})^{n_{20}}Y_{o_2}}{(-s_{\alpha_2}^2 + c_{\alpha_2}^2c_{\beta_2})}$$
$$\frac{\partial^{n_{20}}Y_{o_2}}{\partial\beta_2} = \frac{Es_{\beta_2}c_{\alpha_2}^2 - 2Es_{\beta_2}c_{\beta_2}(1+3s_{\alpha_2}^2) + 2^{n_{20}}Z_{o_2}c_{\alpha_2}s_{\alpha_2}s_{\beta_2} + 2c_{\alpha_2}^2s_{\beta_2}{}^{n_{20}}Y_{o_2}}{2(-s_{\alpha_2}^2 + c_{\alpha_2}^2c_{\beta_2})}$$

The angular velocity of n_{21} relative to $\{n_{20}\}$ of the 3-SPR PM can be expressed as:

$${}^{n_{20}}_{n_{21}}\boldsymbol{\omega} = {}^{n_{20}}\boldsymbol{R}_{\alpha_2}\dot{\alpha}_2 + {}^{n_{20}}\boldsymbol{R}_{\beta_2}\dot{\beta}_2 + {}^{n_{20}}\boldsymbol{R}_{\lambda_2}\dot{\lambda}_2 = \mathbf{J}_{\omega_2}\begin{bmatrix}\dot{\alpha}_2\\ \dot{\beta}_2\\ {}^{n_{20}}\dot{Z}_{o_2}\end{bmatrix},\ \mathbf{J}_{\omega_2} = \begin{bmatrix}1+c_{\beta_2} & 0 & 0\\ s_{\alpha_2}s_{\beta_2} & c_{\alpha_2} & 0\\ -c_{\alpha_2}s_{\beta_2} & s_{\alpha_2} & 0\end{bmatrix},$$

$${}^{n_{20}}\boldsymbol{R}_{\alpha_2} = \begin{bmatrix}1\\0\\0\end{bmatrix},\ {}^{n_{20}}\boldsymbol{R}_{\beta_2} = \begin{bmatrix}0\\ c_{\alpha_2}\\ s_{\alpha_2}\end{bmatrix},\ {}^{n_{20}}\boldsymbol{R}_{\lambda_2} = \begin{bmatrix}c_{\beta_2}\\ s_{\alpha_2}s_{\beta_2}\\ -c_{\alpha_2}s_{\beta_2}\end{bmatrix} \tag{23b}$$

From Eqs. (23a) and (23b) it leads to,

$$\begin{bmatrix}{}^{n_{20}}\boldsymbol{v}_{o_2}\\ {}^{n_{20}}_{n_{21}}\boldsymbol{\omega}\end{bmatrix} = \mathbf{J}_{o_2}\begin{bmatrix}\dot{\alpha}_2\\ \dot{\beta}_2\\ {}^{n_{20}}\dot{Z}_{o_2}\end{bmatrix},\ \mathbf{J}_{o_2} = \begin{bmatrix}\mathbf{J}_{v_2}\\ \mathbf{J}_{\omega_2}\end{bmatrix} \tag{23c}$$

Here, $\mathbf{J}o_2$ is a 6×3 form velocity decoupling matrix of the 3-SPR PM.

Based on the geometrical approach for determining the constrained wrenches, one constrained force which is parallel to R_{2j} and passes through S joint can be determined in each SPR leg. As the constrained forces/torques do no work to the moving platform n_{21}, it leads to [14]

$$\mathbf{J}_{\beta 2}\begin{bmatrix}{}^{n_{20}}\boldsymbol{v}_{o2}\\ {}^{n_{20}}_{n_{21}}\boldsymbol{\omega}\end{bmatrix} = \boldsymbol{0}_{3\times 1},\ \mathbf{J}_{\beta_2} = \begin{bmatrix}{}^{n_{20}}\boldsymbol{f}_{21}^T & ({}^{n_{20}}\boldsymbol{t}_{21}\times{}^{n_{20}}\boldsymbol{f}_{21})^T\\ {}^{n_{20}}\boldsymbol{f}_{22}^T & ({}^{n_{20}}\boldsymbol{t}_{22}\times{}^{n_{20}}\boldsymbol{f}_{22})^T\\ {}^{n_{20}}\boldsymbol{f}_{32}^T & ({}^{n_{20}}\boldsymbol{t}_{32}\times{}^{n_{20}}\boldsymbol{f}_{32})^T\end{bmatrix}, \tag{24}$$
$${}^{n_{20}}\boldsymbol{f}_{1j} = {}^{n_{20}}\boldsymbol{R}_{2j1},\ {}^{n_{20}}\boldsymbol{t}_{1j} = {}^{n_{20}}\boldsymbol{o}_2 - {}^{n_{20}}\boldsymbol{A}_{2j}$$

Here, $\mathbf{J}_{\beta 2}$ is a 3×6 form velocity constraint matrix for the 3-SPR PM.

4.3 Velocity analysis of the (3-RRS)+(3-SPR) S-PM

Referring to Fig. 3, the vector loop for the j-th link $O_1A_{1j}C_{1j}B_{1j}o_1$ for the 3-RRS PM can be expressed as:

$${}^{n_{10}}\boldsymbol{O}_1 + {}^{n_{10}}\boldsymbol{d}_{j_1} + {}^{n_{10}}\boldsymbol{d}_{j_2} = {}^{n_{10}}\boldsymbol{o}_1 + {}^{n_{10}}\boldsymbol{e}_{1j},$$
$${}^{n_{10}}\boldsymbol{d}_{j_1} = {}^{n_{10}}\boldsymbol{C}_{1j} - {}^{n_{10}}\boldsymbol{A}_{1j},\ {}^{n_{10}}\boldsymbol{d}_{j_2} = {}^{n_{10}}\boldsymbol{B}_{1j} - {}^{n_{10}}\boldsymbol{C}_{1j},\ {}^{n_{10}}\boldsymbol{e}_{1j} = {}^{n_{10}}\boldsymbol{B}_{1j} - {}^{n_{10}}\boldsymbol{o}_1. \tag{25a}$$

Based on the rules of vector derivation, by differentiating both sides of Eq. (25a),

$${}^{n_{10}}\boldsymbol{\omega}_{d_{j1}}\times{}^{n_{10}}\boldsymbol{d}_{j_1} + {}^{n_{10}}\boldsymbol{\omega}_{d_{j2}}\times{}^{n_{10}}\boldsymbol{d}_{j_2} = {}^{n_{10}}\boldsymbol{v}_{o_1} + {}^{n_{10}}_{n_{11}}\boldsymbol{\omega}\times{}^{n_{10}}\boldsymbol{e}_{1j},$$
$$\boldsymbol{\omega}_{d_{jk}} = \omega_{d_{jk}}{}^{n_{10}}\boldsymbol{R}_{1j1} = \dot{\theta}_{jk}{}^{n_{10}}\boldsymbol{R}_{1j1}, \tag{25b}$$

where $\boldsymbol{\omega}_{d_{jk}}$ and $\omega_{d_{jk}}$ denote the vector and the scalar of the angular velocity of d_{jk}, respectively. $\dot{\theta}_{jk}$ denotes the velocity of θ_{jk}.

Dot multiplying both sides of Eq. (25b) with ${}^{n_{10}}\mathbf{d}_{j_2}$,

$$\omega_{d_{j1}} = \dot{\theta}_{j1} = \frac{({}^{n_{10}}\boldsymbol{v}_{o_1} + {}^{n_{10}}_{n_{11}}\boldsymbol{\omega}\times{}^{n_{10}}\boldsymbol{e}_{1j})\cdot{}^{n_{10}}\boldsymbol{d}_{j2}}{({}^{n_{10}}\boldsymbol{R}_{1j1}\times{}^{n_{10}}\boldsymbol{d}_{j1})\cdot{}^{n_{10}}\boldsymbol{d}_{j_2}}$$
$$= \begin{bmatrix}\dfrac{{}^{n_{10}}\boldsymbol{d}_{j2}^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{1j1}\times{}^{n_{10}}\boldsymbol{d}_{j1})\cdot{}^{n_{10}}\boldsymbol{d}_{j_2}} & \dfrac{({}^{n_{10}}\boldsymbol{e}_{1j}\times{}^{n_{10}}\boldsymbol{d}_{j2})^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{1j1}\times{}^{n_{10}}\boldsymbol{d}_{j1})\cdot{}^{n_{10}}\boldsymbol{d}_{j_2}}\end{bmatrix}\begin{bmatrix}{}^{n_{10}}\boldsymbol{v}_{o1}\\ {}^{n_{10}}_{n_{11}}\boldsymbol{\omega}\end{bmatrix}. \tag{26a}$$

Dot multiplying both sides of Eq. (25b) with ${}^{n_{10}}\boldsymbol{d}_{j_1}$,

$$\omega_{d_{j2}} = \dot{\theta}_{j1}+\dot{\theta}_{j2} = \begin{bmatrix} \dfrac{{}^{n_{10}}\boldsymbol{d}_{j1}^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{1j2}\times{}^{n_{10}}\boldsymbol{d}_{j2})\cdot{}^{n_{10}}\boldsymbol{d}_{j_1}} & \dfrac{({}^{n_{10}}\boldsymbol{e}_{1j}\times{}^{n_{10}}\boldsymbol{d}_{j1})^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{1j2}\times{}^{n_{10}}\boldsymbol{d}_{j2})\cdot{}^{n_{10}}\boldsymbol{d}_{j_1}}\end{bmatrix}\begin{bmatrix}{}^{n_{10}}\boldsymbol{v}_{o1}\\ {}^{n_{10}}_{n_{11}}\boldsymbol{\omega}\end{bmatrix},$$

$$\dot{\theta}_{j2} = \omega_{d_{j2}} - \dot{\theta}_{j1}. \qquad (26b)$$

From Eq. (26a) it leads to,

$$\boldsymbol{v}_{r_1} = \mathbf{J}_{\alpha_1}\begin{bmatrix}{}^{n_{10}}\boldsymbol{v}_{o1}\\ {}^{n_{10}}_{n_{11}}\boldsymbol{\omega}\end{bmatrix},\ \mathbf{J}_{\alpha_1} = \begin{bmatrix} \dfrac{{}^{n_{10}}\boldsymbol{d}_{12}^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{111}\times{}^{n_{10}}\boldsymbol{d}_{11})\cdot{}^{n_{10}}\boldsymbol{d}_{12}} & \dfrac{({}^{n_{10}}\boldsymbol{e}_{11}\times{}^{n_{10}}\boldsymbol{d}_{12})^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{111}\times{}^{n_{10}}\boldsymbol{d}_{11})\cdot{}^{n_{10}}\boldsymbol{d}_{12}} \\ \dfrac{{}^{n_{10}}\boldsymbol{d}_{j2}^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{121}\times{}^{n_{10}}\boldsymbol{d}_{21})\cdot{}^{n_{10}}\boldsymbol{d}_{22}} & \dfrac{({}^{n_{10}}\boldsymbol{e}_{12}\times{}^{n_{10}}\boldsymbol{d}_{22})^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{121}\times{}^{n_{10}}\boldsymbol{d}_{21})\cdot{}^{n_{10}}\boldsymbol{d}_{22}} \\ \dfrac{{}^{n_{10}}\boldsymbol{d}_{j2}^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{131}\times{}^{n_{10}}\boldsymbol{d}_{31})\cdot{}^{n_{10}}\boldsymbol{d}_{32}} & \dfrac{({}^{n_{10}}\boldsymbol{e}_{13}\times{}^{n_{10}}\boldsymbol{d}_{32})^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{131}\times{}^{n_{10}}\boldsymbol{d}_{31})\cdot{}^{n_{10}}\boldsymbol{d}_{31}} \end{bmatrix},$$

$$\boldsymbol{v}_{r_1} = \begin{bmatrix}\omega_{d_{j1}}\\ \omega_{d_{j2}}\\ \omega_{d_{j3}}\end{bmatrix} = \begin{bmatrix}\dot{\theta}_{11}\\ \dot{\theta}_{22}\\ \dot{\theta}_{13}\end{bmatrix}. \qquad (27a)$$

From Eq. (27a), the actuation velocity of 3-RRS PM can be solved.

The actuation velocity of 3-SPR PM can be expressed as [14]

$$\boldsymbol{v}_{r2} = \mathbf{J}_{\alpha 2}\begin{bmatrix}{}^{n_{20}}\boldsymbol{v}_{o2}\\ {}^{n_{20}}_{n_{21}}\boldsymbol{\omega}\end{bmatrix},\ \mathbf{J}_{\alpha 2} = \begin{bmatrix}{}^{n_{20}}\boldsymbol{\delta}_{21}^{T} & ({}^{n_{20}}\boldsymbol{e}_{21}\times{}^{n_{20}}\boldsymbol{\delta}_{21})^{T}\\ {}^{n_{20}}\boldsymbol{\delta}_{22}^{T} & ({}^{n_{20}}\boldsymbol{e}_{22}\times{}^{n_{20}}\boldsymbol{\delta}_{22})^{T}\\ {}^{n_{20}}\boldsymbol{\delta}_{23}^{T} & ({}^{n_{20}}\boldsymbol{e}_{23}\times{}^{n_{20}}\boldsymbol{\delta}_{23})^{T}\end{bmatrix},\ \boldsymbol{v}_{r2} = \begin{bmatrix}v_{r_{21}}\\ v_{r_{22}}\\ v_{r_{23}}\end{bmatrix}$$

$${}^{n_{20}}\boldsymbol{\delta}_{2j} = \frac{{}^{n_{20}}\boldsymbol{B}_{2j} - {}^{n_{20}}\boldsymbol{A}_{2j}}{\left|{}^{n_{20}}\boldsymbol{B}_{2j} - {}^{n_{20}}\boldsymbol{A}_{2j}\right|},\ {}^{n_{20}}\boldsymbol{e}_{2j} = {}^{n_{20}}\boldsymbol{B}_{2j} - {}^{n_{20}}\boldsymbol{o}_{2}$$

$$(j = 1, 2, 3). \qquad (27b)$$

Let $\boldsymbol{g} = [g_x \ \ g_y \ \ g_z]^T$, $\boldsymbol{h} = [h_x \ \ h_y \ \ h_z]^T$ be two arbitrary vectors, $S(\boldsymbol{g})$ be a skew-symmetric matrix defined as:

$$S(\boldsymbol{g}) = \begin{bmatrix}0 & -g_z & g_y\\ g_z & 0 & -g_x\\ -g_y & g_x & 0\end{bmatrix},\ S(\boldsymbol{g}) = -S(\boldsymbol{g})^T,\ \boldsymbol{g}\times\boldsymbol{h} = S(\boldsymbol{g})\boldsymbol{h}. \qquad (28)$$

The velocity of the terminal platform can be expressed as [20]:

$$\begin{bmatrix}{}^{n_{10}}\boldsymbol{v}_{o_2}\\ {}^{n_{10}}_{n_{21}}\boldsymbol{\omega}\end{bmatrix} = \mathbf{K}_1\begin{bmatrix}{}^{n_{10}}\boldsymbol{v}_{o_1}\\ {}^{n_{10}}_{n_{11}}\boldsymbol{\omega}\end{bmatrix} + \mathbf{K}_2\begin{bmatrix}{}^{n_{20}}\boldsymbol{v}_{o_2}\\ {}^{n_{20}}_{n_{21}}\boldsymbol{\omega}\end{bmatrix},\ \mathbf{K}_1 = \begin{bmatrix}\mathbf{E}_{3\times 3} & -S({}^{n_{10}}_{n_{20}}\mathrm{R}\ {}^{n_{20}}\boldsymbol{o}_2)\\ 0_{3\times 3} & \mathbf{E}_{3\times 3}\end{bmatrix},$$

$$\mathbf{K}_2 = \begin{bmatrix}{}^{n_{10}}_{n_{20}}\mathrm{R} & 0_{3\times 3}\\ 0_{3\times 3} & {}^{n_{10}}_{n_{20}}\mathrm{R}\end{bmatrix},\ \mathbf{E}_{3\times 3} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}. \qquad (29)$$

From Eqs. (22), (23c) and (29),

$$\begin{bmatrix}\dot{\alpha}_2\\ \dot{\beta}_2\\ {}^{n_{20}}\dot{Z}_{20}\end{bmatrix} = (\mathbf{J}_{\beta_1}\mathbf{K}_1^{-1}\mathbf{K}_2\mathbf{J}_{o_2})^{-1}\mathrm{J}_{\beta_1}\mathrm{K}_1^{-1}\begin{bmatrix}{}^{n_{10}}\boldsymbol{v}_{o_2}\\ {}^{n_{10}}_{n_{21}}\boldsymbol{\omega}\end{bmatrix}. \qquad (30a)$$

Multiplying both sides of Eq. (30a) by $\mathbf{J}_{o2}$,

$$\begin{bmatrix}{}^{n_{20}}\boldsymbol{v}_{o_2}\\ {}^{n_{20}}_{n_{21}}\boldsymbol{\omega}\end{bmatrix} = \mathbf{J}_{s2}\begin{bmatrix}{}^{n_{10}}\boldsymbol{v}_{o_2}\\ {}^{n_{10}}_{n_{21}}\boldsymbol{\omega}\end{bmatrix},\ \mathbf{J}_{s_2} = \mathbf{J}_{o_2}(\mathbf{J}_{\beta_1}\mathbf{K}_1^{-1}\mathbf{K}_2\mathbf{J}_{o_2})^{-1}\mathbf{J}_{\beta_1}\mathbf{K}_1^{-1}. \qquad (30b)$$

From Eqs. (27a), (27b) and (30a),

$$\boldsymbol{v}_{r_2} = \mathrm{J}_{\alpha_2}\mathrm{J}_{o_2}(\mathrm{J}_{\beta_1}\mathrm{K}_1^{-1}\mathrm{K}_2\mathrm{J}_{o_2})^{-1}\mathrm{J}_{\beta_1}\mathrm{K}_1^{-1}\begin{bmatrix}{}^{n_{10}}\boldsymbol{v}_{o_2}\\ {}^{n_{10}}_{n_{21}}\boldsymbol{\omega}\end{bmatrix}. \qquad (31a)$$

In the same way, we can obtain

$$\boldsymbol{v}_{r_1} = \mathbf{J}_{\alpha_1}\mathbf{J}_{o_1}(\mathbf{J}_{\beta_2}\mathbf{K}_2^{-1}\mathbf{K}_1\mathbf{J}_{o_1})^{-1}\mathbf{J}_{\beta_2}\mathbf{K}_2^{-1}\begin{bmatrix}{}^{n_{10}}\boldsymbol{v}_{o_2}\\ {}^{n_{10}}_{n_{21}}\boldsymbol{\omega}\end{bmatrix} \qquad (31b)$$

$$\begin{bmatrix}{}^{n_{10}}\boldsymbol{v}_{o_1}\\ {}^{n_{10}}_{n_{11}}\boldsymbol{\omega}\end{bmatrix} = \mathbf{J}_{s1}\begin{bmatrix}{}^{n_{10}}\boldsymbol{v}_{o_2}\\ {}^{n_{10}}_{n_{21}}\boldsymbol{\omega}\end{bmatrix},\ \mathbf{J}_{s1} = \mathbf{J}_{o1}(\mathbf{J}_{\beta 2}\mathbf{K}_2^{-1}\mathbf{K}_1\mathbf{J}_{o1})^{-1}\mathbf{J}_{\beta 2}\mathbf{K}_2^{-1}. \qquad (31c)$$

From Eqs. (31a) and (31b),

$$\boldsymbol{v}_r = \mathbf{J}\begin{bmatrix}{}^{n_{10}}\boldsymbol{v}_{o_2}\\ {}^{n_{10}}_{n_{21}}\boldsymbol{\omega}\end{bmatrix},\ \ \mathbf{J} = \begin{bmatrix}\mathbf{J}_{\alpha_1}\mathbf{J}_{o_1}(\mathbf{J}_{\beta_2}\mathbf{K}_2^{-1}\mathbf{K}_1\mathbf{J}_{o_1})^{-1}\mathbf{J}_{\beta_2}\mathbf{K}_2^{-1}\\ \mathbf{J}_{\alpha_2}\mathbf{J}_{o_2}(\mathbf{J}_{\beta_1}\mathbf{K}_1^{-1}\mathbf{K}_2\mathbf{J}_{o_2})^{-1}\mathbf{J}_{\beta_1}\mathbf{K}_1^{-1}\end{bmatrix},\ \boldsymbol{v}_r = \begin{bmatrix}\boldsymbol{v}_{r_1}\\ \boldsymbol{v}_{r_2}\end{bmatrix}. \qquad (32)$$

Here, $\mathbf{J}$ is the inverse Jacobian for the S-PMs.

5. Inverse acceleration of the (3-RRS)+(3-SPR) S-PM

For the 3-RRS PM, by differentiating both sides of Eq. (21c) with respect to time,

$$\begin{bmatrix}{}^{n_{10}}\boldsymbol{a}_{o_1}\\ {}^{n_{10}}_{n_{11}}\boldsymbol{\varepsilon}\end{bmatrix} = \mathbf{J}_{o_1}\begin{bmatrix}\ddot{\alpha}_1\\ \ddot{\beta}_1\\ {}^{n_{10}}\ddot{Z}_{o_1}\end{bmatrix} + \begin{bmatrix}\dot{\alpha}_1 & \dot{\beta}_1 & \dot{Z}_{o_1}\end{bmatrix}\mathbf{h}_1\begin{bmatrix}\dot{\alpha}_1\\ \dot{\beta}_1\\ {}^{n_{10}}\dot{Z}_{o_1}\end{bmatrix}, \qquad (33a)$$

$$\mathbf{h}_1 = \begin{bmatrix}\mathbf{h}_{11} & \mathbf{h}_{12} & \mathbf{h}_{13} & \mathbf{h}_{14} & \mathbf{h}_{15} & \mathbf{h}_{16}\end{bmatrix}^T$$

$$\mathbf{h}_{11} = \begin{bmatrix}e_1 s_{\alpha_1}s_{\beta_1} & -e_1 c_{\alpha_1}c_{\beta_1} & 0\\ -e_1 c_{\alpha_1}c_{\beta_1} & e_1 s_{\alpha_1}s_{\beta_1} & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$\mathbf{h}_{p2} = \begin{bmatrix}-e_1 c_{2\alpha_1}(1+c_{\beta_1}) & -e_1 s_{\beta_1}(s_{2\alpha_1}+1)/2 & 0\\ e_1 c_{\alpha_1}s_{\alpha_1}s_{\beta_1} & e_1 c_{\beta_1}(s_{\alpha_1}^2+1)/2 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$\mathbf{h}_{13} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\ \mathbf{h}_{14} = \begin{bmatrix}0 & 0 & 0\\ -s_{\beta_1} & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$\mathbf{h}_{15}=\begin{bmatrix} c_{\alpha_1}s_{\beta_1} & -s_{\alpha_1} & 0 \\ s_{\alpha_1}c_{\beta_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \mathbf{h}_{16}=\begin{bmatrix} s_{\alpha_1}s_{\beta_1} & c_{\alpha_1} & 0 \\ -c_{\alpha_1}c_{\beta_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For the 3-SPR PM, by differentiating both sides of Eq. (23c) with respect to time,

$$\begin{bmatrix} {}^{n_{20}}\boldsymbol{a}_{o_2} \\ {}^{n_{20}}_{n_{21}}\boldsymbol{\varepsilon} \end{bmatrix}=\mathbf{J}_{o_2}\begin{bmatrix} \dot{\alpha}_2 \\ \dot{\beta}_2 \\ {}^{n_{20}}\dot{Z}_{o_2} \end{bmatrix}+\begin{bmatrix} \dot{\alpha}_2 & \dot{\beta}_2 & {}^{n_{q0}}\dot{Z}_{o_2} \end{bmatrix}\mathbf{h}_2\begin{bmatrix} \dot{\alpha}_2 \\ \dot{\beta}_2 \\ {}^{n_{20}}\dot{Z}_{o_2} \end{bmatrix}, \tag{33b}$$

$$\mathbf{h}_2=\begin{bmatrix} \mathbf{h}_{21} & \mathbf{h}_{22} & \mathbf{h}_{23} & \mathbf{h}_{24} & \mathbf{h}_{25} & \mathbf{h}_{26} \end{bmatrix}^T$$

where

$$\mathbf{h}_{21}=\begin{bmatrix} \dfrac{\partial^{2n_{20}}X_{o_2}}{\partial\alpha_2\partial\alpha_2} & \dfrac{\partial^{2n_{20}}X_{o_2}}{\partial\alpha_2\partial\beta_2} & \dfrac{\partial^{2n_{20}}X_{o_2}}{\partial\alpha_2\partial Z_{o_2}} \\ \dfrac{\partial^{2n_{20}}X_{o_2}}{\partial\beta_2\partial\alpha_2} & \dfrac{\partial^{2n_{20}}X_{o_2}}{\partial\beta_2\partial\beta_2} & \dfrac{\partial^{2n_{20}}X_{o_2}}{\partial\beta_2\partial^{n_{20}}Z_{o_2}} \\ \dfrac{\partial^{2n_{20}}X_{o_2}}{\partial^{n_{20}}Z_{o_q}\partial\alpha_2} & \dfrac{\partial^{2n_{20}}X_{o_2}}{\partial^{n_{q0}}Z_{o_q}\partial\beta_q} & \dfrac{\partial^{2n_{20}}X_{o_2}}{\partial^{n_{20}}Z_{o_2}\partial^{n_{20}}Z_{o_2}} \end{bmatrix},$$

$$\mathbf{h}_{22}=\begin{bmatrix} \dfrac{\partial^{2n_{20}}Y_{o_2}}{\partial\alpha_2\partial\alpha_2} & \dfrac{\partial^{2n_{20}}Y_{o_2}}{\partial\alpha_2\partial\beta_2} & \dfrac{\partial^{2n_{20}}Y_{o_2}}{\partial\alpha_2\partial Z_{o_2}} \\ \dfrac{\partial^{2n_{20}}Y_{o_2}}{\partial\beta_2\partial\alpha_2} & \dfrac{\partial^{2n_{20}}Y_{o_2}}{\partial\beta_2\partial\beta_2} & \dfrac{\partial^{2n_{20}}Y_{o_2}}{\partial\beta_2\partial Z_{o_2}} \\ \dfrac{\partial^{2n_{20}}Y_{o_2}}{\partial^{n_{20}}Z_{o_2}\partial\alpha_2} & \dfrac{\partial^{2n_{20}}Y_{o_2}}{\partial^{n_{20}}Z_{o_2}\partial\beta_2} & \dfrac{\partial^{2n_{20}}Y_{o_2}}{\partial^{n_{20}}Z_{o_2}\partial^{n_{20}}Z_{o_2}} \end{bmatrix},$$

$$\mathbf{h}_{23}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \mathbf{h}_{24}=\begin{bmatrix} 0 & 0 & 0 \\ -s_{\beta_2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{h}_{25}=\begin{bmatrix} c_{\alpha_2}s_{\beta_2} & -s_{\alpha_2} & 0 \\ s_{\alpha_2}c_{\beta_2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \mathbf{h}_{26}=\begin{bmatrix} s_{\alpha_2}s_{\beta_2} & c_{\alpha_2} & 0 \\ -c_{\alpha_2}c_{\beta_2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial^2X_{o_2}}{\partial\alpha_2\partial\alpha_2}=\frac{\begin{array}{c}-Es_{\alpha_2}s_{\beta_2}(3c^2_{\alpha_2}-6s^2_{\alpha_2}-9s^2_{\alpha_2}c_{\beta_2}-c_{\beta_2})-2Z_{o_2}s_{\beta_2}c_{\alpha_2}\\ +(1+c_{\beta_2})(4c_{2\alpha_2}X_{o_2}+8s_{\alpha_2}c_{\alpha_2}\dfrac{\partial X_{o_2}}{\partial\alpha_2}-9Ec_{\alpha_2}s_{\beta_2}s_{2\alpha_2})\end{array}}{2(-s^2_{\alpha_2}+c^2_{\alpha_2}c_{\beta_2})},$$

$$\frac{\partial^2X_{o_2}}{\partial\alpha_2\partial\beta_2}=\frac{\partial^2X_{o_2}}{\partial\beta_2\partial\alpha_2}=\frac{\begin{array}{c}3Ec_{\alpha_2}c_{\beta_2}(c^2_{\alpha_2}-2s^2_{\alpha_2})-Ec_{\alpha_2}c_{2\beta_2}(9s^2_{\alpha_2}+1)-2Z_{o_2}c_{\beta_2}s_{\alpha_2}\\ +4s_{\alpha_2}c_{\alpha_2}(1+c_{\beta_2})\dfrac{\partial X_{o_2}}{\partial\beta_2}+2c^2_{\alpha_2}s_{\beta_2}\dfrac{\partial X_{o_2}}{\partial\alpha_2}-4s_{\alpha_2}c_{\alpha_2}s_{\beta_2}X_{o_2}\end{array}}{2(-s^2_{\alpha_2}+c^2_{\alpha_2}c_{\beta_2})},$$

$$\frac{\partial^2X_{o_2}}{\partial\beta_2\partial\beta_2}=\frac{\begin{array}{c}-3Es_{\alpha_2}s_{\beta_2}c^2_{\alpha_2}+2Es_{\alpha_2}(3s^2_{\alpha_2}+1)s_{2\beta_2}\\ -2Z_{o_2}s_{\beta_2}c_{\alpha_2}+2c^2_{\alpha_2}c_{\beta_2}X_{o_2}+4c^2_{\alpha_2}s_{\beta_2}\dfrac{\partial X_{o_2}}{\partial\beta_2}\end{array}}{2(-s^2_{\alpha}+c^2_{\alpha}c_{\beta})},$$

$$\frac{\partial^2X_{o_2}}{\partial\alpha_2\partial Z_{o_2}}=\frac{\partial^2X_{o_2}}{\partial Z_{o_2}\partial\alpha_2}=\frac{-s_{\beta_2}s_{\alpha_2}+2s_{\alpha_2}c_{\alpha_2}(1+c_{\beta_2})(\partial X_{o_2}/\partial Z_{o_2})}{-s^2_{\alpha_2}+c^2_{\alpha_2}c_{\beta_2}},$$

$$\frac{\partial^2X_{o_2}}{\partial\beta_2\partial Z_{o_2}}=\frac{\partial^2X_{o_2}}{\partial Z_{o_2}\partial\beta_2}=\frac{c_{\beta_2}c_{\alpha_2}c^2_{\alpha}s_{\beta}(\partial X_{o_2}/\partial Z_{o_2})}{-s^2_{\alpha_2}+c^2_{\alpha_2}c_{\beta_2}},\ \frac{\partial^2X_{o_2}}{\partial Z_{o_2}\partial Z_{o_2}}=0$$

$$\frac{\partial^2Y_{o_2}}{\partial\alpha_2\partial\alpha_2}=\frac{Ec_{2\alpha_2}(c_{\beta_2}+c^2_{\beta_2}-2s^2_{\beta_2})+2(Z_{o_2}s_{2\alpha_2}+c_{2\alpha_2}Y_{o_2}+s_{2\alpha_2}\dfrac{\partial Y_{o_2}}{\partial\alpha_2})(1+c_{\beta_2})}{(-s^2_{\alpha_2}+c^2_{\alpha_2}c_{\beta_2})},$$

$$\frac{\partial^2Y_{o_2}}{\partial\alpha_2\partial\beta_2}=\frac{\partial^2Y_{o_2}}{\partial\beta_2\partial\alpha_2}=\frac{\begin{array}{c}-Es_{\alpha_2}c_{\alpha_2}(s_{\beta_2}+3s_{2\beta_2})+Z_{o_2}c_{2\alpha_2}s_{\beta_2}\\ -s_{2\alpha_2}s_{\beta_2}Y_{o_2}+s_{2\alpha_2}(1+c_{\beta_2})\dfrac{\partial Y_{o_2}}{\partial\beta_2}+c^2_{\alpha_2}s_{\beta_2}\dfrac{\partial Y_{o_2}}{\partial\alpha_2}\end{array}}{(-s^2_{\alpha_2}+c^2_{\alpha_2}c_{\beta_2})},$$

$$\frac{\partial^2Y_{o_2}}{\partial\alpha_2\partial Z_{o_2}}=\frac{\partial^2Y_{o_2}}{\partial Z_{o_2}\partial\alpha_2}=\frac{-c_{2\alpha_2}(1+c_{\beta_2})+2s_{\alpha_2}c_{\alpha_2}(1+c_{\beta_2})\dfrac{\partial Y_{o_2}}{\partial Z_{o_2}}}{(-s^2_{\alpha_2}+c^2_{\alpha_2}c_{\beta_2})},$$

$$\frac{\partial^2Y_{o_2}}{\partial\beta_2\partial\beta_2}=\frac{\begin{array}{c}Ec_{\beta_2}c^2_{\alpha_2}-2Ec_{2\beta_2}(1+3s^2_{\alpha_2})+2Z_{o_2}c_{\alpha_2}s_{\alpha_2}c_{\beta_2}\\ +2c^2_{\alpha_2}c_{\beta_2}Y_{o_2}+4c^2_{\alpha_2}s_{\beta_2}\dfrac{\partial Y_{o_2}}{\partial\beta_2}\end{array}}{2(-s^2_{\alpha_2}+c^2_{\alpha_2}c_{\beta_2})},$$

$$\frac{\partial^2Y_{o_2}}{\partial\beta_2\partial Z_{o_2}}=\frac{\partial^2Y_{o_2}}{\partial Z_{o_2}\partial\beta_2}=\frac{c_{\alpha_2}s_{\alpha_2}s_{\beta_2}+c^2_{\alpha_2}s_{\beta_2}\dfrac{\partial Y_{o_2}}{\partial Z_{o_2}}}{(-s^2_{\alpha_2}+c^2_{\alpha_2}c_{\beta_2})},\ \frac{\partial^2Y_{o_2}}{\partial Z_{o_2}\partial Z_{o_2}}=0.$$

By differentiating both sides of Eqs. (22) and (24) with respect to time,

$$\boldsymbol{0}_{3\times1}=\mathbf{J}_{\beta_1}\begin{bmatrix} {}^{n_{10}}\boldsymbol{a}_{o_1} \\ {}^{n_{10}}_{n_{11}}\boldsymbol{\varepsilon} \end{bmatrix}+\begin{bmatrix} {}^{n_{10}}\boldsymbol{v}^T_{o_1} & {}^{n_{10}}_{n_{11}}\boldsymbol{\omega}^T \end{bmatrix}\mathbf{H}_{\beta_1}\begin{bmatrix} {}^{n_{10}}\boldsymbol{v}_{oi} \\ {}^{n_{10}}_{n_{11}}\boldsymbol{\omega} \end{bmatrix},$$

$$\mathbf{H}_{\beta_1}=\begin{bmatrix} \mathbf{H}_{\beta_{11}} & \mathbf{H}_{\beta_{12}} & \mathbf{H}_{\beta_{13}} \end{bmatrix}^T,\ \mathbf{H}_{\beta_{1j}}=\begin{bmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & S({}^{n_{10}}\boldsymbol{d}_{1j})S({}^{n_{10}}\boldsymbol{R}_{1j1}) \end{bmatrix}_{6\times6} \tag{34a}$$

$$\boldsymbol{0}_{3\times1}=\mathbf{J}_{\beta_2}\begin{bmatrix} {}^{n_{20}}\boldsymbol{a}_{o_2} \\ {}^{n_{20}}_{n_{21}}\boldsymbol{\varepsilon} \end{bmatrix}+\begin{bmatrix} {}^{n_{10}}\boldsymbol{v}^T_{o_2} & {}^{n_{20}}_{n_{21}}\boldsymbol{\omega}^T \end{bmatrix}\mathbf{H}_{\beta_2}\begin{bmatrix} {}^{n_{20}}\boldsymbol{v}_{o2} \\ {}^{n_{20}}_{n_{21}}\boldsymbol{\omega} \end{bmatrix},$$

$$\mathbf{H}_{\beta_2}=\begin{bmatrix} \mathbf{H}_{\beta_{21}} & \mathbf{H}_{\beta_{22}} & \mathbf{H}_{\beta_{23}} \end{bmatrix}^{\mathrm{T}},\ \mathbf{H}_{\beta_{2j}}=\begin{bmatrix} \mathbf{0}_{3\times3} & -S({}^{n_{20}}\boldsymbol{R}_{2j1}) \\ S({}^{n_{20}}\boldsymbol{R}_{2j1}) & -S({}^{n_{20}}\boldsymbol{R}_{2j})S({}^{n_{20}}\boldsymbol{d}_{2j1}) \end{bmatrix}_{6\times6}. \tag{34b}$$

Here, $\mathbf{H}_{\beta ij}(i=1,2;j=1,2,3)$ is a 6×6 form matrix.
From Eq. (27a) it leads to,

$$\boldsymbol{a}_{r_1}=\begin{bmatrix} a_{r_{11}} \\ a_{r_{12}} \\ a_{r_{13}} \end{bmatrix}=\mathbf{J}_{\alpha_1}\begin{bmatrix} {}^{n_{10}}\boldsymbol{a}_{o1} \\ {}^{n_{10}}_{n_{11}}\boldsymbol{\varepsilon} \end{bmatrix}+\begin{bmatrix} {}^{n_{10}}\boldsymbol{v}^{\mathrm{T}}_{o1} & {}^{n_{10}}_{n_{11}}\boldsymbol{\omega}^{\mathrm{T}} \end{bmatrix}\mathbf{H}_{\alpha_{1j}}\begin{bmatrix} {}^{n_{10}}\boldsymbol{v}_{o1} \\ {}^{n_{10}}_{n_{11}}\boldsymbol{\omega} \end{bmatrix}, \tag{35}$$

$$\mathbf{H}_{\alpha_1}=\begin{bmatrix} \mathbf{H}_{\alpha_{11}} & \mathbf{H}_{\alpha_{12}} & \mathbf{H}_{\alpha_{13}} \end{bmatrix}$$

$$\mathbf{H}_{\alpha_{1j}}=\frac{\begin{array}{c}\mathbf{J}^{\mathrm{T}}_{\mathrm{E}_\omega}\mathrm{S}({}^{n_{10}}\boldsymbol{d}_{j2})\mathrm{S}({}^{n_{10}}\boldsymbol{e}_{1j})\ \mathbf{J}_{\mathrm{E}_\omega}-\mathbf{J}^{\mathrm{T}}_{\mathrm{E}_v}\ \mathrm{S}({}^{n_{10}}\boldsymbol{d}_{j2})\ \mathbf{J}_{\omega_{dj2}}-\mathbf{J}^{\mathrm{T}}_{\mathrm{E}_\omega}\mathrm{S}({}^{n_{10}}\boldsymbol{e}_{1j})\mathrm{S}({}^{n_{10}}\boldsymbol{d}_{j2})\ \mathbf{J}_{\omega_{dj2}}\\ +\mathbf{J}^T_{\omega_{dj1}}\ \mathrm{S}({}^{n_{10}}\boldsymbol{d}_{j1})\mathrm{S}({}^{n_{10}}\boldsymbol{d}_{j2})\mathbf{J}_{\omega_{dj2}}-\mathbf{J}^T_{\omega_{dj1}}\ \mathrm{S}({}^{n_{10}}\boldsymbol{d}_{j2})\mathrm{S}({}^{n_{10}}\boldsymbol{d}_{j1})\mathbf{J}_{\omega_{dj1}}\end{array}}{{}^{n_{10}}\boldsymbol{R}_{1j1}\cdot({}^{n_{10}}\boldsymbol{d}_{j1}\times{}^{n_{10}}\boldsymbol{d}_{j2})}$$

$$\mathbf{J}_{\mathrm{E}_v}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},\ \mathbf{J}_{\mathrm{E}_\omega}=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

For the 3-SPR PM, the scalar accelerations a_{ri} of r_{2j} (j = 1, 2, 3) have been derived as [14]:

$$\boldsymbol{a}_{r_2}=\begin{bmatrix} a_{r_{21}} \\ a_{r_{22}} \\ a_{r_{23}} \end{bmatrix}=\mathbf{J}_{\alpha_2}\begin{bmatrix} {}^{n_{20}}\boldsymbol{a}_{o_2} \\ {}^{n_{20}}_{n_{21}}\boldsymbol{\varepsilon} \end{bmatrix}+\begin{bmatrix} {}^{n_{20}}\boldsymbol{v}^T_{o_2} & {}^{n_{20}}_{n_{21}}\boldsymbol{\omega}^T \end{bmatrix}\mathbf{H}_{\alpha_2}\begin{bmatrix} {}^{n_{20}}\boldsymbol{v}_{o_2} \\ {}^{n_{20}}_{n_{21}}\boldsymbol{\omega} \end{bmatrix},\ \mathbf{H}_{\alpha_2}=\begin{bmatrix} \mathbf{H}_{\alpha_{21}} & \mathbf{H}_{\alpha_{22}} & \mathbf{H}_{\alpha_{23}} \end{bmatrix},$$

$$\mathbf{H}_{\alpha_{2j}} = \frac{1}{r_{2j}} \begin{bmatrix} -S({}^{n_{20}}\boldsymbol{\delta}_{2j})^2 & S({}^{n_{20}}\boldsymbol{\delta}_{2j})^2 S({}^{n_{20}}\boldsymbol{e}_{2j}) \\ -S({}^{n_{20}}\boldsymbol{e}_{2j}) S({}^{n_{20}}\boldsymbol{\delta}_{2j})^2 & r_{2j} S({}^{n_{20}}\boldsymbol{e}_{2j}) S({}^{n_{20}}\boldsymbol{\delta}_{2j}) + S({}^{n_{20}}\boldsymbol{e}_{2j}) S({}^{n_{20}}\boldsymbol{\delta}_{2j})^2 S({}^{n_{20}}\boldsymbol{e}_{2j}) \end{bmatrix}. \tag{36}$$

The acceleration of the terminal platform can be expressed as [20]:

$$\begin{bmatrix} {}^{n_{10}}\boldsymbol{a}_{o_2} \\ {}^{n_{10}}_{n_{21}}\boldsymbol{\varepsilon} \end{bmatrix} = \mathbf{K}_1 \begin{bmatrix} {}^{n_{10}}\boldsymbol{a}_{o_1} \\ {}^{n_{10}}_{n_{11}}\boldsymbol{\varepsilon} \end{bmatrix} + \mathbf{K}_2 \begin{bmatrix} {}^{n_{20}}\boldsymbol{a}_{o_2} \\ {}^{n_{20}}_{n_{21}}\boldsymbol{\varepsilon} \end{bmatrix} + \boldsymbol{L},$$

$$\boldsymbol{L} = \begin{bmatrix} 2S({}^{n_{10}}_{n_{11}}\boldsymbol{\omega}) {}^{n_{10}}_{n_{11}}\mathbf{R} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & S({}^{n_{10}}_{n_{11}}\boldsymbol{\omega}) {}^{n_{10}}_{n_{11}}\mathbf{R} \end{bmatrix} \begin{bmatrix} {}^{n_{20}}\boldsymbol{v}_{o_2} \\ {}^{n_{20}}_{n_{21}}\boldsymbol{\omega} \end{bmatrix} + \begin{bmatrix} S({}^{n_{10}}_{n_{11}}\boldsymbol{\omega}) S({}^{n_{10}}_{n_{11}}\boldsymbol{\omega}) {}^{n_{10}}_{n_{11}}\mathbf{R} {}^{n_{20}}\boldsymbol{o}_{o_2} \\ \boldsymbol{0}_{3\times1} \end{bmatrix}. \tag{37}$$

From Eqs. (33a), (33b), (34a), (34b), (36) and (37),

$$\begin{bmatrix} \ddot{\alpha}_1 \\ \ddot{\beta}_1 \\ \ddot{Z}_{o_1} \end{bmatrix} = (\mathbf{J}_{\beta_2} \mathbf{K}_2^{-1} \mathbf{K}_1 \mathbf{J}_{o_1})^{-1} \left\{ \mathbf{J}_{\beta_2} \mathbf{K}_2^{-1} \left(\begin{bmatrix} {}^{n_{10}}\boldsymbol{a}_{o_2} \\ {}^{n_{10}}_{n_{21}}\boldsymbol{\varepsilon} \end{bmatrix} - \boldsymbol{L} \right) - \boldsymbol{g}_2 - \mathbf{J}_{\beta_2} \mathbf{K}_2^{-1} \mathbf{J}_1 \begin{bmatrix} \dot{\alpha}_1 & \dot{\beta}_1 & \dot{Z}_{o_1} \end{bmatrix} \mathbf{h}_1 \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\beta}_1 \\ \dot{Z}_{o_1} \end{bmatrix}) \right\}$$

$$\begin{bmatrix} \ddot{\alpha}_2 \\ \ddot{\beta}_2 \\ \ddot{Z}_{o_2} \end{bmatrix} = (\mathbf{J}_{\beta_1} \mathbf{K}_1^{-1} \mathbf{K}_2 \mathbf{J}_{o_2})^{-1} \left\{ \mathbf{J}_{\beta_1} \mathbf{K}_1^{-1} \left(\begin{bmatrix} {}^{n_{10}}\boldsymbol{a}_{o_2} \\ {}^{n_{10}}_{n_{21}}\boldsymbol{\varepsilon} \end{bmatrix} - \boldsymbol{L} \right) - \boldsymbol{g}_1 - \mathbf{J}_{\beta_1} \mathbf{K}_1^{-1} \mathbf{J}_2 \begin{bmatrix} \dot{\alpha}_2 & \dot{\beta}_2 & \dot{Z}_{o_2} \end{bmatrix} \mathbf{h}_2 \begin{bmatrix} \dot{\alpha}_2 \\ \dot{\beta}_2 \\ \dot{Z}_{o_2} \end{bmatrix}) \right\}$$

$$\boldsymbol{g}_i = -\begin{bmatrix} {}^{n_{i0}}\boldsymbol{v}_{o_i}^T & {}^{n_{i0}}_{n_{i1}}\boldsymbol{\omega}^T \end{bmatrix} \mathbf{H}_{\beta_i} \begin{bmatrix} {}^{n_{i0}}\boldsymbol{v}_{o_i} \\ {}^{n_{i0}}_{n_{i1}}\boldsymbol{\omega} \end{bmatrix}. \tag{38}$$

When the velocity and the acceleration of n_{21} relative to n_{10} are given, by substituting Eq. (38) into Eqs. (33a) and (33b), ${}^{n_{i0}}\boldsymbol{a}_{o_i}$ and ${}^{n_{i0}}_{n_{i1}}\boldsymbol{\varepsilon}$ can be obtained. Then, by substituting this result into Eqs. (35) and (36), the inverse acceleration of this S-PM can be derived.

6. Inverse dynamics of the S-PM

The inverse dynamics analysis [23] is to determine the required forces of actuators from the given kinematics of the terminal platform in given poses.

Let $\boldsymbol{\omega}_{l_{ij}}$, $\boldsymbol{\varepsilon}_{l_{ij}}$ be the angular velocity and angular acceleration of the lower link in j-th leg of the i-th PM, respectively. Let $\boldsymbol{v}_{l_{ij}}$, $\boldsymbol{a}_{l_{ij}}$ and be the linear velocity and linear acceleration of the mass-center of lower link in j-th leg of the i-th PM, respectively. Let $\boldsymbol{\omega}_{u_{ij}}$ and $\boldsymbol{\varepsilon}_{u_{ij}}$ be the angular velocity and angular acceleration of the upper link in j-th leg of the i-th PM, respectively. Let $\boldsymbol{v}_{u_{ij}}$ and $\boldsymbol{a}_{u_{ij}}$ be the linear velocity and linear acceleration of the mass-center of upper link in j-th leg of the i-th PM, respectively.

From Eq. (26a), the angular velocity of d_{j1} can be expressed as:

$${}^{n_{10}}\boldsymbol{\omega}_{l_{1j}} = \dot{\theta}_{j1} {}^{n_{10}}\boldsymbol{R}_{1j1} = \begin{bmatrix} \dfrac{{}^{n_{10}}\boldsymbol{R}_{1j1} {}^{n_{10}}\boldsymbol{d}_{j2}^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{1j1} \times {}^{n_{10}}\boldsymbol{d}_{j1}) \cdot {}^{n_{10}}\boldsymbol{d}_{j_2}} & \dfrac{{}^{n_{10}}\boldsymbol{R}_{1j1}({}^{n_{10}}\boldsymbol{e}_{1j} \times {}^{n_{10}}\boldsymbol{d}_{j2})^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{1j1} \times {}^{n_{10}}\boldsymbol{d}_{j1}) \cdot {}^{n_{10}}\boldsymbol{d}_{j_2}} \end{bmatrix} \begin{bmatrix} {}^{n_{10}}\boldsymbol{v}_{o1} \\ {}^{n_{10}}_{n_{11}}\boldsymbol{\omega} \end{bmatrix} \tag{39a}$$

$$\mathbf{J}_{\omega_{l_{1j}}} = \begin{bmatrix} \dfrac{{}^{n_{10}}\boldsymbol{R}_{1j1} {}^{n_{10}}\boldsymbol{d}_{j2}^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{1j1} \times {}^{n_{10}}\boldsymbol{d}_{j1}) \cdot {}^{n_{10}}\boldsymbol{d}_{j_2}} & \dfrac{{}^{n_{10}}\boldsymbol{R}_{1j1}({}^{n_{10}}\boldsymbol{e}_{1j} \times {}^{n_{10}}\boldsymbol{d}_{j2})^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{1j1} \times {}^{n_{10}}\boldsymbol{d}_{j1}) \cdot {}^{n_{10}}\boldsymbol{d}_{j_2}} \end{bmatrix}.$$

From Eq. (26b), the angular velocity of d_{j2} can be expressed as:

$${}^{n_{10}}\boldsymbol{\omega}_{u_{1j}} = \begin{bmatrix} \dfrac{{}^{n_{10}}\boldsymbol{R}_{1j2} {}^{n_{10}}\boldsymbol{d}_{j1}^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{1j2} \times {}^{n_{10}}\boldsymbol{d}_{j2}) \cdot {}^{n_{10}}\boldsymbol{d}_{j_1}} & \dfrac{{}^{n_{10}}\boldsymbol{R}_{1j2}({}^{n_{10}}\boldsymbol{e}_{1j} \times {}^{n_{10}}\boldsymbol{d}_{j1})^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{1j2} \times {}^{n_{10}}\boldsymbol{d}_{j2}) \cdot {}^{n_{10}}\boldsymbol{d}_{j_1}} \end{bmatrix} \begin{bmatrix} {}^{n_{10}}\boldsymbol{v}_{o1} \\ {}^{n_{10}}_{n_{11}}\boldsymbol{\omega} \end{bmatrix} \tag{39b}$$

$$\mathbf{J}_{\omega_{u_{1j}}} = \begin{bmatrix} \dfrac{{}^{n_{10}}\boldsymbol{R}_{1j2} {}^{n_{10}}\boldsymbol{d}_{j1}^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{1j2} \times {}^{n_{10}}\boldsymbol{d}_{j2}) \cdot {}^{n_{10}}\boldsymbol{d}_{j_1}} & \dfrac{{}^{n_{10}}\boldsymbol{R}_{1j2}({}^{n_{10}}\boldsymbol{e}_{1j} \times {}^{n_{10}}\boldsymbol{d}_{j1})^{\mathrm{T}}}{({}^{n_{10}}\boldsymbol{R}_{1j2} \times {}^{n_{10}}\boldsymbol{d}_{j2}) \cdot {}^{n_{10}}\boldsymbol{d}_{j_1}} \end{bmatrix}.$$

For the RRS leg, the velocity of the mass-center of d_{j1} can be expressed as

$${}^{n_{10}}\boldsymbol{v}_{l_{1j}} = {}^{n_{10}}\boldsymbol{\omega}_{l_{1j}} \times {}^{n_{10}}\boldsymbol{d}_{j1} / 2 = -\frac{S({}^{n_{10}}\boldsymbol{d}_{j1}) \mathbf{J}_{\omega_{l_{1j}}}}{2} \begin{bmatrix} {}^{n_{10}}\boldsymbol{v}_{o_1} \\ {}^{n_{10}}_{n_{11}}\boldsymbol{\omega} \end{bmatrix}. \tag{40}$$

From Eqs. (39a) and (40),

$$\begin{bmatrix} {}^{n_{10}}\boldsymbol{v}_{l_{1j}} \\ {}^{n_{10}}\boldsymbol{\omega}_{l_{1j}} \end{bmatrix} = \mathbf{J}_{l_{1j}} \begin{bmatrix} {}^{n_{10}}\boldsymbol{v}_{o_1} \\ {}^{n_{10}}_{n_{11}}\boldsymbol{\omega} \end{bmatrix}, \ \mathbf{J}_{l_{1j}} = \begin{bmatrix} -S({}^{n_{10}}\boldsymbol{d}_{j1}) \mathbf{J}_{\omega_{l_{1j}}} / 2 \\ \mathbf{J}_{\omega_{l_{1j}}} \end{bmatrix}. \tag{41}$$

For the RRS leg, the linear acceleration of the mass-center of d_{j1} can be expressed as

$${}^{n_{10}}\boldsymbol{a}_{l_{1j}} = {}^{n_{10}}\boldsymbol{\varepsilon}_{l_{1j}} \times {}^{n_{10}}\boldsymbol{d}_{j1} / 2 + {}^{n_{10}}\boldsymbol{\omega}_{l_{1j}} \times ({}^{n_{10}}\boldsymbol{\omega}_{l_{1j}} \times {}^{n_{10}}\boldsymbol{d}_{j1})/2. \tag{42}$$

For the RRS leg, the velocity of the mass-center of d_{j2} can be expressed as

$$\begin{aligned} {}^{n_{10}}\boldsymbol{v}_{u_{1j}} &= {}^{n_{10}}\boldsymbol{\omega}_{u_{1j}} \times {}^{n_{10}}\boldsymbol{d}_{j1} + {}^{n_{10}}\boldsymbol{\omega}_{u_{1j}} \times {}^{n_{10}}\boldsymbol{d}_{j2} / 2 \\ &= [-S({}^{n_{10}}\boldsymbol{d}_{j1}) \mathbf{J}_{\omega_{l_{1j}}} - S({}^{n_{10}}\boldsymbol{d}_{j2}) \mathbf{J}_{\omega_{u_{1j}}} / 2] \begin{bmatrix} {}^{n_{10}}\boldsymbol{v}_{o_1} \\ {}^{n_{10}}_{n_{11}}\boldsymbol{\omega} \end{bmatrix}. \end{aligned} \tag{43a}$$

From Eqs. (39a) and (43a),

$$\begin{bmatrix} {}^{n_{10}}\boldsymbol{v}_{u_{1j}} \\ {}^{n_{10}}\boldsymbol{\omega}_{u_{1j}} \end{bmatrix} = \mathbf{J}_{u_{1j}} \begin{bmatrix} {}^{n_{10}}\boldsymbol{v}_{o_1} \\ {}^{n_{10}}_{n_{11}}\boldsymbol{\omega} \end{bmatrix}, \ \mathbf{J}_{u_{1j}} = \begin{bmatrix} -S({}^{n_{10}}\boldsymbol{d}_{j1}) \mathbf{J}_{\omega_{l_{1j}}} - S({}^{n_{10}}\boldsymbol{d}_{j2}) \mathbf{J}_{\omega_{u_{1j}}} / 2 \\ \mathbf{J}_{\omega_{u_{1j}}} \end{bmatrix}. \tag{43b}$$

For the RRS leg, the linear acceleration of the mass-center of d_{j2} can be expressed as

$$\begin{aligned} {}^{n_{10}}\boldsymbol{a}_{u_{1j}} &= {}^{n_{10}}\boldsymbol{\varepsilon}_{l_{1j}} \times {}^{n_{10}}\boldsymbol{d}_{j1} + {}^{n_{10}}\boldsymbol{\omega}_{l_{1j}} \times ({}^{n_{10}}\boldsymbol{\omega}_{l_{1j}} \times {}^{n_{10}}\boldsymbol{d}_{j1}) \\ &+ {}^{n_{10}}\boldsymbol{\varepsilon}_{u_{1j}} \times {}^{n_{10}}\boldsymbol{d}_{j2} / 2 + {}^{n_{10}}\boldsymbol{\omega}_{u_{1j}} \times ({}^{n_{10}}\boldsymbol{\omega}_{u_{1j}} \times {}^{n_{10}}\boldsymbol{d}_{j2}) / 2. \end{aligned} \tag{43c}$$

Let $\boldsymbol{v}_{2j}$ denote the velocity vector of the three vertices of the moving platform relative to the corresponding base, $\boldsymbol{\omega}_{r2j}$ denotes the angular velocity of r_{2j} for the 3-SPR PM. The velocity of three vertices of the moving platform relative to the corresponding base can be expressed as:

$$^{n20}\boldsymbol{v}_{2j} = {}^{n20}\boldsymbol{\omega}_{r_{2j}} \times r_{2j}\, {}^{n20}\boldsymbol{\delta}_{2j} + {}^{n20}\boldsymbol{\delta}_{2j} v_{r_{2j}}. \tag{44a}$$

Cross-multiplying both sides of Eq. (44a) by $^{n20}\boldsymbol{\delta}_{2j}$, it leads to

$$^{n20}\boldsymbol{\delta}_{2j} \times {}^{n20}\boldsymbol{v}_{2j} = r_{2j}\, {}^{n20}\boldsymbol{\omega}_{r_{2j}} - r_{2j}\, {}^{n20}\boldsymbol{\delta}_{2j}({}^{n20}\boldsymbol{\omega}_{r_i 2} \cdot {}^{n20}\boldsymbol{\delta}_{2j}) \tag{44b}$$

For the SPR-type leg, the angular velocity satisfy:

$$^{n20}\boldsymbol{\omega}_{r_{2j}} + \omega_{R_{2j1}}\, {}^{n20}\boldsymbol{R}_{2j1} = {}^{n20}_{n21}\boldsymbol{\omega} \tag{45a}$$

where ω_{R2j1} is the velocity of joint R_{2j1}. $^{n20}\boldsymbol{\omega}_{r_{2j}}$ is the angular velocity vector of r_{2j}.

Since $r_{2j} \perp R_{2j}$, dot-multiplying both sides of Eq. (45a) by $^{n20}\boldsymbol{\delta}_{2j}$,

$$^{n20}\boldsymbol{\omega}_{r_{2j}} \cdot {}^{n20}\boldsymbol{\delta}_{2j} = {}^{n20}_{n21}\boldsymbol{\omega} \cdot {}^{n20}\boldsymbol{\delta}_{2j}. \tag{45b}$$

From Eqs. (44b) and (45b) it leads to,

$$^{n20}\boldsymbol{\omega}_{r_{2j}} = \frac{^{n20}\boldsymbol{\delta}_{2j} \times {}^{n20}\boldsymbol{v}_{2j} + r_{2j}\, {}^{n20}\boldsymbol{\delta}_{2j}({}^{n20}_{n21}\boldsymbol{\omega} \cdot {}^{n20}\boldsymbol{\delta}_{2j})}{r_{2j}}. \tag{46a}$$

Eq. (46a) can be expressed as:

$$^{n20}\boldsymbol{\omega}_{r_{2j}} = \mathbf{J}_{\omega_{r_{2j}}} \begin{bmatrix} ^{n20}\boldsymbol{v}_{o_2} \\ ^{n20}_{n21}\boldsymbol{\omega} \end{bmatrix},$$
$$\mathbf{J}_{\omega_{r_{2j}}} = \frac{1}{r_{2j}}\left[S({}^{n20}\boldsymbol{\delta}_{2j}) \quad -S({}^{n20}\boldsymbol{\delta}_{2j})S({}^{n20}\boldsymbol{e}_{2j}) + r_{2j}\, {}^{n20}\boldsymbol{\delta}_{2j}\, {}^{n20}\boldsymbol{\delta}_{2j}^T \right]. \tag{46b}$$

Differentiating both sides of Eq. (46a) respect to time,

$$^{n20}\boldsymbol{\varepsilon}_{r_{2j}} = \frac{\begin{array}{l} ^{n20}\dot{\boldsymbol{\delta}}_{2j} \times {}^{n20}\boldsymbol{v}_{2j} + {}^{n20}\boldsymbol{\delta}_{2j} \times {}^{n20}\dot{\boldsymbol{v}}_{2j} + v_{r_{2j}}\, {}^{n20}\boldsymbol{\delta}_{2j}\, {}^{n20}\boldsymbol{\delta}_{2j}^T\, {}^{n20}_{n21}\boldsymbol{\omega} \\ + r_{2j}({}^{n20}\boldsymbol{\delta}_{2j}\, {}^{n20}\dot{\boldsymbol{\delta}}_{2j}^T + {}^{n20}\dot{\boldsymbol{\delta}}_{2j}\, {}^{n20}\boldsymbol{\delta}_{2j}^T)\, {}^{n20}_{n21}\boldsymbol{\omega} + r_{2j}\, {}^{n20}\boldsymbol{\delta}_{ij}\, {}^{n20}\boldsymbol{\delta}_{ij}^T\, {}^{n20}_{n21}\boldsymbol{\varepsilon} - v_{r_{2j}}\, {}^{n20}\boldsymbol{\omega}_{r_{2j}} \end{array}}{r_{2j}} \tag{47}$$

where $^{n20}\dot{\boldsymbol{\delta}}_{2j} = {}^{n20}\boldsymbol{\omega}_{r_{2j}} \times {}^{n20}\boldsymbol{\delta}_{2j}$,

$^{n20}\dot{\boldsymbol{v}}_{2j} = {}^{n20}\boldsymbol{a}_{o_2} + {}^{n20}_{n21}\boldsymbol{\varepsilon} \times {}^{n20}\boldsymbol{e}_{2j} + {}^{n20}_{n21}\boldsymbol{\omega} \times ({}^{n20}_{n21}\boldsymbol{\omega} \times {}^{n20}\boldsymbol{e}_{2j})$.

For the 3-SPR PM, Let r_{l2j} be the distance from the bottom to the mass-center of the cylinder and r_{u2j} be the distance form the mass-center to the top of the piston in the j-th leg.

For the 3-SPR PM, the velocity of the mass-center of the j-th cylinder can be expressed as

$$^{n20}\boldsymbol{v}_{l_{2j}} = r_{l_{2j}}\, {}^{n20}\boldsymbol{\omega}_{r_{2j}} \times {}^{n20}\boldsymbol{\delta}_{2j} = -r_{l_{2j}} S(\boldsymbol{\delta}_{2j})\, \mathbf{J}_{\omega_{r_{2j}}} \begin{bmatrix} ^{n20}\boldsymbol{v}_{o_2} \\ ^{n20}_{n21}\boldsymbol{\omega} \end{bmatrix} \tag{48a}$$

Thus, the velocity relation between the mass center of the j-th cylinder and the moving platform of the 3-SPR PM can be expressed as

$$\begin{bmatrix} ^{n20}\boldsymbol{v}_{l_{2j}} \\ ^{n20}\boldsymbol{\omega}_{l_{2j}} \end{bmatrix} = \mathbf{J}_{l_{2j}} \begin{bmatrix} ^{n20}\boldsymbol{v}_{o_2} \\ ^{n20}_{n21}\boldsymbol{\omega} \end{bmatrix}, \ \mathbf{J}_{l_{2j}} = \begin{bmatrix} -r_{l_{2j}} S({}^{n20}\boldsymbol{\delta}_{2j})\, \mathbf{J}_{\omega_{r_{2j}}} \\ \mathbf{J}_{\omega_{r_{2j}}} \end{bmatrix}. \tag{48b}$$

For the SPR leg, the linear acceleration of the mass-center of the j-th cylinder can be expressed as

$$^{n20}\boldsymbol{a}_{l_{2j}} = {}^{n20}_{n21}\boldsymbol{\varepsilon}_{r_{2j}} \times {}^{n20}\boldsymbol{\delta}_{l_{2j}} r_{l_{2j}} + {}^{n20}_{n21}\boldsymbol{\omega}_{r_{2j}} \times ({}^{n20}_{n21}\boldsymbol{\omega}_{r_{2j}} \times {}^{n20}\boldsymbol{\delta}_{2j}) r_{l_{2j}}. \tag{48c}$$

For the SPR leg, the velocity of the mass-center of the j-th piston can be expressed as

$$\begin{aligned} ^{n20}\boldsymbol{v}_{u_{2j}} &= {}^{n20}_{n21}\boldsymbol{\omega}_{r_{2j}} \times {}^{n20}\boldsymbol{\delta}_{2j}(r_{2j} - r_{u_{2j}}) + {}^{n20}\boldsymbol{\delta}_{2j}\dot{r}_{2j} \\ &= \left\{ -(r_{2j} - r_{u_{2j}}) S({}^{n20}\boldsymbol{\delta}_{2j})\, \mathbf{J}_{\omega_{r_{2j}}} + [\, {}^{n20}\boldsymbol{\delta}_{2j}\, {}^{n20}\boldsymbol{\delta}_{2j}^{\mathrm{T}} \quad -{}^{n20}\boldsymbol{\delta}_{2j}\, {}^{n20}\boldsymbol{\delta}_{2j}^{\mathrm{T}} S({}^{n20}\boldsymbol{e}_{2j})] \right\} \begin{bmatrix} ^{n20}\boldsymbol{v}_{o_2} \\ ^{n20}_{n21}\boldsymbol{\omega} \end{bmatrix}. \end{aligned} \tag{49a}$$

From Eqs. (46b) and (49a),

$$\begin{bmatrix} ^{n20}\boldsymbol{v}_{u_{2j}} \\ ^{n20}\boldsymbol{\omega}_{r_{2j}} \end{bmatrix} = \mathbf{J}_{u_{2j}} \begin{bmatrix} ^{n20}\boldsymbol{v}_{o_2} \\ ^{n20}_{n21}\boldsymbol{\omega} \end{bmatrix},$$
$$\mathbf{J}_{u_{2j}} = \begin{bmatrix} -(r_{2j} - r_{u_{2j}}) S({}^{n20}\boldsymbol{\delta}_{2j})\, \mathbf{J}_{\omega_{r_{2j}}} + [\, {}^{n20}\boldsymbol{\delta}_{2j}\, {}^{n20}\boldsymbol{\delta}_{2j}^{\mathrm{T}} \quad -{}^{n20}\boldsymbol{\delta}_{2j}\, {}^{n20}\boldsymbol{\delta}_{2j}^{\mathrm{T}} S({}^{n20}\boldsymbol{e}_{2j})] \\ \mathbf{J}_{\omega_{r_{2j}}} \end{bmatrix}. \tag{49b}$$

For the SPR leg, the linear acceleration of the mass-center of the j-th piston can be expressed as:

$$\begin{aligned} ^{n20}\boldsymbol{a}_{u_{2j}} &= {}^{n20}_{n21}\boldsymbol{\varepsilon}_{r_{2j}} \times {}^{n20}\boldsymbol{\delta}_{2j}(r_{2j} - r_{u_{2j}}) + {}^{n20}_{n21}\boldsymbol{\omega}_{r_{2j}} \times ({}^{n20}_{n21}\boldsymbol{\omega}_{r_{2j}} \times {}^{n_{10}}\boldsymbol{\delta}_{2j})(r_{2j} - r_{u_{2j}}) \\ &+ {}^{n20}\boldsymbol{\delta}_{2j}\ddot{r}_{2j} + 2({}^{n20}_{n21}\boldsymbol{\omega}_{r_{2j}} \times {}^{n20}\boldsymbol{\delta}_{2j})\dot{r}_{2j} \end{aligned} \tag{49c}$$

For the 3-SPR PM, the following equation is satisfied:

$$^{n20}\boldsymbol{\omega}_{l_{2j}} = {}^{n20}\boldsymbol{\omega}_{u_{2j}} = {}^{n20}\boldsymbol{\omega}_{r_{2j}},\ {}^{n20}\boldsymbol{\varepsilon}_{l_{2j}} = {}^{n20}\boldsymbol{\varepsilon}_{u_{2j}} = {}^{n20}\boldsymbol{\varepsilon}_{r_{2j}}. \tag{49d}$$

Let m_{o_i}, $\boldsymbol{I}_{o_i}$, $\boldsymbol{f}_{o_i}$, $\boldsymbol{n}_{o_i}$ and $\boldsymbol{G}_{o_i}$ be the mass, inertia matrix, inertia force, inertia torque, and the gravity of the moving platform for i-th PM. Let $\boldsymbol{F}_{o2}$, $\boldsymbol{T}_{o2}$ be the workloads applied onto $n21$ at o_2. Let $m_{l_{ij}}$, $\boldsymbol{I}_{l_{ij}}$, $\boldsymbol{f}_{l_{ij}}$, $\boldsymbol{n}_{l_{ij}}$ and $\boldsymbol{G}_{l_{ij}}$ (i = 1, 2; j = 1, 2, 3) be the mass, inertia matrix, inertia force, inertia torque, and the gravity of the lower link in j-th leg of the i-th PM, respectively. Let $m_{u_{ij}}$, $\boldsymbol{I}_{u_{ij}}$, $\boldsymbol{f}_{u_{ij}}$, $\boldsymbol{n}_{u_{ij}}$ and $\boldsymbol{G}_{u_{ij}}$ be the mass, inertia matrix, inertia force, inertia torque, and the gravity of the upper link in j-th leg of the i-th PM, respectively.

The inertia force, torque, and the gravity can be derived as follows:

$$\begin{aligned}
&{}^{n_{i0}}\boldsymbol{f}_{l_{ij}} = -m_{l_{ij}}{}^{n_{i0}}\boldsymbol{a}_{l_{ij}},\ {}^{n_{i0}}\boldsymbol{G}_{l_{ij}} = m_{l_{ij}}\boldsymbol{g},\\
&{}^{n_{i0}}\boldsymbol{n}_{l_{ij}} = -{}^{n_{i0}}\boldsymbol{I}_{l_{ij}}{}^{n_{i0}}\boldsymbol{\varepsilon}_{l_{ij}} - {}^{n_{i0}}\boldsymbol{\omega}_{l_{ij}} \times ({}^{n_{i0}}\boldsymbol{I}_{l_{ij}}{}^{n_{i0}}\boldsymbol{\omega}_{l_{ij}}),\\
&{}^{n_{i0}}\boldsymbol{f}_{u_{ij}} = -m_{u_{ij}}{}^{n_{i0}}\boldsymbol{a}_{u_{ij}},\ {}^{n_{i0}}\boldsymbol{G}_{u_{ij}} = m_{u_{ij}}\boldsymbol{g},\\
&{}^{n_{i0}}\boldsymbol{n}_{u_{ij}} = -{}^{n_{i0}}\boldsymbol{I}_{u_{ij}}{}^{n_{i0}}\boldsymbol{\varepsilon}_{u_{ij}} - {}^{n_{i0}}\boldsymbol{\omega}_{u_{ij}} \times ({}^{n_{i0}}\boldsymbol{I}_{u_{ij}}{}^{n_{i0}}\boldsymbol{\omega}_{u_{ij}}),\\
&{}^{n_{i0}}\mathbf{f}_{o_i} = -m_{o_i}{}^{n_{i0}}\mathbf{a}_{o_i},\ {}^{n_{i0}}\mathbf{G}_{oi} = m_{o_i}\mathbf{g},\\
&{}^{n_{i0}}\mathbf{n}_{o_i} = -{}^{n_{i0}}\mathbf{I}_{o_i}{}^{n_{i0}}\boldsymbol{\varepsilon}_{o_i} - {}^{n_{i0}}\boldsymbol{\omega}_{o_i} \times ({}^{n_{i0}}\mathbf{I}_{o_i}{}^{A}\boldsymbol{\omega}_{o_i}),\ {}^{n_{i0}}\mathbf{I}_{o_1} = {}^{n_{i0}}_{n_{i1}}\mathrm{R}\,{}^{n_{i0}}\mathbf{I}_{o_i}\\
&{}^{n_{i0}}\boldsymbol{I}_{l_{ij}} = {}^{n_{i0}}\mathbf{R}\,{}^{n_{i0}}\boldsymbol{I}_{l_{ij}},\ {}^{n_{i0}}\boldsymbol{I}_{u_{ij}} = {}^{n_{i0}}_{ij}\mathbf{R}\,{}^{ij}\boldsymbol{I}_{u_{ij}},\\
&{}^{n_{10}}\boldsymbol{I}_{o_1} = {}^{n_{10}}_{n_{11}}\mathbf{R}\,{}^{n_{10}}\boldsymbol{I}_{o_1},\ {}^{n_{10}}\boldsymbol{I}_{o_2} = {}^{n_{10}}_{n_{21}}\mathbf{R}\,{}^{n_{21}}\boldsymbol{I}_{o_2},\\
&{}^{n_{i0}}_{ij}\mathbf{R} = [\boldsymbol{R}_{ij1}\quad \boldsymbol{\delta}_{ij}\times\boldsymbol{R}_{ij1}\quad \boldsymbol{\delta}_{ij}]
\end{aligned} \tag{50}$$

where ${}^{n_{i0}}_{ij}\mathbf{R}$ denotes the rotational matrix of $\{ij\}$ relative to $\{n_{i0}\}$. $\{ij\}$ is a coordinate frame with $\boldsymbol{R}_{ij1}$, $\boldsymbol{\delta}_{ij}\times\boldsymbol{R}_{ij1}$ and $\boldsymbol{\delta}_{ij}$ are the diction vectors corresponding to their three orthogonal coordinate axes, which are used to express the inertia matrices.

Let F_{r2j} be the active force applied on rij. Based on the principle of virtue work,

$$\begin{aligned}
&\boldsymbol{F}_r^{\mathrm{T}}\boldsymbol{v}_r + \sum_{i=1}^{2}\sum_{j=1}^{3}\left(\begin{bmatrix}{}^{n_{i0}}\boldsymbol{f}_{l_{ij}}^{\mathrm{T}} + {}^{n_{i0}}\boldsymbol{G}_{l_{ij}}^{\mathrm{T}} & {}^{n_{i0}}\boldsymbol{n}_{l_{ij}}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}{}^{n_{i0}}\boldsymbol{v}_{l_{ij}}\\ {}^{n_{i0}}\boldsymbol{\omega}_{r_{ij}}\end{bmatrix} + \begin{bmatrix}{}^{n_{i0}}\boldsymbol{f}_{u_{ij}}^{\mathrm{T}} + {}^{n_{i0}}\boldsymbol{G}_{u_{ij}}^{\mathrm{T}} & {}^{n_{i0}}\boldsymbol{n}_{u_{ij}}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}{}^{n_{i0}}\boldsymbol{v}_{u_{ij}}\\ {}^{n_{i0}}\boldsymbol{\omega}_{r_{ij}}\end{bmatrix}\right)\\
&+\begin{bmatrix}{}^{n_{10}}\boldsymbol{f}_{o_1}^{\mathrm{T}} + {}^{n_{10}}\boldsymbol{G}_{o_1}^{\mathrm{T}} & {}^{n_{10}}\boldsymbol{n}_{o_1}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}{}^{n_{10}}\boldsymbol{v}_{o_1}\\ {}^{n_{10}}_{n_{11}}\boldsymbol{\omega}\end{bmatrix} + \begin{bmatrix}{}^{n_{10}}\boldsymbol{f}_{o_2}^{\mathrm{T}} + {}^{n_{10}}\boldsymbol{G}_{o_2}^{\mathrm{T}} & {}^{n_{10}}\boldsymbol{n}_{o_2}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}{}^{n_{20}}\boldsymbol{v}_{o_2}\\ {}^{n_{20}}_{n_{21}}\boldsymbol{\omega}\end{bmatrix} = 0,\\
&\boldsymbol{F}_r = \begin{bmatrix}T_{\theta_{11}} & T_{\theta_{12}} & T_{\theta_{13}} & F_{r_{21}} & F_{r_{22}} & F_{r_{23}}\end{bmatrix}^{\mathrm{T}}.
\end{aligned} \tag{51a}$$

From Eqs. (30b), (31c), (41), (43b), (48b), (49b) and (51a),

$$\begin{aligned}
\boldsymbol{F}_r = -(\mathbf{J}^{-1})^{\mathrm{T}}\Bigg\{&\sum_{i=1}^{2}\sum_{j=1}^{3}\left(\mathbf{J}_{si}^{T}\mathbf{J}_{l_{ij}}^{T}\begin{bmatrix}{}^{n_{i0}}\boldsymbol{f}_{l_{ij}} + {}^{n_{i0}}\boldsymbol{G}_{l_{ij}}\\ {}^{n_{i0}}\boldsymbol{n}_{l_{ij}}\end{bmatrix} + \mathbf{J}_{si}^{T}\mathbf{J}_{u_{ij}}^{T}\begin{bmatrix}{}^{n_{i0}}\boldsymbol{f}_{u_{ij}} + {}^{n_{i0}}\boldsymbol{G}_{u_{ij}}\\ {}^{n_{i0}}\boldsymbol{n}_{u_{ij}}\end{bmatrix}\right)\\
&+\mathbf{J}_{s1}^{T}\begin{bmatrix}{}^{n_{10}}\boldsymbol{f}_{o_1} + {}^{n_{10}}\boldsymbol{G}_{o_1}\\ {}^{n_{10}}\boldsymbol{n}_{o_1}\end{bmatrix} + \begin{bmatrix}{}^{n_{10}}\boldsymbol{f}_{o_2} + {}^{n_{10}}\boldsymbol{G}_{o_2}\\ {}^{n_{10}}\boldsymbol{n}_{o_2}\end{bmatrix}\Bigg\}.
\end{aligned} \tag{51b}$$

From Eq. (51b), the inverse dynamics of (3-RRS)+(3-SPR) S-PM can be solved.

7. Workspace

In this section, the workspace of the (3-RRS)+(3-SPR) S-PM is constructed using CAD variation geometry approach [20] and Matlab software. Generally, the workspace is constructed by a family of similar spatial boundary surfaces. For the (3-RRS)+(3-SPR) S-PM, the points of the boundary surfaces can be achieved when four actuators reach their minimum or maximum extensions and the other two actuators change in the range of extensions. The construction steps of the workspace are as follows:

Step 1. Construct the simulation mechanism of the (3-RRS)+(3-SPR) S-PM in CAD software [20].

Step 2. Set E_{10} = 1.20 mm, e_{11} = E_{20} = 0.80 m, e_{21} = 0.60 m in the simulation mechanism. Set $(\theta_{1i})_{\min}$ = 95°, $(\theta_{1j})_{\max}$ = 125°,$(r_{2i})_{\min}$ = 1.0 m,$(r_{2i})_{\max}$ = 1.5 m, $\delta\theta$ = 5°, δr = 0.05 m.

Step 3. Set θ_{13} = $(\theta_{1i})_{\max}$, r_{21} = r_{22} = r_{23} = $(r_{2i})_{\max}$. Set θ_{11} = $(\theta_{1j})_{\min}$ +$(j-1)\delta\theta$ (j = 1, . . . , w_1),where w_1 = $[(\theta_{11})_{\max}$ - $(\theta_{11})_{\min}]/\delta\theta$.

Step 4. Set j = 1 and increase θ_{12} by $\delta\theta$ at each increment from $(\theta_{12})_{\min}$ to $(\theta_{12})_{\max}$. Solve the position components (${}^{n10}X_{o2}$ ${}^{n10}Y_{o2}$ ${}^{n10}Z_{o2}$) using the simulation mechanism.

Step 5. Repeat the steps 4, except that set j = 2, . . . , w_1, other points of the boundary surface can be obtained.

Step 6. Repeat the steps 2-5, except set the other different four of the six actuators to reach their limited values and by varying the remaining two from the minimum extension to the maximum extension, respectively.

Step 7. Based on the points obtained from the above steps, construct the workspace boundary surfaces using the command for drawing surfaces in Matlab software.

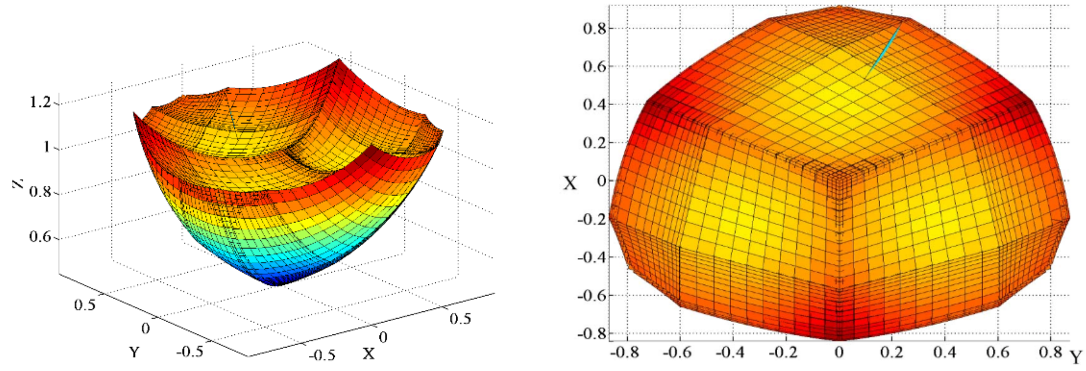
The workspace of the (3-RRS)+(3-SPR) S-PM is constructed as shown in Fig. 4.

8. Analytic solved example

In this section, the inverse dynamics of the (3-RRS)+(3-SPR) S-PM is computed by using the established dynamics model. Set the dimension parameters of the (3-RRS)+(3-SPR) S-PM as: E_1 = 1.2/q m, E_2 = e_1 = 0.8/qm, e_2 = 0.6/qm. Let the rotation of n_{21} relative to n_{10} formed by XYX Euler rotations, where α, β and λ are three Euler angles parameters about corresponding axes.

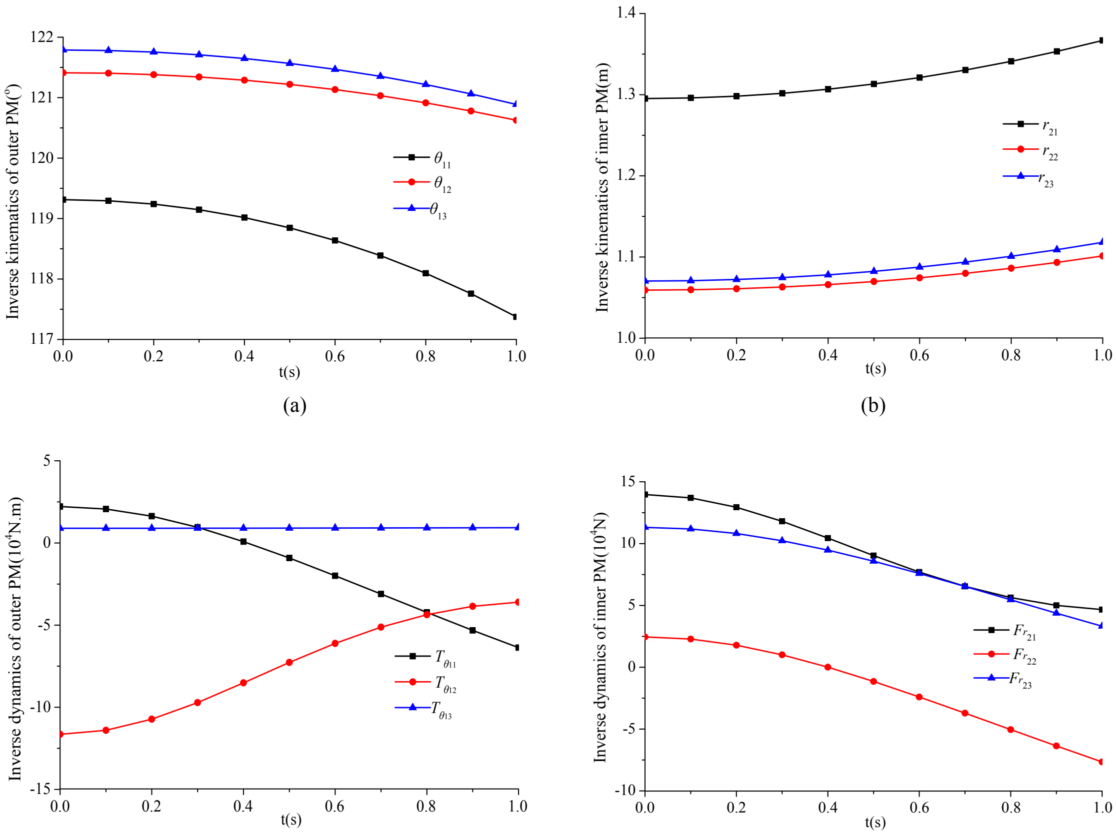
Set the mass and inertial parameters as: m_{o1} = 112.47Kg, m_{o2} = 48.54 Kg, m_{l11} = m_{l12} = m_{l13} = 47.62 Kg, m_{l21} = m_{l22} = m_{l23} = 12.54 Kg, m_{u11} = m_{u12} = m_{u13} = 12.54 Kg, m_{u21} = m_{u22} = m_{u23} = 9.75 Kg, ${}^{n10}I_{o1}$ = diag[1.14 1.14 2.16] $Kg{\cdot}m^2$, ${}^{n10}I_{o2}$ = diag[0.83 0.83 1.64] $Kg{\cdot}m^2$, ${}^{11}I_{l11}$ = ${}^{12}I_{l12}$ = ${}^{13}I_{l13}$ = diag[6.052 6.052 0.037] $Kg{\cdot}m^2$, ${}^{21}I_{l21}$ = ${}^{22}I_{l22}$ = ${}^{23}I_{l23}$ = diag[2.886 2.886 0.004] $Kg{\cdot}m^2$,${}^{11}I_{u11}$ = ${}^{12}I_{u12}$ = ${}^{13}I_{u13}$ = diag[6.052 6.052 0.037] $Kg{\cdot}m^2$, ${}^{21}I_{u21}$ = ${}^{22}I_{u22}$ = ${}^{23}I_{u23}$ = diag[0.475 0.475 0.006] $Kg{\cdot}m^2$.

Set the workloads applied onto n_{21} at o_2 as: $\boldsymbol{F}_{o2}$ = $[-20\ -30\ -60]^{\mathrm{T}}$, $\boldsymbol{T}_{o2}$ = $[-30\ -30\ 100]^{\mathrm{T}}$. Support the independent parameters (${}^{n10}X_{o2}$, ${}^{n10}Y_{o2}$, ${}^{n10}Z_{o2}$, α, β, λ) varying according constant accelerations with (-0.015 m/s^2 0.015 m/s^2 -0.02 m/s^2 0°/s^2



(a) The isometric view (b) The top view

Fig. 4. Workspace of the (3-RRS)+(3-SPR) S-PM.



(a) (b)

(c) (d)

Fig. 5. (a) Inverse kinematics of outer PM; (b) inverse kinematics of inner PM; (c) inverse dynamics of outer PM; (d) inverse dynamics of inner PM.

$0\,^{o}/s^2$ $0\,^{o}/s^2$) begin at initial pose (-0.19 m 0.16 m 0.45 m 5.8^o 10.2^o 2.6^o) from immobile state. The inverse kinematics are solved as shown in Figs. 5(a) and (b), the inverse dynamics are solved as shown in Figs. 5(c) and (d).

From the analytic solved example, it can be seen that when the displacement, velocity, and acceleration of the terminal platform are varied smoothly, the inverse velocity, acceleration and dynamics are varied smoothly in a large range. It implies that the proposed series-parallel dynamics simulator has good kinematics and dynamics characteristics.

9. Conclusion

The main contribution of this paper lies in the concept design and the establishment of inverse Jacobian, velocity, acceleration, dynamics and workspace of the series-parallel dynamics simulator formed by the 3-RRS PM and 3-SPR PM. The designed series-parallel dynamics simulator uses the outer and inner layout. This concept has high rotation motion ability and the advantage of compacted structure and small space-occupancy. By choosing the proper position parameters and geometrical constraints, the inverse position solutions in close form are derived. The formulae for solving the inverse velocity, acceleration and dynamics are derived in compact forms by skillfully integrating the kinematics, constraint and coupling information of the single PMs into the S-PM. The workspace of the (3-RRS)+(3-SPR) series-parallel dynamics simulator is constructed by CAD variation geometry approach. The result shows that this series-parallel dynamics simulator has a symmetric and large workspace.

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Bo Hu was born in 1982 in Hubei, P.R. China. He got his B.S. degree at Hubei University of Technology in Wuhan, P.R. China, in 2004, and Ph.D. at School of Mechanical Engineering, Yanshan University in Qinhuangdao, P.R. China, in 2010. He has been an Associate Professor at School of Mechanical Engineering, Yanshan University since 2013. His major reaserch focus on kinematics and dynamics of robotic systems. He has authored/co-authored more than 40 regular papers published in several journals approaching these topics.