Springer

KSME

Journal of Mechanical Science and Technology 29 (3) (2015) 1207~1215
www.springerlink.com/content/1738-494x
DOI 10.1007/s12206-015-0234-7

# Application of the differential transformation method for nonlocal vibration analysis of functionally graded nanobeams †

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(Manuscript Received July 22, 2014; Revised October 27, 2014; Accepted December 9, 2014)

**Abstract**

In this study, the applicability of differential transformation method (DTM) in investigations on vibrational characteristics of functionally graded (FG) size-dependent nanobeams is examined. The material properties of FG nanobeam vary over the thickness based on the power law. The nonlocal Eringen theory, which takes into account the effect of small size, enables the present model to be effective in the analysis and design of nanosensors and nanoactuators. Governing equations are derived through Hamilton's principle. The obtained results exactly match the results of the presented Navier-based analytical solution as well as those available in literature. The DTM is also demonstrated to have high precision and computational efficiency in the vibration analysis of FG nanobeams. The detailed mathematical derivations are presented and numerical investigations performed with emphasis placed on investigating the effects of several parameters, such as small scale effects, volume fraction index, mode number, and thickness ratio on the normalized natural frequencies of the FG nanobeams. The study also shows explicitly that vibrations of FG nanobeams are significantly influenced by these effects. Numerical results are presented to serve as benchmarks for future analyses of FG nanobeams.

*Keywords*: Differential transformation method; Functionally graded material; Nanobeam; Nonlocal elasticity; Vibration

## 1. Introduction

Functionally graded materials (FGMs), which are microscopically heterogeneous and typically made from isotropic components, such as metals and ceramics, were initially designed to serve as thermal barrier materials for aerospace structures and fusion reactors [1]. In comparison with traditional composites, FGMs possess various advantages, including ensuring smooth transition of stress distributions, minimization or elimination of stress concentration, and increased bonding strength along the interface of two dissimilar materials. Over the past two decades, FGMs have had wide applications in modern industries including, aerospace, mechanical, electronics, optics, chemical, biomedical, nuclear, and civil engineering to name a few. These engineering applications have also resulted in intensive research attention on FGMs with focus mainly on its static, dynamic and vibration characteristics of FG structures [2].

Nanoscale engineering materials have displayed significant mechanical, electrical, and thermal performances superior to conventional structural materials. These materials have attracted considerable interest in modern science and technology after the invention of carbon nanotubes (CNTs) by Iijima [3], such as in micro/nano electromechanical systems (MEMS/NEMS). Nanostructures have been used in a number of areas, including communications, machinery, information technology, and biotechnology.

Thus far, three main methods have been utilized to investigate the mechanical behaviors of nanostructures, namely, atomistic model [4], semi-continuum, and continuum models [5]. However, both atomistic and semi-continuum models are computationally expensive and not suitable for analyzing large scale systems. Therefore, considerable efforts have been exerted to develop and calibrate continuum structural models for CNTs analysis. The inherent size effects at nanoscale often cause the mechanical characteristics of nanostructures to behave in a significantly different manner than at the macroscopic scale. Such effects are essential for nanoscale materials or structures and have substantial influence on nanoinstruments [6]. Generally, theoretical studies on size effects at nanoscale are conducted mostly through surface effects, strain gradients in elasticity and plasticity, and nonlocal stress field theory [7]. Choi et al. [8] first investigated the surface

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† Recommended by Chief-in-Editor Maenghyo Cho

effects on nanoscale thin films. Cho et al. [9] presented a continuum-based bridging model of a nanoscale thin film that considered surface effects, whereas Kim et al. [10] introduced molecular dynamics-based continuum models for linear elasticity analysis of nanofilms and nanowires with anisotropic surface effects. Ko et al. [11] also investigated the quality factor in clamping loss of nano-cantilever resonators. Unfortunately, classical continuum theories are deemed to fail for these nanostructures because the length dimensions at nanoscale are often sufficiently small, and call the applicability of classical continuum theories into question. Consequently, classical continuum models need to be extended to consider the nanoscale effects, which can be achieved through the use of the nonlocal elasticity theory proposed by Eringen [7], which considers the size-dependent effect. According to this theory, the stress state at a reference point is considered a function of the strain states of all points in the body.

In recent years, nanobeams and CNTs have had a wide variety of potential applications, such as sensors, actuators, transistors, probes, and resonators in NEMSs [12]. The application of FG materials is in the areas of MEMS and NEMS such as atomic force microscopes and electrically actuated MEMS and NEMS. The dimensions of these structural devices do not usually exceed micron scale, making size-dependent analysis necessary when studying FGMs [13].

With the quick growth of nanostructures, FGMs have also been used extensively in MEMS/NEMS, such as thin films, micro switches, and micro piezoactuators [14]. Understanding the mechanical properties and vibration behavior of FGMs are significantly important in the design and manufacture of FG MEMS/NEMS because of the high sensitivity of MEMS/NEMS to external stimulations. Thus, establishing an accurate model of FG nanobeams is a key factor in a successful MEMS/NEMS design. Asghari et al. [15] studied the free vibration of FGM Euler–Bernoulli microbeams that was extended to consider a size-dependent Timoshenko beam based on modified couple stress. The dynamic characteristics of FG beams with power law material graduation in the axial or transversal directions were examined by Alshorbagy et al. [16]. Ke and Wang [17] exploited the effect of size on the dynamic stability of FG Timoshenko microbeams. The modified couple stress theory employed on nonlinear free vibration of FG microbeams based on von Karman geometric nonlinearity was presented by Ke et al. [18]. The study revealed that both linear and nonlinear frequencies increased significantly when the thickness of the FGM microbeam reached a scale comparable to the material length scale parameter. Eltaher et al. [19, 20] presented a finite element formulation for free vibration analysis of FG nanobeams based on the nonlocal Euler beam theory. Their study also exploited the size-dependent static-buckling behavior of FG nanobeams based on the nonlocal continuum model [21]. Using nonlocal Timoshenko and Euler–Bernoulli beam theories, Simsek and Yurtcu [22] conducted an analytical investigation of the bending and buckling of FG nanobeams. Most previous literature focused on modeling micro/nano-beams based on the assumptions that the material is homogeneous. Very few studies are available for FGM micro/nanoscale structures. Previous literature also shows the wide use of DTM to solve a vast range of problems in different fields of engineering. To the best knowledge of the authors, thus far, no research efforts have been devoted to discovering a solution for the vibration of FG nanobeams by employing DTM.

Motivated by these facts, in the current study, DTM is applied in analyzing the vibration characteristics of size-dependent FG nanobeams. The superiority of DTM is its simplicity, excellent precision, and dependence on Taylor series expansion as well as the lesser time it takes to solve the polynomial series. DTMs differ from the traditional high order Taylor's series method, which requires a symbolic competition of the necessary derivatives of data functions. The Taylor series method takes a computationally long time for large orders. DTM makes it possible to obtain highly accurate results or exact solutions for differential equations.

In this study, the non-classical beam model within the framework of the Euler–Bernoulli beam theory is developed for FG nanobeams. Governing equations and boundary conditions for the free vibration of a nonlocal FG beam were derived via Hamilton principle. The detailed mathematical derivations are presented and numerical investigations performed, with emphasis on investigating the effects of several parameters, including size effects, constituent volume fractions, mode number, slenderness ratios, and small scale on vibration characteristics of FG nanobeams. Comparisons with the results from existing literature are provided and good agreement between the results of this article and those available in literature validate the presented approach. Numerical results are presented to serve as benchmarks for the application and design of nanoelectronic and nanodrive devices, nano-oscillators, and nanosensors in which nanobeams act as basic elements.

## 2. Theory and formulation

### *2.1 Power-law functionally graded material (P-FGM) beam*

The material properties of an FGM nanobeam are assumed to vary according to the power law on spatial coordinates. The coordinate system for an FG nanobeam is shown in Fig. 1.

According to the rule of mixture, effective material properties, $P$ (Young's modulus $E_f$, shear modulus $G_f$, and mass density $\rho_f$) can be expressed as [22]

$$P_f = P_c V_c + P_m V_m , \tag{1}$$

where $P_m$, $P_c$, $V_m$, and $V_c$ are the material properties and the volume fractions of the metal and ceramic constituents are related by

$$V_c + V_m = 1 . \tag{2a}$$

The volume fraction of the ceramic constituent of the beam

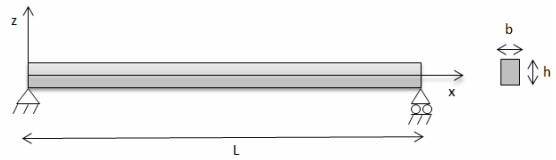



Fig. 1. Typical FG beam with Cartesian coordinates.

is assumed to be given by

$$V_c = (\frac{z}{h} + \frac{1}{2})^P , \tag{2b}$$

where $p$ is the non-negative variable parameter (power-law exponent) that determines the material distribution through the thickness of the beam. Based on Eqs. (1) and (2), the effective material properties of the FG nanobeam can be expressed as follows:

$$P_f(z) = \left(P_c - P_m\right)\left(\frac{z}{h} + \frac{1}{2}\right)^p + P_m . \tag{3}$$

According to this distribution, the bottom surface (z = -h/2) of an FG beam is pure metal, whereas its top surface (z = h/2) is pure ceramics.

### *2.2 Kinematic relations*

Using the Euler–Bernoulli beam model, the displacement field at any point of the beam can be written as follows:

$$u_x\left(x,z,t\right) = u\left(x,t\right) - z\frac{\partial w(x,t)}{\partial x} , \tag{4a}$$

$$u_z(x,z,t) = w(x,t) \tag{4b}$$

where $t$ is time, and $u$ and $w$ are displacement components of the mid-plane along the x and z directions, respectively. By assuming small deformations, the only nonzero strain of the Euler–Bernoulli beam theory is obtained as

$$\varepsilon_{xx} = \varepsilon_{xx}^0 - zk^0, \quad \varepsilon_{xx}^0 = \frac{\partial u(x,t)}{\partial x}, \quad k^0 = \frac{\partial^2 w(x,t)}{\partial x^2} , \tag{5}$$

where $\varepsilon_{xx}^0$ is the extensional strain and $k^0$ is the bending strain. Based on Hamilton's principle, which states that the motion of an elastic structure during time interval $t_1 < t < t_2$ is such that the time integral of the total dynamics potential is extremum [23], we have

$$\int_0^t \delta(U - T + W_{ext})\, dt = 0 , \tag{6}$$

where $U$ is strain energy, $T$ is kinetic energy, and $W_{ext}$ is work done by external forces. The virtual strain energy can be calculated as follows:

$$\delta U = \int_v \sigma_{ij}\delta\varepsilon_{ij} dV = \int_v (\sigma_{xx}\delta\varepsilon_{xx})\, dV . \tag{7}$$

Substituting Eq. (5) into Eq. (7) yields

$$\delta U = \int_0^L (N(\delta\varepsilon_{xx}^0) - M(\delta k^0))\, dx , \tag{8}$$

where $N$, $M$ are the axial force and bending moment, respectively. The stress resultants used in Eq. (8) are defined as

$$N = \int_A \sigma_{xx} dA, \quad M = \int_A \sigma_{xx} z dA . \tag{9}$$

The kinetic energy for Euler-Bernoulli beam is written as

$$T = \frac{1}{2}\int_0^L \int_A \rho(z)((\frac{\partial u_x}{\partial t})^2 + (\frac{\partial u_z}{\partial t})^2)\, dA dx . \tag{10}$$

The virtual kinetic energy is

$$\begin{aligned} \delta T = \int_0^L \Big[ & I_0(\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t}) \\ & - I_1(\frac{\partial u}{\partial t}\frac{\partial^2 \delta w}{\partial t \partial x} + \frac{\partial \delta u}{\partial t}\frac{\partial^2 w}{\partial t \partial x}) + I_2 \frac{\partial^2 w}{\partial t \partial x}\frac{\partial^2 \delta w}{\partial t \partial x}\Big] dx \end{aligned} , \tag{11}$$

where $(I_0, I_1, I_2)$ are the mass moment of inertias, which are defined as

$$(I_0, I_1, I_2) = \int_A \rho(z)(1, z, z^2)\, dA . \tag{12}$$

The first variation of the work done by external forces can be written in the form of

$$\delta W_{ext} = \int_0^L \left(f(x)\delta u + q(x)\delta w\right) dx , \tag{13}$$

where $f(x)$ and $q(x)$ are external axial and transverse load distributions along the length of the beam, respectively. By Substituting Eqs. (8), (11) and (13) into Eq. (6) and setting the coefficients of $\delta u$, $\delta w$, and $\delta \partial w/\partial x$ to zero, the following Euler–Lagrange equations can be obtained:

$$\frac{\partial N}{\partial x} + f = I_0\frac{\partial^2 u}{\partial t^2} - I_1\frac{\partial^3 w}{\partial x \partial t^2} \tag{14a}$$

$$\frac{\partial^2 M}{\partial x^2} + q = I_0\frac{\partial^2 w}{\partial t^2} + I_1\frac{\partial^3 u}{\partial x \partial t^2} - I_2\frac{\partial^4 w}{\partial x^2 \partial t^2} . \tag{14b}$$

Under the following boundary conditions,

$$N = 0 \quad \text{or} \quad u = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \tag{15a}$$

$$\frac{\partial M}{\partial x} - I_1 \frac{\partial^2 u}{\partial t^2} + I_2 \frac{\partial^3 w}{\partial x \partial t^2} = 0 \quad \text{or}$$

$$w = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \tag{15b}$$

$$M = 0 \quad \text{or} \quad \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L\,. \tag{15c}$$

### *2.3 Nonlocal elasticity model for FG nanobeam*

The Eringen nonlocal elasticity model [7] holds that stress at a reference point $x$ in a body is considered as a function of strains of all points in the near region. This assumption is in agreement with experimental observations of atomic theory and lattice dynamics in phonon scattering in which for a homogeneous and isotropic elastic solid. The nonlocal stress-tensor components $\sigma_{ij}$ at any point $x$ in the body can be expressed as

$$\sigma_{ij}(x) = \int_{\Omega} \alpha(|x' - x|, \tau)\, t_{ij}(x')\, d\Omega(x')\,, \tag{16}$$

where $t_{ij}(x')$ are components of the classical local stress tensor at point $x$ related to the components of the linear strain tensor $\varepsilon_{kl}$ through the conventional constitutive relations for a Hookean material, that is,

$$t_{ij} = C_{ijkl}\varepsilon_{kl}\,. \tag{17}$$

The meaning of Eq. (16) is that nonlocal stress at point $x$ is the weighted average of local stress of all points in the neighborhood of $x$, the size of which is related to nonlocal kernel $\alpha(|x' - x|, \tau)$. Here, $|x' - x|$ is the Euclidean distance and $\tau$ is a constant given by

$$\tau = \frac{e_0\, a}{l} \tag{18}$$

that represents the ratio between a characteristic internal length, a, (such as lattice parameter, C–C bond length and granular distance) and a characteristic external one, l, (e.g. crack length, wavelength) through an adjusting constant, $e_0$, dependent on each material. The magnitude of $e_0$ is determined experimentally or approximated by matching the dispersion curves of the plane waves with those of atomic lattice dynamics. According to Ref. [7], for a class of physically admissible kernel $\alpha(|x' - x|, \tau)$, it is possible to represent the integral constitutive relations given by Eq. (16) in an equivalent differential form as

$$(1 - (e_0 a)\nabla^2)\sigma_{kl} = t_{kl}\,, \tag{19}$$

where $\nabla^2$ is the Laplacian operator. Thus, the scale length $e_0 a$ takes into account the effect of size on the response of nanostructures. For an elastic material in the one-dimensional case, nonlocal constitutive relations may be simplified as [7]:

$$\sigma(x) - (e_0\, a)^2 \frac{\partial^2 \sigma(x)}{\partial x^2} = E\varepsilon(x)\,, \tag{20}$$

where $\sigma$ and $\varepsilon$ are the nonlocal stress and strain, respectively and $E$ is the Young's modulus. For Euler–Bernoulli nonlocal FG beam, Eq. (20) can be written as

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E(z)\varepsilon_{xx}\,, \tag{21}$$

where ( $\mu = (e_0\, a)^2$ ). We can obtain the force-strain by integrating Eq. (21) over the cross-section area of the beam. The moment-strain of the nonlocal Euler-Bernoulli FG beam theory can be obtained as follows:

$$N - \mu \frac{\partial^2 N}{\partial x^2} = A_{xx} \frac{\partial u}{\partial x} - B_{xx} \frac{\partial^2 w}{\partial x^2}\,, \tag{22}$$

$$M - \mu \frac{\partial^2 M}{\partial x^2} = B_{xx} \frac{\partial u}{\partial x} - C_{xx} \frac{\partial^2 w}{\partial x^2}\,. \tag{23}$$

The cross-sectional rigidities are defined as

$$(A_{xx}, B_{xx}, C_{xx}) = \int_A E(z)(1, z, z^2)\, dA\,. \tag{24}$$

The explicit relation of nonlocal normal force can be derived by substituting for the second derivative of $N$ from Eq. (14a) into Eq. (22) as follows:

$$N = A_{xx} \frac{\partial u}{\partial x} - B_{xx} \frac{\partial^2 w}{\partial x^2} + \mu\left(I_0 \frac{\partial^3 u}{\partial x \partial t^2} - I_1 \frac{\partial^4 w}{\partial x^2 \partial t^2} - \frac{\partial f}{\partial x}\right). \tag{25}$$

The explicit relation of the nonlocal bending moment can be derived by substituting for the second derivative of $M$ from Eq. (14b) into Eq. (23) as follows:

$$\begin{aligned} M &= B_{xx} \frac{\partial u}{\partial x} - C_{xx} \frac{\partial^2 w}{\partial x^2} \\ &+ \mu\left(I_0 \frac{\partial^2 w}{\partial t^2} + I_1 \frac{\partial^3 u}{\partial x \partial t^2} - I_2 \frac{\partial^4 w}{\partial x^2 \partial t^2} - q\right) \end{aligned}\,. \tag{26}$$

The nonlocal governing equations of the Euler-Bernoulli FG nanobeam in terms of displacement can be derived by substituting for $N$ and $M$ from Eqs. (25) and (26), respectively, into Eq. (14) as follows:

$$\begin{aligned} &A_{\mathrm{xx}} \frac{\partial^2 u}{\partial x^2} - B_{\mathrm{xx}} \frac{\partial^3 w}{\partial x^3} + \mu\left(I_0 \frac{\partial^4 u}{\partial t^2 \partial x^2} - I_1 \frac{\partial^5 w}{\partial t^2 \partial x^3} - \frac{\partial^2 f}{\partial x^2}\right) \\ &- I_0 \frac{\partial^2 u}{\partial t^2} + I_1 \frac{\partial^3 w}{\partial t^2 \partial x} + f = 0 \end{aligned} \tag{27a}$$

$$B_{xx}\frac{\partial^3 u}{\partial x^3}-C_{xx}\frac{\partial^4 w}{\partial x^4}+$$
$$\mu(I_0\frac{\partial^4 w}{\partial t^2\partial x^2}+I_1\frac{\partial^5 u}{\partial t^2\partial x^3}-I_2\frac{\partial^6 w}{\partial t^2\partial x^4}-\frac{\partial^2 q}{\partial x^2}). \quad (27b)$$
$$-I_0\frac{\partial^2 w}{\partial t^2}-I_1\frac{\partial^3 u}{\partial t^2\partial x}+I_2\frac{\partial^4 w}{\partial t^2\partial x^2}+q=0$$

## 3. Solution method

### *3.1 Analytical solution*

In this study, an analytical solution of the governing equations for free vibration of a simple supported FG nanobeam is presented based on the Navier method. The displacement functions are expressed as products of undetermined coefficients and known trigonometric functions to satisfy the governing equations and conditions at $x = 0$, L. The displacement fields are assumed to be in the form of the following:

$$u(x,t)=\sum_{n=1}^{\infty}U_n\cos(\frac{n\pi}{L}x)e^{i\omega_n t}, \quad (28)$$

$$w(x,t)=\sum_{n=1}^{\infty}W_n\sin(\frac{n\pi}{L}x)e^{i\omega_n t}, \quad (29)$$

where ($U_n$, $W_n$) are the unknown Fourier coefficients to be determined for each n value. Boundary conditions for simple supported beam are as follows:

$$u(0)=0,\ \frac{\partial u}{\partial x}(L)=0$$
$$w(0)=w(L)=0,\ \frac{\partial^2 w}{\partial x^2}(0)=\frac{\partial^2 w}{\partial x^2}(L)=0. \quad (30)$$

Substituting Eqs. (28) and (29) into Eqs. (27a) and (27b) respectively, leads to Eqs. (31) and (32):

$$(-A_{xx}(\frac{n\pi}{L})^2+I_0(1+\mu(\frac{n\pi}{L})^2)\omega_n{}^2)U_n$$
$$+(B_{xx}(\frac{n\pi}{L})^3-I_1((\frac{n\pi}{L})+\mu(\frac{n\pi}{L})^3)\omega_n{}^2)W_n=0 \quad (31)$$
$$(B_{xx}(\frac{n\pi}{L})^3-I_1((\frac{n\pi}{L})+\mu(\frac{n\pi}{L})^3)\omega_n{}^2)U_n$$
$$+(-C_{xx}(\frac{n\pi}{L})^4+I_0(1+\mu(\frac{n\pi}{L})^2)\omega_n{}^2 \quad (32)$$
$$+I_2((\frac{n\pi}{L})^2+\mu(\frac{n\pi}{L})^4)\omega_n{}^2)W_n=0.$$

By setting the determinant of the coefficient matrix of the above equations, a quadratic polynomial for $\omega_n^2$ can be obtained. We can find $\omega_n$ by setting this polynomial to zero.

### *3.2 Differential transformation method*

Several common numerical methods used to solve the initial-and/or boundary value problems that occur in engineering domain, include the finite element method (FEM), Galerkin method, and finite difference (FD) method. FEM and FD method for higher-order modes require to a significant number of grid points. The solution methods for all these points require considerable CPU time. DTM has several benefits because it is one of the most useful techniques for solving the differential solutions with small calculation errors and ability to solve nonlinear equations with boundary conditions. Abdel-Halim Hassan [24] applied DTM on eigenvalues and normalized eigenfunctions. Wang [25] presented an axial vibration analysis of stepped bars utilizing DTM. DTM has been proven to be a good computational tool for various engineering problems. Through the DTM technique, the ordinary and partial differential equations can be transformed into algebraic equations, from which a closed-form series solution can be obtained easily. In this method, certain transformation rules are applied to both the governing differential equations of motion and boundary conditions of the system to transform them into a set of algebraic equations as presented in Table 1. The solutions to these algebraic equations yield the desired results of the problem. The basic definitions and application procedure of this method can be introduced as follows. The transformation equation of function $f(x)$ can be defined as [24]

Table 1. Transformation rules of one-dimensional DTM [24].

| Original function | Transformed function |
|---|---|
| $f(x)=g(x)\pm h(x)$ | $F(k)=G(k)\pm H(k)$ |
| $f(x)=\lambda g(x)$ | $F(k)=\lambda G(k)$ |
| $f(x)=g(x)h(x)$ | $F(k)=\sum_{l=0}^{k}G(k-l)H(l)$ |
| $f(x)=\frac{d^n g(x)}{dx^n}$ | $F(k)=\frac{(k+n)!}{k!}G(k+n)$ |
| $f(x)=x^n$ | $F(k)=\delta(k-n)=\begin{cases}1 & k=n\\ n & k\neq n\end{cases}$ |

$$F[k]=\frac{1}{k!}(\frac{d^k f(x)}{dx^k})_{x=x_0}, \quad (33)$$

where $f(x)$ is the original function and $F[k]$ is the transformed function. The inverse transformation is defined as

$$f(x)=\sum_{k=0}^{\infty}(x-x_0)^k F[k]. \quad (34)$$

Combining Eqs. (33) and (34), we can obtain

$$f(x)=\sum_{k=0}^{\infty}\frac{(x-x_0)^k}{k!}(\frac{d^k f(x)}{dx^k})_{x=x_0}. \quad (35)$$

In actual application, function $f(x)$ is expressed by a finite series and Eq. (35) can be written as

$$f(x)=\sum_{k=0}^{N}\frac{(x-x_0)^k}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0}, \tag{36}$$

which implies that the term in relation Eq. (37) is negligible:

$$f(x)=\sum_{k=N+1}^{\infty}\frac{(x-x_0)^k}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0}. \tag{37}$$

Generally, deriving an analytical solution for Eqs. (27a) and (27b) can be difficult because of non-homogeneity. In this circumstance, the DTM is employed to translate these equations into a set of ordinary equations. A sinusoidal variation of $u(x,t)$ and $w(x,t)$ with a circular natural frequency $\omega$ is assumed and the functions are approximated as follows:

$$u(x,t)=u(x)e^{i\omega t}, \tag{38a}$$

$$w(x,t)=w(x)e^{i\omega t}. \tag{38b}$$

Substituting Eqs. (38a) and (38b) into Eqs. (27a) and (27b), the equations of motion can be rewritten as follows:

$$\begin{aligned}&A_{xx}\frac{\partial^2 u}{\partial x^2}-B_{xx}\frac{\partial^3 w}{\partial x^3}+\mu\left(-I_0\omega^2\frac{\partial^2 u}{\partial x^2}+\right.\\&\left.I_1\omega^2\frac{\partial^3 w}{\partial x^3}-\frac{\partial^2 f}{\partial x^2}\right)+I_0\omega^2 u-I_1\omega^2\frac{\partial w}{\partial x}+f=0\end{aligned} \tag{39a}$$

$$\begin{aligned}&B_{xx}\frac{\partial^3 u}{\partial x^3}-C_{xx}\frac{\partial^4 w}{\partial x^4}+\mu(-I_0\omega^2\frac{\partial^2 w}{\partial x^2}-I_1\omega^2\frac{\partial^3 u}{\partial x^3}\\&+I_2\omega^2\frac{\partial^4 w}{\partial x^4}-\frac{\partial^2 q}{\partial x^2})+I_0\omega^2 w+I_1\omega^2\frac{\partial u}{\partial x}-I_2\omega^2\frac{\partial^2 w}{\partial x^2}+q=0.\end{aligned} \tag{39b}$$

According to the basic transformation operations introduced in Table 1, the transformed form of governing equations Eqs. (27a) and (27b) around $x_0=0$ may be obtained as follows:

$$\begin{aligned}&A_{xx}(k+1)(k+2)U[k+2]\\&-B_{xx}(k+1)(k+2)(k+3)W[k+3]\\&-I_0\omega^2\left(-U[k]+\mu(k+1)(k+2)U[k+2]\right)\\&-I_1\omega^2(-\mu(k+1)(k+2)(k+3)W[k+3]\\&+(k+1)W[k+1])=0\\&B_{xx}(k+1)(k+2)(k+3)U[k+3]\\&-C_{xx}(k+1)(k+2)(k+3)(k+4)W[k+4]\\&-I_0\omega^2\left(-W[k]+\mu(k+1)(k+2)W[k+2]\right)\\&-I_1\omega^2(-(k+1)U[k+1]\\&+\mu(k+1)(k+2)(k+3)U[k+3])\\&-I_2\omega^2(-\mu(k+1)(k+2)(k+3)(k+4)W[k+4]\\&+(k+1)(k+2)W[k+2])=0\end{aligned}, \tag{40}$$

where $U[k]$ and $W[k]$ are the transformed functions of $u$ and $w$, respectively. By employing DTM theorems, the simple supported edge condition can be given as

$$\begin{aligned}&W[0]=0,\ W[2]=0,\ U[1]=0\\&\sum_{k=0}^{\infty}W[k]=0,\ \sum_{k=0}^{\infty}k(k-1)W[k]=0,\ \sum_{k=0}^{\infty}kU[k]=0.\end{aligned} \tag{41}$$

Using Eqs. (40) and (41) together with the transformed boundary condition Eq. (42), we can arrive at the following eigenvalue problem:

$$\begin{bmatrix}A_{11}(\omega) & A_{12}(\omega) & A_{13}(\omega)\\ A_{21}(\omega) & A_{22}(\omega) & A_{23}(\omega)\\ A_{31}(\omega) & A_{32}(\omega) & A_{33}(\omega)\end{bmatrix}[C]=0, \tag{42a}$$

where $[C]$ corresponds to the missing boundary conditions at $x = 0$. For non-trivial solutions of Eq. (43a), the determinant of the coefficient matrix should be equal to zero:

$$\begin{vmatrix}A_{11}(\omega) & A_{12}(\omega) & A_{13}(\omega)\\ A_{21}(\omega) & A_{22}(\omega) & A_{23}(\omega)\\ A_{31}(\omega) & A_{32}(\omega) & A_{33}(\omega)\end{vmatrix}=0. \tag{42b}$$

The frequency equation of the FG nanobeam, which is in the form of a transcendental eigenvalue equation and in terms of free vibration frequency, can be obtained by setting the determinant of the coefficient matrix of Eq. (43b) equal to zero. The eigen or mode function that describes the instantaneous deflected shape of the beam for a given frequency may then be obtained by using Eq. (34) as described in detail in Malik et al. [26]. Following an identical procedure, we can obtain the frequency equations and mode functions for all types of boundary conditions of FG nanobeams. The solution of Eq. (43b) is simply a polynomial root finding problem. In the present study, the Newton–Raphson method is used to solve the governing equation of non-dimensional natural frequencies. Solving Eq. (43b), the ith estimated eigenvalue for nth iteration ($\omega=\omega_i^{(n)}$) may be obtained. The total number of iterations is related to the accuracy of calculations that can be determined by using the following equation:

$$\left|\omega_i^{(n)}-\omega_i^{(n-1)}\right|<\varepsilon. \tag{43}$$

In this study, $\varepsilon$ = 0.0001 is considered in the process of determining eigenvalues, which resulted in a four-digit precision in the estimated eigenvalues.

## 4. Numerical results and discussion

A numerical testing of the procedure and parametric studies are performed to establish the validity and usefulness of the DTM approach. The effect of FG distribution, nonlocality effect, and thickness ratios on the natural frequencies of the

Table 2. Material properties of FGM constituents [15].

| Properties | Steel | Alumina ($Al_2O_3$) |
|---|---|---|
| $E$ | 210 (Gpa) | 390 (Gpa) |
| $\rho$ | 7800 ($Kg/m^3$) | 3960 ($Kg/m^3$) |

Table 3. Convergence study for the first three natural frequencies of the simple supported FG nanobeam ($L/h = 50$, $\mu = 1*10^{-12}$).

| Method | $k$ | $p = 5$ | | |
|---|---|---|---|---|
| | | $\bar{\omega}_1$ | $\bar{\omega}_2$ | $\bar{\omega}_3$ |
| Present DTM | 15 | 5.6690 | - | - |
| | 17 | 5.6690 | 20.4365 | - |
| | 19 | 5.6690 | 20.0842 | - |
| | 21 | 5.6690 | 20.1195 | - |
| | 23 | 5.6690 | 20.1161 | - |
| | 25 | 5.6690 | 20.1163 | - |
| | 27 | 5.6690 | 20.1163 | 38.9367 |
| | 29 | 5.6690 | 20.1163 | 38.8627 |
| | 31 | 5.6690 | 20.1163 | 38.8706 |
| | 33 | 5.6690 | 20.1163 | 38.8698 |
| | 35 | 5.6690 | 20.1163 | 38.8699 |
| | 37 | 5.6690 | 20.1163 | 38.8699 |
| | 39 | 5.6690 | 20.1163 | 38.8699 |
| Present analytical | | 5.6689 | 20.1158 | 38.8699 |

FG nanobeam were determined. The FG nanobeam is composed of steel and alumina ($Al_2O_3$), and its properties are provided in Table 2. The table shows that the steel material is assumed to be isotropic. Although single crystal material was dominant in the nanoscale, isotropicness could be achieved in the microscale film in polycrystalline configuration [10]. The bottom surface of the beam is pure steel, whereas its top surface is pure alumina. The beam geometry has the following dimensions: L (length) = 10,000 nm, b (width) = 1000 nm, and h (thickness) = 100 nm. The relationship described in Eq. (45) is performed to calculate the non-dimensional natural frequencies:

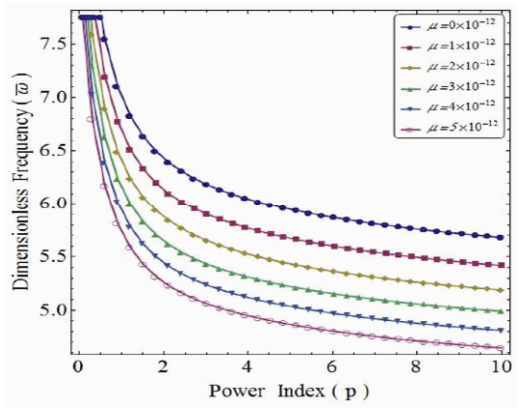
$$\bar{\omega} = \omega L^2 \sqrt{\rho A / EI}\,, \tag{45}$$

where $I = bh^3/12$ is the moment of inertia of the cross-section of the beam.

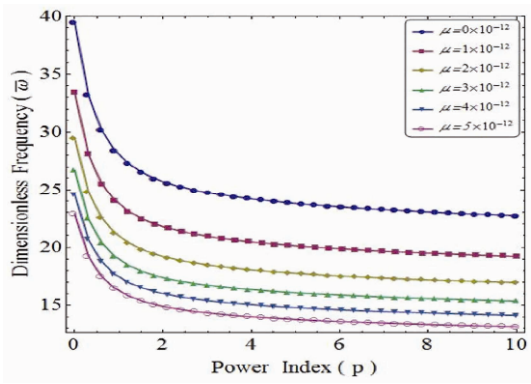
Table 3 shows the convergence study of DTM for the first three frequencies of the nanobeam. After a certain number of iterations, eigenvalues converged to a value with good precision. The results in Table 3 clearly show the high convergence rate of the method, and it may be deduced that $k$ = 35 led to accurate results. The accuracy of the natural frequencies can be predicted by the present method by comparing the results of the present study (both analytical and DTM-based solutions) and the results presented by Eltaher et al. [19] that were obtained through FEM for FG nanobeams with different FG distribution indices, length-to-thickness ratios, and nonlocal parameters as presented in Table 4.

Table 4. Comparison of non-dimensional fundamental natural frequencies ($\bar{\omega}_1 = \omega_1 L^2 \sqrt{\rho A / EI}$) of simple supported FG nanobeams (b = 1000 nm, L = 10,000 nm, h = 100 nm).

| L/h | $\mu*10^{-12}$ | $p = 0.1$ | | |
|---|---|---|---|---|
| | | FEM eltaher et al. [19] | Present | |
| | | | DTM | Analytical |
| 20 | 0 | 9.2129 | 9.1887 | 9.1887 |
| | 1 | 8.7879 | 8.7663 | 8.7663 |
| | 2 | 8.4166 | 8.3972 | 8.3972 |
| | 3 | 8.0887 | 8.0712 | 8.0712 |
| | 4 | 7.7964 | 7.7804 | 7.7804 |
| | 5 | 7.5336 | 7.5189 | 7.5189 |
| 50 | 0 | 9.2045 | 9.1968 | 9.1968 |
| | 1 | 8.7815 | 8.7740 | 8.7740 |
| | 2 | 8.4116 | 8.4047 | 8.4047 |
| | 3 | 8.0848 | 8.0783 | 8.0783 |
| | 4 | 7.7934 | 7.7873 | 7.7873 |
| | 5 | 7.5313 | 7.5256 | 7.5256 |
| 100 | 0 | 9.2038 | 9.1980 | 9.1980 |
| | 1 | 8.7806 | 8.7752 | 8.7752 |
| | 2 | 8.4109 | 8.4057 | 8.4057 |
| | 3 | 8.0842 | 8.0793 | 8.0793 |
| | 4 | 7.7929 | 7.7883 | 7.7882 |
| | 5 | 7.5310 | 7.5265 | 7.5265 |
| L/h | $\mu*10^{-12}$ | $p = 5$ | | |
| | | FEM eltaher et al. [19] | Present | |
| | | | DTM | Analytical |
| 20 | 0 | 6.0025 | 5.9373 | 5.9370 |
| | 1 | 5.7256 | 5.6643 | 5.6641 |
| | 2 | 5.4837 | 5.4258 | 5.4256 |
| | 3 | 5.2702 | 5.2152 | 5.2149 |
| | 4 | 5.0797 | 5.0273 | 5.0271 |
| | 5 | 4.9086 | 4.8583 | 4.8581 |
| 50 | 0 | 5.9990 | 5.9421 | 5.9421 |
| | 1 | 5.7218 | 5.6690 | 5.6689 |
| | 2 | 5.4808 | 5.4303 | 5.4303 |
| | 3 | 5.2679 | 5.2195 | 5.2194 |
| | 4 | 5.0780 | 5.0314 | 5.0314 |
| | 5 | 4.9072 | 4.8623 | 4.8623 |
| 100 | 0 | 5.9970 | 5.9428 | 5.9428 |
| | 1 | 5.7212 | 5.6696 | 5.6696 |
| | 2 | 5.4803 | 5.4309 | 5.4309 |
| | 3 | 5.2675 | 5.2201 | 5.2201 |
| | 4 | 5.0777 | 5.0320 | 5.0320 |
| | 5 | 4.9071 | 4.8629 | 4.8629 |




(a)




(b)

Fig. 2. Variations in the (a) first and; (b) second dimensionless frequencies of the simple supported FG nanobeam with material gradation for various nonlocality parameters (L/h = 100).

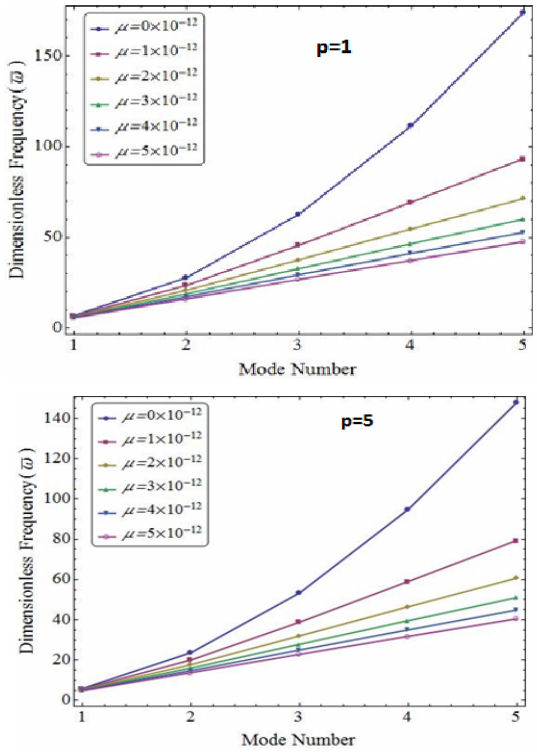



Fig. 3. Effects of nonlocality parameters on the dimensionless frequency of FG nanobeams for various mode numbers and with different material gradation indices ( $p$ = 1, 5) (L/h = 50).

The excellent agreement between the presented DTM results and the analytical based solution as well as the results provided by FEM [19] can be noticed clearly. Thus, the proposed method of solution is validated. The classical isotropic beam theory is rendered when both material gradation index and nonlocal parameter vanish.

The effects of slenderness ratios on dimensionless frequency are presented in Table 4. When the slenderness ratio of the FG nanobeam decreased (thickness reduces), the frequencies increased. Fixing the nonlocal parameter and varying the material distribution parameter resulted in a decrease in fundamental frequencies, which can be attributed to the increase in the ceramics phase constituent as well as stiffness of the beam. However, increasing the nonlocal parameter caused a decrease in the fundamental frequency, for a constant material gradation index. For the case on hand, changing the nonlocal parameter from 0 to 5 resulted in a decrease in fundamental frequency parameter of about 22% when L/h = 20. This result indicated that the effect of the nonlocal small scale parameter softened the nanobeam.

The qualitative effects of the nonlocal parameter and material index on the first two dimensionless frequencies of the simple supported FG nanobeams are shown in Fig. 2. The first and second dimensionless frequencies of the simple supported FG nanobeam decreased as the material index parameter increased from 0 to 10. Increasing the nonlocal parameter from 0 to 5 also resulted in a decrease in the fundamental frequency parameters of the FG nanobeam. Fig. 3 demonstrates the variations in mode number with changes in the nonlocality parameter at constant slenderness ratios (L/h = 50) for different material distributions. The influence of nonlocality parameter on the non-dimensional frequency increased for higher mode numbers. The influence of nonlocality parameter was also unaffected by the change in material distribution.

## 5. Conclusions

In this paper, a semi-analytical DTM is employed in the vibration analysis of size-dependent FG nanobeams. This method provides a semi-analytical solution that considers the influence of various parameters, such as small scale effects, volume fraction index, mode number, and thickness ratio on the normalized natural frequencies of the FG nanobeams. The vibration behavior of FG nanobeams based on Euler–Bernoulli beam theory and Eringen nonlocal constitutive equations is investigated. A Navier-based analytical model is also employed to solve the governing equations derived through Hamilton's principle. The good agreement between the results of this article and those available in literature validate the presented approach. Numerical results demonstrate

that the small scale effects play an important role on the vibrational behavior of the FG nanobeam. Thus, the nonlocality effects should be reflected in the study of dynamic behavior of nanostructures. The vibrational characteristics of FG nanobeams can also be enhanced through the selection of appropriate values of the power-law indices. Thus, it can be concluded that the applied DTM approach provides accurate results and can be easily used for vibration analysis of FG nanobeams.

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