

Influence of slip velocity in Herschel-Bulkley fluid flow between parallel plates - A mathematical study†

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Abstract

This theoretical study investigates three types of basic flows of viscous incompressible Herschel-Bulkley fluid such as (i) plane Couette flow, (ii) Poiseuille flow and (iii) generalized Couette flow with slip velocity at the boundary. The analytic solutions to the nonlinear boundary value problems have been obtained. The effects of various physical parameters on the velocity, flow rate, wall shear stress and frictional resistance to flow are analyzed through appropriate graphs. It is observed that in plane Poiseuille flow and generalized Couette flow, the velocity and flow rate of the fluid increase considerably with the increase of the slip parameter, power law index, pressure gradient. The fluid velocity is significantly higher in plane Poiseuille flow than in plane Couette flow. The wall shear stress and frictional resistance to flow decrease considerably with the increase of the power law index and increase significantly with the increase of the yield stress of the fluid. The wall shear stress and frictional resistance to flow are considerably higher in plane Poiseuille flow than in generalized Couette flow.

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Keywords: Herschel-Bulkley fluid; Plane Couette flow; Poiseuille flow; Generalized Couette flow; Slip velocity

1. Introduction

Recently, non-Newtonian fluid flows attracted several researchers due to its growing applications in various fields of science and engineering, such as bio-fluid mechanics, biomedical engineering, chemical engineering, food processing technology, polymer extrusion processes, drilling operations, metallurgy etc [1]. Herschel-Bulkley (H-B) fluid is classified as non-Newtonian fluid with yield stress, since the relationship between the shear stress and rate of strain of this fluid model is nonlinear [2, 3]. Due to its distinct physical properties, it finds applications in the manufacturing of bioengineering products, paints, synthetic lubricants and also in the biological fluid flows such as blood, synovial fluids etc [4-6].

The studies on the incompressible fluid flow with slip velocity at the boundary of the flow region was proposed by Navier [7] in 1823 and he propounded that the tangential velocity of the fluid at any point on the surface of the solid boundary is proportional to the tangential stress acting at that point. The contributions of Basset [8] and O'Neil et al. [9] are some sample studies for the existence of the slip velocity at the boundary of the flow region. It is well understood that in

most of the flow problems studied at the micro scale, the assumption of no-slip velocity at the boundary is true only if the surface of the boundary is very rough, since the no-slip velocity conditions in this case cannot not derived from the first principle [10, 11]. Goldstein [12] and Batchelor [13] reported that the use of no-slip velocity at the boundary stems from the necessity that the theoretical predictions need to agree with the experimental observations. But, if the surface of the boundary is smooth, then the usual assumption of no-slip velocity at the boundary is not valid, indeed the fluid slips at the smooth boundary as this is more evident from the studies of Lauga et al. [10], Denn [14], Potente et al. [15], Mitsoulis et al. [16]. Denn [14] propounded that in molecular studies such as in polymer melts and polymer extrusion process, at the boundary the slip effects are found at the macro scale which leads to flow instabilities.

The aforementioned wealthy background on the slip velocity at the boundary of fluid flow attracted the attention of several researchers to do further investigations in this field with applications to various disciplines of engineering and technology [17-20]. The analytical solutions of shear flow problems with slip velocity at the boundary play a major role not only in solving the relevant industrial problems, but also to understand the complexity of many flow problems. Indeed there are many analytical solutions available in the literature for slip

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shear flow problems and of these, some of them are very trivial and others involve more complex rheological parameters [21-26]. Although the simple shear flow solutions seem to be trivial, they are very useful for the better understanding of more complex solutions and even these solutions lead to novel methods to solve advanced level problems [27-29].

Wu et al. [30] obtained analytical solutions to the pressure driven unsteady flow of Newtonian fluid in micro-tubes with slip velocity as defined by Navier [7]. In an analytical study on the flow through a pipe, Mathews and Hill [24] used the sip boundary conditions proposed by Thompson and Troian [31], while Yang ang Zhau [23] used the slip velocity condition of Navier [7] in their theoretical studies to the squeeze flow in a pipe. Hakeem et al. [32] investigated the effects of slip velocity in the peristaltic motion of power law fluid through an inclined tube. Chen and Zhu [33] performed analytical studies on the Couette-Poiseuille flow of Bingham fluid between two parallel porous plates. Abelman et al. [34] obtained numerical solutions for their computational study on the steady Couette flow of thermodynamic third grad fluid through porous medium, using rotating frame of reference. Noor et al. [35] investigated the effects of slip velocity in the mixed convection stagnation flow of micropolar nanofluid along a vertically stetching surface. Devakar and Ramesh [36] analytically in vestigated the Couette flow, Poiseuille flow and generalized Couette flow of Casson fluid between parallel plates.

 The (i) plane Couette flow (ii) plane Poiseuille flow and (iii) generalized Couette flow of viscous incompressible fluid are some of the simple shear flows in the axial (horizontal) direction and although the analytical solutions of these flows are simple, these solutions are the building blocks for the easy understanding of more complex fluid flows. Since the Her schel-Bulkley (H-B) fluid model has one more parameter namely the power law index *n* than the Casson fluid model, it is hoped that the use of H-B fluid model rather than Casson fluid model in these types of flows would yield more information on flow characteristics. As mentioned by Chaturani [37] and Tu and Deville [2], Casson fluid model can be used for flow at moderate shear rates, whereas, the H-B fluid model can still be used at low shear rates. Hence, it is appropriate to density, D/Dt is the material derivative; $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is the model the incompressible fluid used in these kinds of flow as H-B fluid model rather than Casson fluid model.

The mathematical modeling of H-B fluid for the (i) plane Couette flow, (ii) plane Poiseuille flow and (iii) generalized Couette flow between parallel plates was not studied by any one so far, to the knowledge of the author. Hence, in this mathematical analysis, we investigate the steady, laminar and fully developed flow of viscous incompressible fluid between parallel plates, considering the flow to be (i) plane Couette flow, (ii) plane Poiseuille flow and (iii) generalized Couette noted that for particular values of the parameters *n* and τ_y , the H-B fluid model reduces to Bingham fluid model, power law in the modeling of industrial engineering problems. When $n =$

1, H-B fluid model reduces to Bingham fluid model, when *τ^y* $= 0$, it reduces to the power law index and when $n = 1$ and τ ^{*y*} = 0, it reduces to Newtonian fluid model. When *n* < 1, the H-B fluid model behaves as shear thinning fluid and when $n < 1$, it exhibits the character of shear thickening fluid. Since the wall shear stress and frictional resistance to flow are important flow measurements which are widely used to analyze the flow patterns and also in the design of flow channels in industrial en gineering. In addition to the studies on the effects of slip velocity on the velocity distribution and volumetric flow rate, the present study also discusses the effects of slip velocity on the wall shear stress and frictional resistance to flow which are not studied by Devakar and Ramesh [35]. Hence, it is felt that the present study will have more scope than the earlier studies in applications point of view. The layout of the paper is given below:

In Sec. 2, the three types of flows such as (i) plane Couette flow, (ii) plane Poiseuille flow and (iii) generalized Couette flow are mathematically formulated and then the analytical solutions to velocity distribution, flow rate, frictional resistance to flow and wall shear stress are obtained for each of the flow considered. The variation of the aforementioned flow quantities with yield stress, slip parameter, power law index and pressure gradient are discussed through appropriate graphs in Sec. 3. The main findings of this investigation are summarized in the conclusion Sec. 4. in Sec. 2, the three types of flows such as (i) plane Couette
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2. Mathematical formulation and solution method

2.1 Basic governing equations

For the laminar and fully developed flow of viscous incom pressible fluid, the equations of continuity and momentum reduce to the following form:

$$
\nabla \cdot \vec{q} = 0 \tag{1}
$$

$$
\overline{\rho} \frac{D \overline{u}_i}{D \overline{t}} = \frac{\partial}{\partial \overline{x}_j} (\overline{\sigma}_y), \quad i = 1, 2, 3
$$
 (2)

tance to riow and sinear stress are obtained for each or
flow considered. The variation of the aforementioned flow
quantities with yield stress, slip parameter, power law index
and pressure gradient are discussed through $\vec{q} = (\overline{u}_1, \overline{u}_2, \overline{u}_3)$ is the velocity vector, $\overline{\rho}$ is the fluid's The material of the variance of the variance of the material order and pressure gradient are discussed through appropriate graphs in Sec. 3. The main findings of this investigation are summarized in the conclusion Sec. 4. Cartesian coordinate system; σ_{ij} is the Cauchy's stress tensor. The modified constitutive equation (in the tensor form) of the generalized Newtonian fluid model which establishes the relationship between the stress and the rate of strain is given below [38, 39]: **Easic governing equations**

For the laminar and fully developed flow of viscous incom-

ssible fluid, the equations of continuity and momentum

luce to the following form:
 $\nabla \cdot \vec{q} = 0$ (1)
 $\vec{p} \frac{\overline{Di}_i}{\overline{Di}} = \frac{\$

$$
\overline{\sigma}_{ii} = -\overline{p}\delta_{ii} + 2\overline{\mu}(\overline{J}_2)\overline{V}_{ii}
$$
 (3)

flow, the flowing fluid is modeled as H-B fluid model. It is is the Kronecker delta; $\overline{\mu}(\overline{J}_2)$ is the variable viscosity of the fluid model and Newtonian fluid model which are widely used stress and for this fluid, the viscosity coefficient $\overline{\mu}(\overline{J}_2)$ is where \bar{p} is the pressure; \bar{J}_2 and \bar{V}_i are the second invariant of the stress tensor and deformation tensor respectively; δ_{ij} reduce to the following form:
 $\nabla \cdot \vec{q} = 0$ (1)
 $\overline{\rho} \frac{D \vec{u}_i}{Dt} = \frac{\partial}{\partial x_j} (\overline{\sigma}_{ij}), i = 1, 2, 3$ (2)

where $\vec{q} = (\overline{u}_i, \overline{u}_i, \overline{u}_j)$ is the velocity vector, $\overline{\rho}$ is the fluid's

density, *DIDt* is the ma generalized Newtonian fluid. Herschel-Bulkley (H-B) fluid is a non-Newtonian (generalized Newtonian) fluid with yield $\overline{\rho} \frac{D u_i}{D \overline{t}} = \frac{\partial}{\partial \overline{x}_i} (\overline{\sigma}_y)$, $i = 1, 2, 3$ (2)
where $\vec{q} = (\overline{u}_i, \overline{u}_i, \overline{u}_i)$ is the velocity vector, $\overline{\rho}$ is the fluid's
density, D/Dt is the material derivative; $(\overline{x}_i, \overline{x}_i, \overline{x}_i)$ is defined as below:

$$
\overline{\mu}(\overline{J}_2) = \overline{K} \left| \overline{J}_2 \right|^{n-1} + \frac{\overline{\tau}_y}{\left| \overline{J}_2 \right|} \tag{4}
$$

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 $(\overline{J}_2) = \overline{K} |\overline{J}_2|^{n-1} + \frac{\overline{\tau}_y}{|\overline{J}_2|}$ (4)
 $\overline{K} = \sqrt[n]{\overline{\mu}_{H}}$; $\overline{\mu}_{H}$ and $\overline{\tau}_y$ are the H-B fluid's coeffi-

of viscosit *D. S. Sankar and U. Lee / Journal of Mechanical Science and Technology 30 (7) (2)*
 \overline{J}_2) = $\overline{K} \left| \overline{J}_2 \right|^{n-1} + \frac{\overline{\tau}_y}{\left| \overline{J}_2 \right|}$ (4) \overline{Y}
 $\overline{K} = \sqrt[n]{\mu_H}$; $\overline{\mu}_H$ and $\overline{\tau}_y$ are the H-B fl where $K = \sqrt[n]{\mu_H}$; $\overline{\mu}_H$ and $\overline{\tau}_y$ are the H-B fluid's coefficient of viscosity and yield stress respectively; *n* is the power law index and

$$
\overline{J}_2 = \frac{1}{2} \overline{V}_{ij} \overline{V}_{ij} = \frac{1}{2} \left[\overline{V}_{11}^2 + \overline{V}_{22}^2 + \overline{V}_{33}^2 + 2 \left(\overline{V}_{12}^2 + \overline{V}_{13}^2 + \overline{V}_{23}^2 \right) \right]
$$
(5)

$$
\overline{V}_{ij} = \frac{1}{2} \left(\frac{\partial \overline{u}_i}{\partial \overline{x}_j} + \frac{\partial \overline{u}_j}{\partial \overline{x}_i} \right), \text{ for } i, j = 1, 2, 3.
$$
 (6)

For the uni-directional steady, laminar and slow flow of viscous incompressible fluid in the horizontal direction in a channel between parallel flat plates, let us use Cartesian coordenote the velocity vector by $\vec{q} = (\overline{u}_1, \overline{u}_2, \overline{u}_3) = (\overline{u}, 0, 0)$. For this flow, the aforesaid governing equations of motion reduce to the following equation:

$$
\frac{d}{d\bar{y}}\left[\left\{\overline{K}\left(\frac{d\overline{u}}{d\overline{y}}\right)^{\frac{1}{n}-1} + \frac{\overline{\tau}_{y}}{\left(\frac{d\overline{u}}{d\overline{y}}\right)}\right\}\left(\frac{d\overline{u}}{d\overline{y}}\right)\right] - \frac{d\overline{p}}{d\overline{x}} = 0.
$$
\n(7)

2.2 Plane Couette flow

Let us consider the steady, laminar and fully developed flow of viscous incompressible H-B fluid in a channel between two infinitely long horizontal parallel plates which are at distant 2*h*. The pressure gradient between the plates is taken as zero, i.e. the pressure \bar{p} between the plates is assumed as dinate system $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (\bar{x}, \bar{y}, \bar{z})$ to analyze the flow. We

denote the velocity vector by $\vec{q} = (\bar{u}_1, \bar{u}_2, \bar{u}_3) = (\bar{u}_1, 0, 0)$. For

this flow, the aforesiad governing equations of motion reduce

to denote the velocity vector by $\vec{q} = (\vec{u}, \vec{u}, \vec{u}) = (\vec{u}, 0, 0)$. For
this flow, the aforesaid governing equations of motion reduce
to the following equation:
the following equation:
 $\frac{d}{d\vec{v}}\left\{\vec{K}\left(\frac{d\vec{u}}{d\vec{y}}$ \bar{U} in the horizontal direction. The fluid flow in the channel is in the \bar{x} direction and is due to the movement of the upper plate. This kind of flow is known as plane Couette flow. The geometry of this flow is shown in Fig. 1(a). Assume that the relative velocity between the plate and fluid is proportional to the plate's shear rate and thus, the governing equation of motion Eq. (7) reduces to **EVENT:**
 EXECUTE: the steady, laminar and fully developed

incompressible H-B fluid in a channel be-

tely long horizontal parallel plates which are

pressure gradient between the plates is assumed as

were plate $\overline{$ **Plane Couette flow**
 g of the steady, laminar and fully developed

or of viscous incompressible H-B fluid in a channel be-

y of viscous incompressible H-B fluid in a channel be-

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stant 2*h*. Th Let us consider the steady, laminar and fully developed
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then two infinitely long horizontal practical patallel plates which are

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Using the consider plane with the baseline and fully developed

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or we consider the w **Example 10** For plate $\overline{y} = \overline{h}$ is lieted used, where the consequence is the lower plate $\overline{y} = h$ is lieted usedenly with constant velocity $\overline{u}(\overline{h}) + \overline{\beta} \left[\left\{ \overline{R} \left(\frac{d\overline{u}}{dy} \right)^{\frac{1}{n}} + \overline{r}_y \left(\frac$ *y* = *A* is given such with considerate viewer by the three such that $\overline{a}(\overline{b})^+ + \overline{r}_y \left(\frac{dy}{d\overline{b}}\right) \left(\frac{dx}{dy}\right) = \overline{d}$
 dy in given the channel is the convention of the upper

direction and is due to the mov ant the lower paix is yearn is not like to the point of the boundary condition of the boundary condition of the boundary condition \overline{x} if $\left[\vec{k}\right]$ $\left[\vec{x}\right]$ $\left[\frac{d\overline{u}}{d\overline{y}}\right]$ $\left[\vec{x}\right]$ $\left[\frac{d\overline{u}}{d\overline{y$ $\vec{v} = \vec{h}$ is relied suddenly with constant velocity
 $\vec{v} = \vec{h}$ is relied flow in the channel is
 $\vec{u}(\vec{h}) + \vec{B} \left[\frac{R}{\epsilon} \left(\frac{d\vec{u}}{dy} \right)^{-1} + \vec{r} \left(\frac{d\vec{y}}{dy} \right) \right] \left(\frac{d\vec{u}}{dy} \right]_{y=\vec{h}} = \vec{U}$

cition

$$
\frac{d}{d\bar{y}} \left[\left\{ \overline{K} \left(\frac{d\overline{u}}{d\overline{y}} \right)^{\frac{1}{n-1}} + \overline{\tau}_y \left(\frac{d\overline{y}}{d\overline{u}} \right) \right\} \left(\frac{d\overline{u}}{d\overline{y}} \right) \right] = 0.
$$
 (8)

The slip boundary conditions of this flow are

$$
\overline{u}\left(-\overline{h}\right) - \overline{\beta} \left[\left\{ \overline{K} \left(\frac{d\overline{u}}{d\overline{y}} \right)^{\frac{1}{n-1}} + \overline{\tau}_{y} \left(\frac{d\overline{y}}{d\overline{u}} \right) \right\} \left(\frac{d\overline{u}}{d\overline{y}} \right) \right]_{\overline{y} = -\overline{h}} = 0 \qquad (9a) \qquad \text{The no} \qquad (9a) \qquad (9a) \text{ and}
$$

Fig. 1. Pictorial description of flow geometries between parallel plates.

(c) Generalized Couette flow
\n1. Pictorial description of flow geometries between parallel plates.
\n
$$
\overline{u}(\overline{h}) + \overline{\beta} \left[\left\{ \overline{K} \left(\frac{d\overline{u}}{d\overline{y}} \right)^{\frac{1}{n-1}} + \overline{\tau}_{y} \left(\frac{d\overline{y}}{d\overline{u}} \right) \right\} \left(\frac{d\overline{u}}{d\overline{y}} \right) \right]_{\overline{y} = \overline{h}} = \overline{U}
$$
\n(9b)
\nere $\overline{\beta}$ is the slip parameter. Let us introduce the following
\nn-dimensional variables:
\n
$$
y = \frac{\overline{y}}{\overline{h}}; u = \frac{\overline{u}}{\overline{U}}; \overline{\tau}_{y} = \frac{\sqrt{\overline{h} \overline{\tau}_{y}}}{\overline{K} \sqrt{\overline{U}}}; \gamma = \frac{\overline{K} \sqrt[n]{U} \overline{\beta}}{\overline{U} \sqrt[n]{\overline{h}}}.
$$
\n(10)
\nUsing the non-dimensional variables Eq. (10) into Eq. (8),
\nget the dimensionless form of the equation of motion as
\n
$$
\frac{d}{dy} \left[\left(\frac{du}{dy} \right)^{\frac{1}{n}} + \overline{\tau}_{y} \right] = 0.
$$
\n(11)
\nThe non-dimensional form of the boundary conditions Eqs.
\nand (9b) are given below:

where $\overline{\beta}$ is the slip parameter. Let us introduce the following non-dimensional variables:

$$
y = \frac{\overline{y}}{\overline{h}}; u = \frac{\overline{u}}{\overline{U}}; \tau_{y} = \frac{\sqrt[n]{h}\,\overline{\tau}_{y}}{\overline{K}\sqrt[n]{\overline{U}}}; \gamma = \frac{\overline{K}\sqrt[n]{U}\,\overline{\beta}}{\overline{U}\sqrt[n]{h}}.
$$
 (10)

Using the non-dimensional variables Eq. (10) into Eq. (8), we get the dimensionless form of the equation of motion as

$$
\frac{d}{dy}\left[\left(\frac{du}{dy}\right)^{\frac{1}{n}} + \tau_y\right] = 0.
$$
\n(11)

0 The non-dimensional form of the boundary conditions Eqs. (9a) and (9b) are given below:

() 1 ¹ ¹ ⁰ *n y ^y du u dy* ^g ^t =- é ù æ ö ê ú - - + = ç ÷ è ø ë û (12a) () 1 ¹ ¹ ¹ *n y ^y du u dy* ^g ^t ⁼ é ù æ ö ê ú + + = ç ÷ è ø ë û . æ ö ç ÷ + = è ø *u A* (- = ¹) ^g (14a) *u A* (1 1 .) = - ^g (14b)

Integration of Eq. (11) yields

L	J _{y=1}	(12b)	the <i>x</i> direction. This kind Positiveulation of Eq. (11) yields	(12b)	the <i>x</i> direction. This kind Positive flow. The geom 1(b). To this flow condition Eq. (7) remains uncha tion Eq. (7) remains uncha and
\n $\left(\frac{du}{dy}\right)^{\frac{1}{n}} + \tau_y = A$ \n	(13)				
\n $\frac{d}{d\theta} \left[\sqrt{\overline{K} \left(\frac{d\overline{u}}{dy}\right)^{\frac{1}{n}} + \overline{\tau}_y \left(\frac{d\overline{u}}{d\overline{u}}\right)^{\frac{1}{n-1}} + \overline{\tau}_y \left(\frac{d\overline{u}}{d\overline{u}}\right)^{\frac{1}{n-1}} + \overline{\tau}_y \left(\frac{d\overline{u}}{d\overline{u}}\right)^{\frac{1}{n-1}}$ \n					
\n Here <i>A</i> is the constant of integration to be determined. Using 4. (14a)\n	\n For this flow, the slip bo 2. (14b)\n				
\n From Eq. (13), one can obtain the expression for fluid ve- city as\n	\n (14b)\n	\n $\overline{u}(-\overline{h}) - \overline{\beta} \left[\sqrt{\overline{K} \left(\frac{d\overline{u}}{d\overline{y}}\right)^{\frac{1}{n-1}} + \overline{\beta} \left(\frac{d\overline{u}}{d\overline{y}}\right)^{\frac{1}{n-1}} + \overline{\beta} \left(\frac{d\overline{u}}{d\overline{y}}\right)^{\frac{1}{n-1}} + \overline{\beta} \left(\frac{d\overline{u}}{d\overline{y}}\right)^{\frac{1}{n-1}}$ \n			
\n Here\n	\n We have <i>β</i> is the slip parameters where <i>β</i> is the slip parameters where <i>β</i> is the slip parameters are defined in Eq. (10				

where *A* is the constant of integration to be determined. Using Eq. (13) in Eqs. (12a) and (12b), we obtain

$$
u(-1) = \gamma A \tag{14a}
$$

$$
u(1) = 1 - \gamma A. \tag{14b}
$$

From Eq. (13), one can obtain the expression for fluid velocity as

$$
u(y) = Ry + B \tag{15}
$$

where

$$
R = \left(A - \tau_y\right)^n\tag{16}
$$

n $\left(\frac{du}{dy}\right)^{\frac{1}{n}} + r_y = A$ (13)

ere *A* is the constant of integration to be determined. Using
 $u(-1) = \gamma A$.

(14a)

For this flow, the s
 $u(1) = 1 - \gamma A$.

(14b)

For this flow, the s

For this flow, the s

For this flo and B is the constant of integration to be determined. Using the boundary conditions Eqs. (14a) and (14b) in Eq. (15), we obtain

$$
u(1) = 1 - \gamma A
$$
 (14a)
\n
$$
u(1) = 1 - \gamma A
$$
 (14b)
\nFrom Eq. (13), one can obtain the expression for fluid ve-
\n
$$
u(y) = Ry + B
$$
 (15) $\overline{u}(\overline{h}) + \overline{B} \left\{ \overline{k} \left(\frac{d\overline{u}}{dy} \right)^{\frac{1}{n}} + \overline{r}_y \left(\frac{d\overline{y}}{dt} \right) \right\}$
\nHere
\n $R = (A - \tau_y)^n$ (16) $\overline{u}(\overline{h}) + \overline{B} \left\{ \overline{k} \left(\frac{d\overline{u}}{dy} \right)^{\frac{1}{n}} + \overline{r}_z \left(\frac{d\overline{y}}{dt} \right) \right\}$
\nwhere β is the slip parameter. Let us
\n $R = (A - \tau_y)^n$ (16) \overline{u} in Eq. (15), we
\n $P = \frac{\overline{y}}{h}; u = \frac{\overline{u}}{\overline{U}}; \tau_y = \frac{\overline{u}}{\overline{K} \sqrt[3]{U}}; \gamma = \frac{\overline{K} \sqrt{U}}{\overline{U} \sqrt[3]{K}}$
\n $R = \frac{1}{2}(1 - 2\gamma A)$ (17) $x = \frac{\overline{x}}{h}; p = \frac{\sqrt{\overline{h}} \overline{p}}{\overline{K} \sqrt[3]{U}}; \gamma = \frac{\overline{K} \sqrt{U}}{\overline{U} \sqrt[3]{K}}$
\n $H(y) = \frac{1}{2}[1 + (1 - 2\gamma A)y].$ (18)
\n $\alpha(y) = \frac{1}{2}[1 + (1 - 2\gamma A)y].$ (18)
\nFrom Eqs. (16) and (17), one can get
\n $2(A - \tau_y)^n + 2\gamma A - 1 = 0$. (19)
\nEq. (19) is a non-linear equation for the unknown A. For a
\neen set of values of τ_u and γ , Eq. (19) can be solved by ap-
\ning Newton-Raphson method. The non-dimensional flow
\n $u(1) + \gamma \left[\$

Substitution of *R* and *B* values in Eq. (15) yields

$$
u(y) = \frac{1}{2} \Big[1 + (1 - 2\gamma A)y \Big].
$$
 (18)

From Eqs. (16) and (17), one can get

$$
2\left(A - \tau_y\right)^n + 2\gamma A - 1 = 0\,. \tag{19}
$$

Eq. (19) is a non-linear equation for the unknown *A*. For a given set of values of τ ^H and *γ*, Eq. (19) can be solved by applying Newton-Raphson method. The non-dimensional flow rate is obtained with the help of Eq. (18) as below: (17) $\vec{h} \cdot \vec{r} = \vec{R} \cdot \sqrt{U}$

Substitution of *R* and *B* values in Eq. (15) yields

The non-dimensional form
 $u(y) = \frac{1}{2} [1 + (1 - 2\gamma A)y]$.

(18)

From Eqs. (16) and (17), one can get
 $2(A - \tau_y)^2 + 2\gamma A - 1 = 0$.
 $2(A - \tau_y)^$

$$
Q = \int_{-1}^{1} u(y) \, dy = 1 \,. \tag{20}
$$

2.3 Plane Poiseuille flow

 (12b) the *x* direction. This kind of flow is generally called as plane Consider the steady, laminar and fully developed flow of viscous incompressible H-B fluid through a channel which is bounded by two infinitely long horizontal parallel plates which are at distant 2*h* and are at rest. It is assumed that the fluid flow is due to the applied constant pressure gradient in Poiseuille flow. The geometry of this flow is depicted in Fig. 1(b). To this flow conditions, the governing equation of motion Eq. (7) remains unchanged and is restated below: *n* (7) (2016) 3203-3218
 iille flow
 iille flow
 iiii c i c i n i c i c i c i c i c i c i c i c i c i c i c i c i c i c i c i c i c i *y and Technology 30 (7) (2016) 3203-3218
 Plane Poiseuille flow

Consider the steady, laminar and fully developed flow of

consince incompressible H-B fluid through a channel which is

anded by two infinitely long ho e and Technology 30 (7) (2016) 3203-3218*
 Plane Poiseuille flow

Consider the steady, laminar and fully developed flow of

cous incompressible H-B fluid through a channel which is

unded by two infinitely long horizon *d Technology 30 (7) (2016) 3203-3218*
 ane Poiseuille flow

sider the steady, laminar and fully developed flow of

sincompressible H-B fluid through a channel which is

at our infinitely long horizontal parallel plates *Consider the steady, lanniar and thuy developed now of considerate seady, lanniar and thuy developed now of non-declination and the step in the ste* by two infinitely long horizontal parameter when
two infinitely long horizontal parameter when the state of distant 2*h* and are at rest. It is assumed that the
s due to the applied constant pressure gradient in
ion. This sis incompressible H-B fluid through a channel rand function
is is incompressible H-B fluid through a channel which is
led by two infinitely long horizontal parallel plates
are at distant 2h and are at rest. It is assumed pressible H-B fluid through a channel which is
two infinitely long horizontal parallel plates
distant 2h and are at rest. It is assumed that the
due to the applied constant pressure gradient in
1. This kind of flow is gen *y* a direction. This kind of flow is generally called as plane
 y a direction. This kind of flow is generally called as plane
 y during the dy flow that of flow is depicted in Fig.

(b). To this flow conditions, the w is due to the applied constant pressure gradient in
ection. This kind of flow is generally called as plane
efforts. The geometry of this flow is depicted in Fig.
this flow conditions, the governing equation of mo-
(7) r

$$
\frac{d}{d\bar{y}} \left[\left\{ \overline{K} \left(\frac{d\overline{u}}{d\overline{y}} \right)^{\frac{1}{n-1}} + \overline{\tau}_y \left(\frac{d\overline{y}}{d\overline{u}} \right) \right\} \left(\frac{d\overline{u}}{d\overline{y}} \right) \right] - \frac{d\overline{p}}{d\overline{x}} = 0. \tag{21}
$$

For this flow, the slip boundary conditions are

e *x* direction. This kind of flow is generally called as plane
\niseuille flow. The geometry of this flow is depicted in Fig.
\nb). To this flow conditions, the governing equation of mo-
\nEq. (7) remains unchanged and is restated below:
\n
$$
\frac{d}{d\overline{y}} \left\{ \overline{K} \left(\frac{d\overline{u}}{d\overline{y}} \right)^{\frac{1}{n-1}} + \overline{\tau}_y \left(\frac{d\overline{y}}{d\overline{u}} \right) \right\} \left(\frac{d\overline{u}}{d\overline{y}} \right) - \frac{d\overline{p}}{d\overline{x}} = 0.
$$
\n(c1)
\nFor this flow, the slip boundary conditions are
\n
$$
\overline{u}(-\overline{h}) - \overline{\beta} \left[\left\{ \overline{K} \left(\frac{d\overline{u}}{d\overline{y}} \right)^{\frac{1}{n-1}} + \overline{\tau}_y \left(\frac{d\overline{y}}{d\overline{u}} \right) \right\} \left(\frac{d\overline{u}}{d\overline{y}} \right) \right]_{\overline{y} = -\overline{h}} = 0
$$
\n(c2a)
\n
$$
\overline{u}(\overline{h}) + \overline{\beta} \left\{ \left\{ \overline{K} \left(\frac{d\overline{u}}{d\overline{y}} \right)^{\frac{1}{n-1}} + \overline{\tau}_y \left(\frac{d\overline{y}}{d\overline{u}} \right) \right\} \left(\frac{d\overline{u}}{d\overline{y}} \right) \right\} = 0
$$
\n(22b)
\nhere β is the slip parameter. Let us introduce the following
\nnensionless variables (the non-dimensional variables that
\nedefined in Eq. (10) are used here too):

$$
\overline{u}\left(\overline{h}\right) + \overline{\beta} \left\lfloor \left\{\overline{K}\left(\frac{d\overline{u}}{d\overline{y}}\right)^{\frac{1}{n-1}} + \overline{\tau}_{y}\left(\frac{d\overline{y}}{d\overline{u}}\right)\right\} \left(\frac{d\overline{u}}{d\overline{y}}\right) \right\rfloor_{\overline{y} = \overline{h}} = 0 \tag{22b}
$$

where β is the slip parameter. Let us introduce the following dimensionless variables (the non-dimensional variables that are defined in Eq. (10) are used here too):

$$
v = \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} + \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} + \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} = 0 \qquad (22a)
$$
\n
$$
v = \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} + \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} + \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} = 0 \qquad (22a)
$$
\n
$$
v = \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} + \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} + \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} = 0 \qquad (22b)
$$
\n
$$
v = \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} + \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} + \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} = 0 \qquad (22b)
$$
\n
$$
v = \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} + \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} + \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} = 0 \qquad (22b)
$$
\n
$$
v = \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} + \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} + \frac{1}{2} \left[\frac{d\theta}{dt} \right]_{-\pi}^{2} = 0 \qquad (22b)
$$
\n
$$
v = \frac{\pi}{h} \left[\frac{u}{\sqrt{h}} \right]_{-\pi}^{2} + \frac{1}{2} \left[\frac{d\theta}{\sqrt{h}} \right]_{-\pi}^{2} + \frac{
$$

The non-dimensional form of the governing equation of motion Eq. (21) and the boundary conditions Eqs. (22a) and (22b) are obtained as below respectively:

$$
\frac{d}{dy}\left[\left(\frac{du}{dy}\right)^{\frac{1}{n}} + \tau_y\right] = -G\tag{24}
$$

$$
u(-1) - \gamma \left[\left(\frac{du}{dy} \right)^{\frac{1}{n}} + \tau_y \right]_{y=-1} = 0 \tag{25a}
$$

$$
h \begin{bmatrix} K \sqrt[n]{U} \end{bmatrix}
$$

The non-dimensional form of the governing equation of
motion Eq. (21) and the boundary conditions Eqs. (22a) and
(22b) are obtained as below respectively:

$$
\frac{d}{dy} \left[\left(\frac{du}{dy} \right)^{\frac{1}{n}} + \tau_y \right] = -G \qquad (24)
$$

$$
u(-1) - \gamma \left[\left(\frac{du}{dy} \right)^{\frac{1}{n}} + \tau_y \right]_{y=1} = 0 \qquad (25a)
$$

$$
u(1) + \gamma \left[\left(\frac{du}{dy} \right)^{\frac{1}{n}} + \tau_{tt} \right]_{y=1} = 0 \qquad (25b)
$$

where $G = -dp / dx$. Integrating Eq. (24), one can get

where $G = -dp/dx$. Integrating Eq. (24), one can get

$$
\left(\frac{du}{dy}\right)^{\frac{1}{n}} + \tau_y = -Gy + C
$$
tié

*D. S. Sankar and U. Lee / Journal of Mechanical Science and Technology 30 (7) (2016) 3203-3

<i>du* $\frac{du}{dy}$ $\int_0^1 + \tau_y = -Gy + C$ (26)
 \int **completes** (25) **completes** (25) **completes** (25) **completes** (25) **completes** (2 *D. S. Sankar and U. Lee / Journal of Mechanical Science and Technology 30 (7) (2016) 3203*
 $\left(\frac{du}{dy}\right)^{\frac{1}{n}} + \tau_y = -Gy + C$ (26)

A = G / Q.

Reperdized Couette flow

Eq. (26) simplifies the boundary conditions Eqs. (25a) *D. S. Sankar and U. Lee / Journal of Mechanical Science and Technology 3*
 $\left(\frac{du}{dy}\right)^{\frac{1}{n}} + \tau_y = -Gy + C$ (26)

A = G / Q.

Reference C is a constant of integration to be determined. The use 2.4 Generalized Completions b where *C* is a constant of integration to be determined. The use of Eq. (26) simplifies the boundary conditions Eqs. (25a) and (25b) to the following form respectively:

$$
u(-1) = \gamma(G + C) \tag{27a}
$$

$$
u(1) = \gamma(G - C). \tag{27b}
$$

Simplifying Eq. (26) and then integrating it, we obtain

$$
u(y) = -\frac{1}{G(n+1)}(C - Gy - \tau_y)^{n+1} + D
$$
 (28)

where *D* is the constant of integration to be determined. Applying the boundary conditions Eqs. (27a) and (27b) in Eq. (28), one can get

$$
(C+G-\tau_y)^{n+1} + (n+1)G\gamma(G+C) - (n+1)GD = 0, \quad (29a)
$$

$$
(C - G - \tau_y)^{n+1} + (n+1)G\gamma(G-C) - (n+1)GD = 0.
$$
 (29b) $\overline{u}(-h) - \beta$

From Eqs. (29a) and (29b), we obtain the following nonlin ear algebraic equation in the unknown *C*.

$$
(C - G - \tau_{y})^{n+1} + (n+1)G\gamma(G - C) - (n+1)GD = 0. \qquad (30)
$$

For a given set of values of the parameters τ_r , γ , *n* and *G*, Eq. (30) can be solved numerically, using the Newton-Raphson method. The non-dimensional flow rate per unit width of the plates is obtained with the help of Eq. (28) as below:

3), one can get
\n
$$
(C+G-\tau_y)^{n+1} + (n+1)G\gamma(G+C) - (n+1)GD = 0
$$
, (29a)
\n $(C-G-\tau_y)^{n+1} + (n+1)G\gamma(G-C) - (n+1)GD = 0$. (29b)
\n $\overline{u}(-\overline{h}) - \overline{\beta} \left\{ \overline{K} \left(\frac{d\overline{u}}{d\overline{y}} \right)^{\frac{1}{n}} + \overline{\tau}_y \left(\frac{d\overline{y}}{d\overline{u}} \right) \right\} \left(\frac{d\overline{u}}{d\overline{y}} \right) \right\} =$
\nFrom Eqs. (29a) and (29b), we obtain the following nonlin-
\nalgebraic equation in the unknown *C*.
\n $(C-G-\tau_y)^{n+1} + (n+1)G\gamma(G-C) - (n+1)GD = 0$. (30)
\nFor a given set of values of the parameters τ_x , γ , *n* and *G*, Eq.
\n(b) can be solved numerically, using the Newton-Raphson
\n(c) $G - \tau_y$
\n(c) $G - \tau_y$
\n(d) \overline{G}
\n(e) is the slip parameter. The dimensionless
\nthe test is obtained with the help of Eq. (28) as below:
\n $Q = \int_{-1}^{1} u(y) dy$
\n $Q = \int_{-1}^{1} u(y) dy$
\n $Q = \int_{-1}^{1} u(y) dy$
\n $Q = \int_{-1}^{1} (f(y)) dy$
\n $Q = \int_{-1}^{1} u(y) dy$
\n $Q = \int_{-1}^{1} (f(y)) dy$
\n $Q = \int_{-1}^{1} u(y) dy$
\n $Q = \int_{-1}^{1} (f(y)) dy$
\n $Q = \int_{-1}^{1} u(y) dy$
\n $Q = \int_{-1}^{1} u(y) dy$
\n $Q = \int_{-1}^{1} u(y) dy$
\n $Q = \int_{-1}^{1} (f(y)) dy$
\n $Q = \int$

The skin friction or wall shear stress at the lower wall (plate) and upper wall (plate) are defined respectively as

$$
\tau_{LW} = \tau\Big|_{y=-1} = \left\{ \left(\frac{du}{dy}\right)^{1/n} \right\}_{y=-1} + \tau_y = C + G \tag{32a}
$$

$$
\tau_{UW} = \tau \Big|_{y=1} = \left\{ \left(\frac{du}{dy} \right)^{1/n} \right\}_{y=1} + \tau_y = C - G \ . \tag{32b}
$$

Note that the subscripts *LW* and *UW* in Eqs. (32a) and (32b) denote the lower wall and upper wall, respectively. The frictional resistance to flow is defined as

$$
\Lambda = G/Q. \tag{33}
$$

2.4 Generalized Couette flow

D. S. Sankar and U. Lee / Journal of Mechanical Science and Technology 30 (7) (2016)
 $\left(\frac{du}{dy}\right)^{\frac{1}{n}} + \tau_y = -Gy + C$

(26)
 $\Lambda = G/Q$.

Let Us a constant of integration to be determined. The use

Eq. (26) simplifies the b *D. S. Sankar and U. Lee / Journal of Mechanical Science and Technology 30 (7) (2016
* $\left(\frac{du}{dy}\right)^{\frac{1}{n}} + \tau_y = -Gy + C$ (26)

Let G (26)

Let G (27b) simplifies the boundary conditions Eqs. (25a) and
 $u(1) = \gamma(G - C)$
 $u(y$ *D. S. Sankar and U. Lee / Journal of Mechanical Science and Technology 30 (7) (2016) 3203-3218

tional resistance to flow is defined as
* $\Delta = G/Q$ *.

ere C is a constant of integration to be determined. The use

Eq. (26) si D. S. Sankar and U. Lee / Journal of Mechanical Science and Technology 30 (7) (20)*

tional resistance to flow is
 $\tau_y = -Gy + C$ (26)

a constant of integration to be determined. The use 2.4 Generalized Couette fisimplifie *D. S. Sankar and U. Lee / Journal of Mechanical Science and Technology 30 (?) (2016) 3203-3218

ional resistance to flow is defined as
* $\Lambda = G/Q$ *.*

is a constant of integration to be determined. The use
 $\Lambda = G/Q$.

(26) si 2. (26) simplifies the boundary conditions Eqs. (25) and

2. (26) simplifies the boundary conditions Eqs. (25) and

1 interventional parameters into the plane Couter flow, laminar, fully developed

1 or of viscous incompr CO is simplifies the boundary conditions Eqs. (25a) and

Let us consider again the steady, laminar, fully developed

to the following form respectively:
 $-1 = \gamma (G + C)$
 $\gamma (G - C)$.

(27a) mitcly both points incompressible H b) to the totowing form respectively:

1 or $V = \frac{1}{f(G + C)} + \frac{1}{f(G)}$ (diversity) and the plane Counter the Ruid between two infi-

(1-1)= $r(G - C)$.

(27a) intervision interpretation that have the state distant 2*b*, the set to the total owing form respectively:
 $\int f(x+G) dx = 0$ (37a) intervalse and instant and the state and state and state and state and state and state that are and state than the state of the bundel place of the state of the s *e and Technology 30 (7) (2016) 3203-3218* 3207

and resistance to flow is defined as
 $\Lambda = G/Q$. (33)
 Generalized Couette flow

Let us consider again the steady, laminar, fully developed

w of viscous incompressible H-B Let us consider again the steady, laminar, fully developed flow of viscous incompressible H-B fluid between two infinitely long horizontal parallel plates that are at distant 2*h*. This flow is similar to the plane Couette flow, but, a constant pressure gradient is also applied in the *x* direction. This kind of flow is called as generalized plane Couette flow. The geometry of this flow is depicted in Fig. 1(c). For the generalized plane Couette flow, the governing equation of motion Eq. (7) remains the same and is restated below: al resistance to flow is defined as
 $\Delta = G/Q$. (33)
 Generalized Couette flow

et us consider again the steady, laminar, fully developed

v of viscous incompressible H-B fluid between two infi-

ly long horizontal parall *d* and resistance to flow is defined as
 $\Delta = G/Q$. (33)
 Generalized Couette flow
 defined Couette flow
 dw of viscous incompressible H-B fluid between two infi-
 dy long horizontal parallel plates that are at d resistance to flow is defined as
 G/Q . (33)

meralized Couette flow

us consider again the steady, laminar, fully developed

of viscous incompressible H-B fluid between two infi-

long horizontal parallel plates that are Let us consider again the steady, laminar, fully developed
w of viscous incompressible H-B fluid between two infi-
ely long horizontal parallel plates that are at distant 2*h*. This
w is similar to the plane Couette flow, cous incompressible H-B fluid between two infi-
horizontal parallel plates that are at distant 2*h*. This
lar to the plane Couette flow, but, a constant pres-
dia is also applied in the *x* direction. This kind of
ed as g us consider again the steady, laminar, fully developed
of viscous incompressible H-B fluid between two infi-
long horizontal parallel plates that are at distant 2*h*. This
is similar to the plane Couette flow, but, a cons Example the Steady, the matter, they concern the sum and properties the B fluid between two infi-
rizontal parallel plates that are at distant 2*h*. This
row is depicted in Fig. 1(c). For the generalized
flow, the governi

$$
\frac{d}{d\overline{y}}\left[\left\{\overline{K}\left(\frac{d\overline{u}}{d\overline{y}}\right)^{\frac{1}{n-1}} + \overline{\tau}_{y}\left(\frac{d\overline{y}}{d\overline{u}}\right)\right\}\left(\frac{d\overline{u}}{d\overline{y}}\right)\right] - \frac{d\overline{p}}{d\overline{x}} = 0.
$$
\n(34)

The slip boundary conditions of this flow are

$$
f(1) = \gamma(G - C).
$$
\n(27b) are gradient is also applied in the *x* direction. This kind of
\nthis form the *x* direction. The generalized
\nsplitting Eq. (26) and then integrating it, we obtain
\n
$$
f(x) = -\frac{1}{G(n+1)}(C - Gy - \tau,)^{n+} + D
$$
\n(28)
\n
$$
F(D) = -\frac{1}{G(n+1)}(C - Gy - \tau,)^{n+} + D
$$
\n(29)
\n
$$
F(D) = -\frac{1}{G(n+1)}(C - Gy - \tau,)^{n+} + D
$$
\n(29)
\n
$$
F(D) = -\frac{1}{G(n+1)}(C - Gy - \tau,)^{n+} + D
$$
\n(29)
\n
$$
F(D) = -\frac{1}{G(n+1)}(C - Gy - \tau,)^{n+} + D
$$
\n(29)
\n
$$
F(D) = -\frac{1}{G(n+1)}(C - Gy - \tau,)^{n+} + D
$$
\n(29)
\n
$$
F(D) = -\frac{1}{G(n+1)}(C - Gy - \tau,)^{n+} + D
$$
\n(29)
\n
$$
F(D) = -\frac{1}{G(n+1)}(C - Gy - \tau,)^{n+} + D
$$
\n(29)
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$$
F(D) = -\frac{1}{G(n+1)}(C - Gy - \tau,)^{n+} + D
$$
\n(29)
\n
$$
F(D) = -\frac{1}{G(n+1)}(C - Gy - \tau,)^{n+} + D
$$
\n(29)
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$$
F(D) = -\frac{1}{G(n+1)}(C - Gy - \tau,)^{n+} + D
$$
\n(29)
\n
$$
F(D) = -\frac{1}{G(n+1)}(C - Gy - \tau,)^{n+} + D
$$
\n(29)
\n
$$
F(D) = -\frac{1}{G(n+1)}(C - Gy - \tau,)^{n+} + D
$$
\n(29)
\n
$$
F(D) = -\frac{1}{G(n+1)}(C - Gy - \tau,)^{n+} + D
$$
\n(20)
\n
$$
F(D) = -\
$$

where β is the slip parameter. The dimensionless variables defined in Eq. (23) are used to obtain the non-dimensional form of Eqs. (34) and (35) as below, respectively:

$$
\frac{d}{dy}\left[\left(\frac{du}{dy}\right)^{1/n} + \tau_y\right] = -G\tag{36}
$$

$$
u(-1) - \gamma \left[\left(\frac{du}{dy} \right)^{1/n} + \tau_H \right]_{y=1} = 0 \tag{37a}
$$

Here β is the slip parameter. The dimensionless variables
\nfind in Eq. (23) are used to obtain the non-dimensional
\nm of Eqs. (34) and (35) as below, respectively:
\n
$$
\frac{d}{dy} \left[\left(\frac{du}{dy} \right)^{1/n} + \tau_y \right] = -G
$$
\n(36)
\n
$$
u(-1) - \gamma \left[\left(\frac{du}{dy} \right)^{1/n} + \tau_y \right]_{y=-1} = 0
$$
\n(37a)
\n
$$
u(1) + \gamma \left[\left(\frac{du}{dy} \right)^{1/n} + \tau_y \right]_{y=1} = 1
$$
\n(37b)
\nhere $G = -\frac{dp}{dx}$. Integrating Eq. (36), one can obtain
\n
$$
\left(\frac{du}{dy} \right)^{1/n} + \tau_y = -Gy + E
$$
\n(38)
\nwhere E is a constant of integration to be determined. Apply-
\ng. Eq. (38) in the boundary conditions Eqs. (37a) and (37b),
\ne can obtain
\n
$$
u(-1) = \gamma (G + E)
$$
\n(39a)

$$
\left(\frac{du}{dy}\right)^{1/n} + \tau_y = -Gy + E\tag{38}
$$

 (32b) where *E* is a constant of integration to be determined. Applying Eq. (38) in the boundary conditions Eqs. (37a) and (37b), one can obtain

$$
u(-1) = \gamma(G + E) \tag{39a}
$$

$$
u(1) = \gamma(G - E) + 1. \tag{39b}
$$

From Eq. (38), one can obtain the expression for velocity as below:

$$
u(y) = -\frac{1}{G(n+1)}(E - Gy - \tau_y)^{n+1} + F
$$
\n(40) for velo
\ntance to

where F is the constant of integration to be determined. Using the boundary conditions Eqs. (39a) and (39b) in Eq. (40), we get

$$
(E+G-\tau_y)^{n+1} + (n+1)G\gamma(G+E)
$$

\n
$$
-(n+1)GF = 0
$$
\n(41a) (ii) pl
\nvisco

$$
(E - G - \tau_y)^{n+1} + (n+1)G(\gamma(G - E) + 1)
$$
 considered 1
onsidered 1
and fully de
tribution, we

From Eqs. (41a) and (41b), one can get the following nonlinear algebraic equation in the unknown *E*:

$$
\left\{ \left(E + G - \tau_y \right)^{n+1} - \left(E - G - \tau_y \right)^{n+1} \right\} + (n+1)G(2\gamma E - 1) = 0.
$$
\n(42)

For a given set of values of the parameters *τy*, *γ*, *n* and *^G*, one can numerically solve the nonlinear Eq. (42) using the Newton-Raphson method. The non-dimensional flow rate per unit width of the plates is obtained with the help of Eq. (40) as below:

+
$$
(n+1)G(2\gamma E-1) = 0
$$
.
\nFor a given set of values of the parameters τ , γ , *n* and *G*, graphs. Total
\ne can numerically solve the nonlinear Eq. (42) using the
\ntwo-metrically solve the nonlinear Eq. (42) using the
\nparameters used in
\nti width of the plates is obtained with the help of Eq. (40) as
\nlower
\nlower
\n $Q = \int_{-1}^{1} u(y) dy$
\n $Q = \int_{-1}^{1} u(y) dy$

The skin friction or wall shear stress at the lower plate and upper plate are defined respectively as

$$
\tau_{LW} = \tau\Big|_{y=-1} = \left\{ \left(\frac{du}{dy}\right)^{1/n} \right\}_{y=-1} + \tau_y = E + G,\tag{44a}
$$

$$
\tau_{UW} = \tau \Big|_{y=1} = \left\{ \left(\frac{du}{dy} \right)^{1/n} \right\}_{y=1} + \tau_y = E - G \tag{44b}
$$

where the subscripts *LW* and *UW* denote the lower wall and defined as

$$
\Lambda = \frac{G}{Q} \,. \tag{45}
$$

As mentioned in the introduction section, for particular val-

B. B. Sankar and U. Lee / Journal of Mechanical Science and Technology 30 (7) (2016) 3203-
 u(1) = $\gamma(G - E) + 1$. (39b) use of the parameters (yield stres

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From Eq. (38), one can ob *y p <i>n s sankar and U. Lee / Journal of Mechanical Science and Technology 30 (7) (2016) 3203-3218*
 w (1) = γ (*G* - *E*) + 1. (39b) us of the parameters (yield stress τ , and power the H-B fluid model red *D. S. Sankar and U. Lee / Journal of Mechanical Science and Technology 30 (7) (2016) 3203-3218

=* $\gamma (G - E) + 1$ *.* (39b) uses of the parameters (yield stress τ_y and power

the H-B fluid model reduces to Newtonian fluid
 (40) for velocity, flow rate, wall shear stress and frictional resis-D. S. Sankar and *U. Lee / Journal of Mechanical Science and Technology 30 (7) (2016) 3203-3218*

(39b) uses of the parameters (yield stress τ_{*s*} and po

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and the expression *ⁿ E G n G G E ^y n G F* D. S. Sankar and U. Lee/Journal of Mechanical Science and Technology 30 (7) (2016) 3203-3218

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= Eq. (38), o *D. S. Sankar and U. Lee / Journal of Mechanical Science and Technology 30 (7) (2016) 3203-3218*
+1. (39b) uses of the parameters (yield stress r, and the expression for velocity as the H-B fluid model reduces to Newtonia ues of the parameters (yield stress τ_y and power law index *n*), the H-B fluid model reduces to Newtonian fluid model, power law fluid model and Bingham fluid model. For each of these reduced fluid models, the non-dimensional form of the gov erning equation of motion, boundary conditions, expressions tance to flow in the (i) plane Couette flow, (ii) plane Poiseuille flow and (iii) generalized Couette flow are summarized in Appendix.

3. Numerical simulation of results

 (41a) (ii) plane Poiseuille flow and (iii) generalized Couette flow of $(E - G - \tau_y)$ + $(n+1)G(\gamma(G - E) + 1)$ (41b) and fully developed. The analytical solutions to velocity dis-1 0 . *ⁿ E G n G G E ^y n G F D. S. Sankar and U. Lee / Journal of Mechanical Science and Technology 30 (7) (2016) 3203-2218

=* $\gamma(G - E) + 1$ *.* (39b) uses of the parameters (yield stress τ_y and power live H-B fluid model reduces to Newtonian fluid m +1. (39b) uss of the parameters (yield stress τ , and

the H-B fluid model reduces to Newtonian

me can obtain the expression for velocity as

law fluid model and Bingham fluid model

reduced fluid models, the non-dimen $\{(E+G-\tau_y)^{n+1}-(E-G-\tau_y)^{n+1}\}\$ such as power law index, yield stress, slip parameter and pres-(*y*) = $-\frac{1}{G(n+1)}(E - Gy - \tau_y)^{n+1} + F$ (40) for velocity, how rate, well shear sitre

e *F* is the constant of integration to be determined. Using there to flow in the (i) plane Couette flow

oundary conditions Eqs. (39a) $\begin{aligned}\n&= -\frac{1}{G(n+1)} \left(E - Gy - \tau_y \right)^{n+1} + F & (40) \quad \text{for velocity, flow rate, wall shear stret, and the potential of Rines are given set of values of the parameters τ_y , and (39b) in Eq. (40), we have the no-dimensional system, and (39c) in Eq. (41a) in Fig. 11.12.13, we have that the initial distribution of results in the following equations. \end{aligned}$ $\mu(y) = -\frac{1}{G(n+1)}(E - 6y - r_y) + F$

(a) the constant of integration to be determined. Using

the constant of integration to be determined. Using

the constant of integration to be determined. Using
 $E + G - \tau_y$, $F = 1$ and (ii) (42) sure gradient are discussed in this section through appropriate a) and (41b), one can get the following stands to flow have been obtained. The values of the arbitrary of the activity stands $\left($ and $\left($ 41b), one can get the following stands A , C and E is equation in the unkn 10 (41b), one can get the following to thus the value of the appearing in these and

to thus the stand of these and the unknown *E*:
 $-G - \tau_{y}$ ¹⁶²¹ $\left\{\frac{42}{42}\right\}$ and a power law mean supplementary using Newton-Raphs $(E - G - \tau_y)^{n+1} + (n+1)G(\gamma(G - E) + 1)$ considered in this study,
 $-(n+1)GF = 0$.

From Eqs. (41a) and (41b), one can get the following stants *A*, *C* and *E* appear

from Eqs. (41a) and (41b), one can get the following stants *A* $-(n+1)GF = 0$.
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Eqs. (41a) and (41b), one can get the following stanks A, C and E appearing in these and

algebraic equation in the 41a) and (41b), one can get the following stants *A*, *C* and *E* appearing in these taic equation in the unknown *E*:

stants *A*, *C* and *E* appearing in these flow quantities with var

stants *A*, *C* and *E* appearin $-(n+1)G(F = 0$.

tribution, wall shear stress, flow rate and f

to flow have been obtained. The values of

to flow have been obtained. The values of

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stants A, C and E appearing and (41b), one can get the following

stants A, C and E appearing in these

uation in the unknown E:

found numerically, using Newton-Raph

and A, C and E appearing in these

and on of these flow quantities with vario

st Three types of basic flows such as (i) plane Couette flow, viscous incompressible H-B fluid between parallel plates are considered in this study, treating the flow as steady, laminar tribution, wall shear stress, flow rate and frictional resistance to flow have been obtained. The values of the arbitrary con stants *A*, *C* and *E* appearing in these analytical solutions are found numerically, using Newton-Raphson method. The variation of these flow quantities with various physical parameters graphs. Matlab program code is developed to compute the data from the analytical solutions obtained for plotting the graphs. To validate the present study, some results are com pared with that of Ramesh and Devakar [36]. The range of parameters used in this study is listed below [36, 40, 41]:

Power law index *n*: $0.75 - 1.5$; Yield stress τ : $0 - 0.2$; Slip parameter γ : 0 - 0.5; Pressure gradient *P*: 4 - 10.

3.1 Velocity distribution

 $(E - G - \tau_y)$ $-(E + G - \tau_y)$ *F*. Couette flow respectively, for different values of the power -1)=0. graphs. Matlab program code is developed to comparing the station the analytical solutions obtained for ply solve the non-dimensional flow rate per parameters τ , γ , n and G , graphs. To validate the presen (42) sure gradient are discussed in this section through
 $\begin{pmatrix} 2rE-1 \end{pmatrix} = 0$. (42) sure gradient are discussed in this section through

data from the analytical solutions obtained is developed to

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data from the analytical solutions

data from the analytical solutions

and G, graphs. To validate the present stu

parameters τ , γ , n and G, graphs. To validate the present stu
 (44a) higher in generalized Couette flow than in plane Poiseuille *y* solve the nonlinear Eq. (42) using the pared with that of Ramesh and Devakar [36]. 2

method. The non-dimensional flow rate per

parameters used in this study is listed below [36,4]

lates is obtained with the help of e can numerically solve the nonlinear Eq. (42) using the pared with that of twoton-Raphson method. The non-dimensional flow rate per

with of the plates is obtained with the help of Eq. (40) as Power law index

ow:
 $Q = \int$ a given set of values of the parameters τ , y , n and G , graphs. To validate the present study, some random Parameters τ , y , n and Devakar [36] and mumerically solve the non-dimensional flow rate per paramet Examples the information and the tell ($\frac{du}{dy}$)¹⁶)
 $\left[\left(\frac{du}{dy}\right)^{1/6}\right]_{y=1} + \tau_y = E - G$

(44b) figher than that of plane couter flow and the filtre position of the position of the state of the state of the state of th (44b) flow, the velocity of the fluid decreases with the increase of upper wall respectively. The frictional resistance to flow is flow region (from $y = 0$ to $y = -1$), whereas in plane $\tau_{Lw} = \tau|_{y=1} = \left\{ \left(\frac{du}{dy} \right)^{1/n} \right\}_{y=1} + \tau_y = E - G$

The sum interesting the signal end and the lower plate are defined respectively as
 $\tau_{Lw} = \tau|_{y=1} = \left\{ \left(\frac{du}{dy} \right)^{1/n} \right\}_{y=1} + \tau_y = E + G$,
 $\tau_{Lw} = \tau|_{z=1} = \left\{ \$ Figs. 2(a)-(c) depict the velocity distribution in (i) plane colaw index with $\gamma = 0.5$, $G = 10$ and $\tau_y = 0.1$. In plane Couette flow, the velocity increases linearly in the *y* direction, whereas in plane Poiseuille flow and generalized Couette flow, para bolic velocity profiles are obtained. For a given set of values of the parameters γ , *G* and τ_y , the fluid velocity is marginally flow and the fluid velocity in these flows is significantly higher than that of plane Couette flow. In the plane Couette the power law index *n* in the lower half of the flow region Power law index *n*: 0.75 - 1.5; Yield stress τ_y : 0 - 0.2; Slip
parameter γ : 0 - 0.5; Pressure gradient *P*: 4 - 10.
3.1 Velocity distribution
Figs. 2(a)-(c) depict the velocity distribution in (i) plane co-
queste the increase of the power law index *n* in the upper half of the **3.1 Velocity distribution**
Figs. 2(a)-(c) depict the velocity distribution in (i) plane co-
quette flow, (ii) plane Poiseuille flow and (iii) generalized
Couette flow, respectively, for different values of the power
Iaw Poiseuille flow and generalized Couette flow, the fluid velocity increases significantly with the increases of the power law index *n*.

Velocity distributions for different values of the pressure gradient in plane Poiseuille flow and generalized Couette flow are shown in Figs. 3(a) and (b), respectively. It is noticed that

Fig. 2. Velocity distribution for different values of power law index *n* with *γ* = 0.5, *G* = 10 and *τy* = 0.1.

Fig. 3. Velocity distribution for different values of pressure gradient *G* with with $\gamma = 0.5$, $n = 1$ and $\tau_y = 0.1$.

the fluid velocity increases significantly with the increase of the pressure gradient. Figs. 4(a)-(c) depict the velocity distribution for different values of the slip parameter γ with $n = 1$, $G = 10$ and $\tau_y = 0.1$ in (i) plane Couette flow, (ii) plane t Poiseuille flow and (iii) generalized Couette flow, respectively. It is seen that the velocity of the fluid increases considerably with the increase of the slip parameter in all the types of flows. The velocity distribution for different values of the yield stress τ ^{*y*} with *n* = 1, *G* = 10 and γ = 0.5 in (i) plane Couette flow, (ii) plane Poiseuille flow and (iii) generalized Couette flow sketched in Figs. 5(a)-(c), respectively. It is observed that in all the types of flows, the fluid velocity decreases very slightly with the increase of the yield stress parameter τ_y when all the (

other parameters are held fixed. When $\tau_y = 0$ and $n = 1$, the Herschel-Bulkley fluid model reduces to Newtonian fluid model. It is of interest to note that the velocity profiles obtained for the Newtonian fluid model in all the three types of flows are in good agreement with Figs. 5(a)-(c) of Ramesh and Devakar [35] and this validates the present study.

3.2 Flow rate

The variation of flow rate with the pressure gradient for different values of *n* and γ with $\tau_y = 0.1$ in plane Poiseuille flow and generalized Couette flow are shown in Figs. 6(a) and (b), respectively. It is observed that for a given set of values of

Fig. 4. Velocity distribution for different values of slip parameter *γ* with *G* = 10, *n* = 1 and *τy* = 0.1.

Fig. 5. Velocity distribution for different values of yield stress τ_y with $G = 10$, $n = 1$ and $\gamma = 0.5$.

Fig. 6. Variation of flow rate with pressure gradient for different values of *n* and *γ* witn *τy* = 0.1.

Fig. 7. Variation of wall shear stress with slip parameter for different values of *n* and *τy* with *G* = 5.

the parameters, the flow rate of the fluid increases slowly with the increase of the pressure gradient *G* from 5 to 7 and then it increases linearly with the further increase of the pressure gradient from 7 to 10. One can note that the flow rate of the fluid increases considerably with the increase of either the power law index *n* or the slip parameter γ . It is also found that for a given set of values of the parameters, the flow rate of the fluid is marginally higher in generalized Couette flow than in plane Poiseuille flow.

3.3 Wall shear stress

The variation of wall shear stress with slip parameter γ for different values of *n* and τ_y with $G = 5$ in plane Poiseuille or flow and generalized Couette flow are shown in Figs. 7(a) and (b). It is seen that the wall shear stress decreases rapidly with the increase of the slip parameter γ from 0 to 0.25 and then it decreases slowly with the further increase of the slip parameter γ from 0.25 to 5. It is also noticed that for a given value of the power law index n , the wall shear stress increases considerably with the increase of the yield stress τ_y of the fluid (and this behavior is reversed when the power law index *n* increases when the yield stress τ _{*v*} is kept as constant. For a

given choice of the parameters, the wall shear stress is considerably higher in plane Poiseuille flow than in generalized Couette flow.

3.4 Frictional resistance to flow

Figs. 8(a) and (b) show the variation of frictional resistance to flow with slip parameter for different values of the power law index *n* with $G = 10$ and $\tau_y = 0.1$ in plane Poiseuille flow and generalized Couette flow, respectively. It is seen that when the fluid is of shear thinning nature ($n \leq 1$), the frictional resistance to flow decreases rapidly (nonlinearly) with the increase of the slip parameter γ from 0 to 0.25 and then it decreases slowly with the increase of the slip parameter γ from 0.25 to 0.5, whereas, if the fluid is of shear thickening nature, then the frictional resistance to flow decreases very slowly (almost constant) with the increase of the slip parameter γ . It also noted the frictional resistance to flow decreases significantly with the increase of the power law index *n* of the fluid when all the other parameters were kept as invariables. One can also notice that for a given set of values of the parameters, the frictional resistance to flow is marginally lower in generalized Couette flow than in plane Poiseuille flow.

Fig. 8. Variation of frictional resistance to flow with slip parameter for different values of power law index with $G = 10$ and $\tau_y = 0.1$.

4. Discussion

4.1 Physical significance of Herschel-Bulkley fluid model

H-B fluid is a non-Newtonian fluid model with three parameters such as the power law index, yield stress and consistency index (coefficient of viscosity). The constitutive equation of H-B fluid model which expresses the relationship between the shear stress and shear rate is defined as below:

$$
\overline{\tau} = \left(-\overline{\mu}\overline{\gamma}\right)^{1/n} + \overline{\tau}_{y},\tag{46}
$$

Example 19 (b) Generation of Society of the Values of the coefficient of shear rate, $\frac{1}{2}$ is the s (in Cartesian plane), *n* is the power law index, $\overline{\mu}_H$ is the trend predicts the fact that the coefficient of viscosity which is shear dependent and $\bar{\tau}$ is the yield stress of the fluid. The shear stress of this fluid model is nonlinearly proportional to the shear rate (rate of strain/ velocity gradient). The nonlinear relationship between the shear stress and shear rate is dependent on all of the aforesaid three fluid parameters.

An advantage of using H-B fluid model for fluid flow is that the flow characteristics of the other fluid models such as Newtonian fluid model, power law fluid model and Bingham fluid model can also be obtained from H-B fluid model by assigning particular values to the fluid parameters. H-B fluid model reduces to Newtonian fluid model when $n = 1$ and $\overline{\tau}_n = 0$, Bingham fluid model when $n = 1$ and $\overline{\tau}_n \neq 0$, and power law fluid model when $n \neq 1$ and $\overline{\tau}_v = 0$. The power law index parameter *n* plays an important role in the major classification of non-Newtonian power law fluids. When $n < 1$, it behaves like shear thinning fluids and when $n > 1$, it exhibits the characteristics of shear thickening fluids. In shear thinning fluids, the fluid viscosity decreases with the increase of the shear stress whereas this behavior is reversed in shear thickening fluids.

Polymer solutions, molten polymers, suspensions like ketchup, whipped cream, blood, paint and nail polish are some examples of shear thinning fluids. When a modern paint is applied on a wall, the shear created by the brush or roller will allow them to thin and wet out the surface evenly. Once applied, the paints regain their higher viscosity which avoids drips and runs.

Corn starch, silica and polyethylene glycol, quicksand and dilatant fluids are some of the well known examples for shear thickening fluids. Cornstarch is a good shear thickening agent used in cooking. When a force is applied to a mixture of cornstarch and water, the cornstarch acts as a solid and resists the force.

Example 1
 Example 10
 Form II Alter CONFIGUATE AND CONFIDE AND CONFIDE AND CONFIDENT (1) A FOR THE VALUES OF DWO WIN SIP parameter (b) Generalized Couette flow

Fig. 8. Variation of fricol metallity of the shear rate of the shear rate of the shear rate of the shear stress in the control of since the shear stress in the shear stress in the shear stres The variation of shear stress with shear rate for different values of the coefficient of viscosity μ and yield stress τ for (i) $n = 0.75$, (ii) $n = 1$ and (iii) $n = 1.25$ is delineated in Figs. 9(a)-(c). Fig. 9(a) exhibits that in shear thinning fluids over any natural notation and prother $h \rightarrow 1$ and $t_y = 0$. The power and the shear stress in function of non-Newtonian power law this s. When $n < 1$, it behaves like shear thrianing fluids and when $n > 1$, it behaves like the increase of the shear rate from $0s^{-1}$ to $0.5s^{-1}$ and then it increases linearly with the increase of the shear rate from $0.5s^{-1}$ to $1s^{-1}$. This trend indicates that the shear stress becomes almost constant with the further increase of the shear stress which confirms the shear thinning behavior of fluid for high shear rates.

Fig. 9(b) indicates that the shear stress increases linearly with the increase of the shear rate of the fluid when the power law index $n = 1$ which is the character of Newtonian fluid when the yield stress $\overline{\tau}_{y} = 0$ and Bingham fluid when the yield stress $\overline{\tau}_y \neq 0$.
Fig. 9(c) illustrates that the shear stress of the shear thicken-

ing fluid increases linearly with the increase of the shear rate from $0s^{-1}$ to $0.5s^{-1}$ and then it increases rapidly (nonlinearly) with the increase of the shear rate from $0.5s^{-1}$ to $1s^{-1}$. This trend predicts the fact that the shear stress increases very rapidly with the further increase of the shear stress which proves the shear thickening characteristic of the fluid for high shear rates. From Figs. $9(a)-(c)$, it is also clear that the shear stress increases considerably with the increase of the coefficient of viscosity $\bar{\mu}_{H}$ and yield stress $\bar{\tau}_{y}$ of the fluid.

The variation of shear stress with shear rate for different values of the power law index *n* with the yield stress $\overline{\tau}_{v}$ = 0.1 (Poise) and viscosity coefficient $\overline{\mu}_{H}$ = 1.5 (Poise second) is depicted in Fig. 10. One can easily understand the shear thinning and shear thickening character of H-B fluid when $n < 1$ and $n > 1$, respectively.

coefficient of viscosity $\bar{\mu}_H$ and yield stress $\bar{\tau}_y$ with (a) $n = 0.75$; (b) Shear $n = 1.0$; (c) $n = 1.25$.

4.2 Some applications of Herschel-Bulkley fluid model

From the Refs. [1-4], it is evident that the plane Poiseuille flow, plane Couette flow and generalized Couette flow are some of the basic flow types which are widely used to understand many complex flow problems. Some physical flows of these kinds with slip at the wall are polymer extrusion process, polymer melting, gross melt fracture, sharkskin, stick-slip, bio-fluid (blood, synovial fluids, lymphatic fluids etc.) flow in clinical devices used for diagnostics [14, 42, 43]. From the above discussion, it is clear that H-B fluid model has the ability to explain these kinds of physical phenomena.

As an example, let us consider the plane Poiseuille flow

Fig. 10. Variation of shear stress with shear rate for different values of the power law index *n* with coefficient of viscosity $\overline{\mu}_H = 1.5$ (*Pascal* × sec *ond*) and yield stress $\overline{\tau}_v = 0$ (*Pascal*).

Fig. 11. Velocity distribution of Herschel-Bulkley fluid model in the plane Poiseuille flow between parallel plates for different values of the power law index *n* and yield stress $\overline{\tau}_{y}$.

(pressure driven flow) of blood (treated as viscous incompressible non-Newtonian H-B fluid) in a channel between parallel plates (clinical devises used for diagnosis of biofluids) which has the characteristics of laminar, axi-symmetric and fully developed unidirectional flow. Fig. 11 exhibits the velocity profiles of H-B fluid for different values of its parameters such as power law index *n* and yield stress $\overline{\tau}_y$. One can notice the difference between the flow patterns of the Fowerlaw fluid with $n = 3$

and the method of the method with $n = 10$

and shear this method with $n = 0$ and the method with $n = 0$ and $n = 16$

properties and also the difference between the flow patterns of the fluids with yield stress () ⁰ *^y* ^t ^¹ and fluids without yield stress (*Praceal s second*) and yield stress τ , = 0 *Premetine fluid with* $n = 2$
 $\frac{2}{3}$ o $\frac{2}{3}$
 \frac **Example 12**
 Example 10
 Example 10 shear thinning fluids have velocity profiles which are almost parabolic. Fig. 9. Variation of shear stress with shear rate for different values of can notice the different of viscosity \overline{u} , and vield stress $\overline{\tau}$ with (a) $n = 0.75$; (b) shear thinning fluid $(n < 1)$ and shear thickening

5. Conclusions

This mathematical analysis brings out many important and useful results in the steady, laminar and fully developed flow of viscous incompressible Herschel-Bulkley fluid between parallel plates, considering three types of flows such as (i) plane Couette flow (ii) plane Poiseuille flow and (iii) generalized coquette flow. The main findings of this theoretical investigation are summarized below:

- ·In plane Couette flow, the fluid velocity increases margin ally with the increase of the slip parameter, power law in-
dex and decreases marginally with the increase of the \overline{y} dex and decreases marginally with the increase of the power law index.
- In plane Poiseuille flow and generalized Couette flow, the δ_{ij} velocity and flow rate of the fluid increase considerably $\overline{\mu}$ with the increase of the slip parameter, power law index, $\overline{\mu}_{H}$ pressure gradient.
- The fluid velocity is significantly higher in plane τ_{v} Poiseuille flow than in plane Couette flow.
- The wall shear stress and frictional resistance to flow of $\overline{\beta}$ the fluid decrease considerably with the increase of the γ power law index n and these are increasing significantly Λ with the increase of the yield stress of the fluid.
- ·The wall shear stress and frictional resistance to flow decrease rapidly with the increase of the slip parameter γ from 0 to 2.5 and then they decrease slowly with the in crease of the slip parameter from 0.25 to 0.5.
- ·The fluid velocity and flow rate are considerably higher in generalized Couette flow than in plane Couette flow.
- ·The wall shear stress and frictional resistance to flow are considerably higher in plane Poiseuille flow than in gen eralized Couette flow.

From the recorded outcomes of this investigation, it is found that there are significant difference between the flow quantities obtained in the generalized Couette flow, plane Poiseuille flow and plane Couette flow. The present study could be applied to analyze some industrial flow problems such as polymer extrusion process and polymer melting process in which the wall slip occurs essentially. Since some physical flow through parallel plates is unsteady, the present steady flow analysis can be extended to unsteady case, this mathematical analysis would be carried out in the near future. From the crooted toute tower

From the recorded outcomes of this investigation, it is found [2] C. Tu and M.

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that there are significant difference between the flow quanti-

its obtained in the generalized Couette flow, plane Poiseuille

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Nomenclature-

Greek symbols

- : Shear rate
- $\bar{\sigma}_{ii}$: Cauchy's stress tensor
	- : Kronecker delta
	- : Variable viscosity of generalized Newtonian fluid
- $\bar{\mu}_{\scriptscriptstyle H}$: Viscosity coefficient of Herschel-Bulkley fluid
- $\overline{\tau}_y$: Yield stress of Herschel-Bulkley fluid
	- *^y* ^t : Dimensionless yield stress of Herschel-Bulkley fluid
- : Dimensional slip parameter
- : Dimensionless slip parameter
- : Frictional resistance to flow
- τ_{LW} : Wall shear stress at the lower wall
- $\tau_{\textit{\tiny{TW}}}$: Wall shear stress at the upper wall

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Appendix

For steady, laminar and fully developed plane Couette flow of viscous incompressible (i) Newtonian fluid, (ii) power law fluid and (iii) Bingham fluid, the non-dimensional form of the governing equation of motion, boundary conditions, expressions for velocity and flow rate are obtained and are summarized in Table A.1.

Table A.1. Governing equations and the analytic solutions to plane Couette flow.

	of viscous incompressible (i) Newtonian fluid, (ii) power law fluid and (iii) Bingham fluid, the non-dimensional form of the	Velocity	$u(y) = \frac{1}{2} \left 1 + \frac{(1 - 2\gamma\tau_y)y}{(1 + \gamma)} \right $	
	governing equation of motion, boundary conditions, expres- sions for velocity and flow rate are obtained and are summa-	Flow rate	$Q = 1$	
rized in Table A.1.	In the case of plane Poiseuille flow and generalized Couette flow of these fluid models, the governing equations of motion,	Poiseuille flow.	Table A.2. Governing equations and the	
	boundary conditions, expression for velocity, flow rate, wall shear stress and frictional resistance to flow are obtained and		Case 1. Newtonian	
	summarized in Tables A.2 and A.3, respectively.		(Reduced from H-B fluid wher	
Table A.1. Governing equations and the analytic solutions to plane Couette flow.		Governing differential equation	$rac{d}{dv}\left(\frac{du}{dv}\right) = -G$	
			where $G = -\frac{dp}{dx}$	
Case 1. Newtonian fluid (Reduced from H-B fluid when $\tau_y = 0$ and $n = 1$)		Boundary	$u(-1)-\gamma\left(\frac{du}{dy}\right)_{x=0} = 0$	
Governing differential equation	$\frac{d}{dy}\left(\frac{du}{dv}\right) = 0$ (A.1)	conditions	$u(1) + \gamma \left(\frac{du}{dy}\right)_{x=1} = 0$	
Boundary conditions	$u(-1)-\gamma\left(\frac{du}{dy}\right)_{x=1}=0$ (A.2a)	Velocity	$u(y) = -\frac{Gy^2}{2} + \frac{1}{2}(2\gamma + 1)$	
	$u(1) + \gamma \left(\frac{du}{dy}\right)_{x=1} = 1$ (A.2b)	Flow rate	$Q = -\frac{G}{3} + (2\gamma + 1)G$	
Velocity	$u(y) = \frac{1}{2} \left 1 + \frac{y}{(1+y)} \right $ (A.3)	Wall shear stress	$\tau_{LW} = \left(\frac{du}{dy}\right)_{U} = G$	
Flow rate	$Q=1$ (A.4)		$\tau_{UW} = \left(\frac{du}{dy}\right) = -G$	
	Case 2. Power law fluid (Reduced from H-B fluid when $\tau_y = 0$)			
Governing differential equation	$\frac{d}{dy}\left(\frac{du}{dy}\right)^{1/n}=0$ (A.5)	Frictional resistance to flow	$\Lambda = G/Q$	
		Case 2. Power law		
Boundary conditions	$u(-1)-\gamma\left \left(\frac{du}{dy}\right)^{1/n}\right =0$ (A.6a) $u(1)+\gamma\left \left(\frac{du}{dy}\right)^{1/n}\right =1$ (A.6b)	Governing	(Reduced from H-B fluid	
		differential equation	$\frac{d}{dv}\left(\frac{du}{dv}\right)^{1/n} = -G$	
Velocity	$u(y) = \frac{1}{2} \left[1 + (1 - 2 \gamma A) y \right]$ (A.7)	Boundary	$u(-1)-\gamma\left[\left(\frac{du}{dy}\right)^{1/n}\right]$	
	where the value of the arbitrary constant A is obtained by	conditions	$u(1)+\gamma\left \left(\frac{du}{dy}\right)^{1/n}\right =0$	
	$2A'' + 2\gamma A - 1 = 0$ (A.8)			
Flow rate	$Q=1$ (A.9)			
	Case 3. Bingham fluid	Velocity	where the value of the arbitra	
Governing differential equation	$\frac{d}{dy}\left(\frac{du}{dy}\right) + \tau_y = 0$ (A.10)			
	solving Eq. (A.8) numerically. (Reduced from H-B fluid when $\tau_v = 0$ and $n = 1$)		$u(y) = -\frac{1}{G(n+1)}(-Gy + C)$ solving the following Eq. (A. { $(C+G)^{n+1} - (C-G)^{n+1}$ } +2(n+1) γ GC = 0	

	Velocity $u(y) = \frac{1}{2} 1 + y $ $(1-2\gamma\tau_v)y$	(A.12)
Flow rate $Q=1$		(A.13)

Table A.2. Governing equations and the analytic solutions to plane Poiseuille flow.

	D. S. Sankar and U. Lee / Journal of Mechanical Science and Technology 30 (7) (2016) 3203~3218			
	And D is obtained form Eq. $(A.25)$. $D = \frac{(C-G)^{n+1} + (n+1)\gamma G(G-C)}{(n+1)G}$	(A.25)	Velocity	$u(y) = -\frac{Cy^{2}}{2} + \frac{y}{(2y+1)} + \frac{(2y+1)G}{2}$
Flow rate	$Q = \frac{(C-G)^{n+2} - (C+G)^{n+2}}{(n+1)(n+2)G^2} + 2D$	(A.26)	Flow rate	$Q = -\frac{G}{2} + [(2\gamma + 1)G + 1]$
Wall shear	$\tau_{LW} = \tau\Big _{y=-1} = \left \left(\frac{du}{dy}\right)^{yn}\right = C + G$	(A.27a)	Wall shear stress	$\tau_{LW} = \tau\big _{y=-1} = \left(\frac{du}{dv}\right) = \frac{1}{(2\gamma+1)} + C$ $\tau_{UW} = \tau _{y=1} = \left(\frac{du}{dv} \right)_{y=1} = \frac{1}{(2\gamma + 1)} - G$
stress	$\tau_{UW} = \tau\Big _{y=1} = \left \left(\frac{du}{dy}\right)^{1/n}\right = C - G$	(A.27b)	Frictional resistance to flow	$\Lambda = G/Q$
Frictional resistance to flow	$\Lambda = G/Q$	(A.28)		Case 2. Power law fluid (Reduced from H-B fluid when τ_v =
	Case 3. Bingham fluid (Reduced from H-B fluid when $\tau_y = 0$ and $n = 1$)		Governing differential equation	$\frac{d}{dv}\left(\frac{du}{dv}\right)^{yn}=-G$
Governing differential equation	$\frac{d}{dv}\left(\frac{du}{dv}\right) + \tau_y = -G$	(A.29)		$u(-1)-\gamma\left \left(\frac{du}{dy}\right)^{1/n}\right =0$
Boundary conditions	$u(-1)-\gamma\left \left(\frac{du}{dv}\right)+\tau_y\right =0$	(A.30a)	Boundary conditions	$u(1) + \gamma \left \left(\frac{du}{dy} \right)^{\gamma/n} \right = 1$
	$u(1) + \gamma \left[\left(\frac{du}{dy} \right) + \tau_y \right]_{-1} = 0$	(A.30b)		$u(y) = -\frac{1}{G(n+1)}(-Gy+E)^{n+1} + F$
Velocity	$u(y) = -\frac{Gy^2}{2} - \frac{\gamma \tau_y y}{(1+\gamma)} + \frac{1}{2}(1+2\gamma)$	(A.31)		where the value of the arbitrary consta solving the following equation numerica
Flow rate	$Q = -\frac{G}{2} + (2\gamma + 1)$	(A.32)	Velocity	$\{(E+G)^{n+1}-(E-G)^{n+1}\}\$ + $(n+1)G(2\gamma E-1)=0$.
Wall shear stress	$\tau_{IW} = \tau$ (A.33a) $=\left \left(\frac{du}{dy}\right)\right $ + $\tau_y = \frac{\tau_y}{(2\gamma+1)} + G$ $\tau_{UW} = \tau$ (A.33b) $=\left \left(\frac{du}{dy}\right)\right _1 + \tau_y = \frac{\tau_y}{(2\gamma+1)} - G$		and F is obtained from the following equal $F = \frac{(E+G)^{n+1} + (n+1)\gamma G(G+E)}{(n+1)G}.$	
			Flow rate	$Q = \frac{(C-G)^{n+2} - (C+G)^{n+2}}{(n+1)(n+2)G^2} + 2D$
Frictional resistance to flow	$\Lambda = G/Q$	(A.34)	Wall shear stress	$\tau_{LW} = \tau\Big _{y=-1} = \left \left(\frac{du}{dy}\right)^{1/n}\right _{x=-1} = E + G$ $\tau_{UW} = \tau\Big _{y=1} = \left \left(\frac{du}{dy}\right)^{yn}\right = E - G$
ized Couette flow.	Table A.3. Governing equations and the analytic solutions to general-		Frictional resistance to flow	$\Lambda = G/Q$
	Case 1. Newtonian fluid (reduced from H-B fluid when $\tau_y = 0$ and $n = 1$)			Case 3. Bingham fluid (Reduced from H-B fluid when $\tau_y = 0$ and
Governing differential equation	$\frac{d}{dv}\left(\frac{du}{dv}\right) = -G$	(A.35)	Governing differential equation	$\frac{d}{dv}\left(\frac{du}{dv}\right) + \tau_y = -G$
Boundary conditions	$u(-1)-\gamma\left(\frac{du}{dv}\right) = 0$ (du)	(A.36a)	Boundary conditions	$u(-1)-\gamma\left \left(\frac{du}{dy}\right)+\tau_y\right =0$ $\lceil (x) \rceil$

Table A.3. Governing equations and the analytic solutions to generalized Couette flow.

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 $\begin{bmatrix} \frac{du}{dy} \end{bmatrix}_{y=1} + \tau_y = \frac{(1+2\tau_y)}{2(1+\gamma)} + G$ (A.53a) includes Mathematical Modeling, Applied Fluid Dynamics, Biomechanics, Differential Equations and Numerical Analysis. ing, Universiti Teknologi Brunei, Brunei. His research interest Biomechanics, Differential Equations and Numerical Analysis.

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