

# Vibration of nonuniform carbon nanotube with attached mass via nonlocal Timoshenko beam theory†

Hai-Li Tang, Zhi-Bin Shen\* and Dao-Kui Li\*

*College of Aerospace Science and Engineering, National University of Defense Technology, Changsha, Hunan 410073, China* 

(Manuscript Received August 28, 2013; Revised May 4, 2014; Accepted May 27, 2014) --

# **Abstract**

This paper studies the vibrational behavior of nonuniform single-walled carbon nanotube (SWCNT) carrying a nanoparticle. A nonuniform cantilever beam with a concentrated mass at the free end is analyzed according to the nonlocal Timoshenko beam theory. A governing equation of a nonuniform SWCNT with attached mass is established. The transfer function method incorporating with the perturbation method is utilized to obtain the resonant frequencies of a vibrating nonlocal cantilever-mass system. The effects of the nonlocal parameter, taper ratio and attached mass on the natural frequencies and frequency shifts are discussed. Obtained results indicate that the sensitivity of the frequency shifts on the attached mass increases when the length-to-diameter ratio decreases. Tapered SWCNT possesses higher fundamental frequencies if the taper ratio becomes larger.

*Keywords*: Nonuniform SWCNT; Mass sensor; Nonlocal Timoshenko beam theory; Transfer function method; Perturbation method

# **1. Introduction**

Carbon nanotubes (CNTs) have attracted increasing attention of researchers since their discovery [1] in 1991, and owing to their unique properties [2] they have potential and evolving applications [3]. Nanoscale mass sensors are based on the fact that the change in the natural frequencies is sensitive to attached masses. Therefore, a key issue of mass detection is to quantify the change in resonant frequencies or frequency shift due to attached masses. Nanoparticle detection including gas detection, virus detection and charge detection requires extra-high mass sensitivity. The vibration frequencies of CNTs reaching THz and their high sensitivity to environment change make it possible to fabricate nanoscale mass sensors [4, 5].

As we know, the geometry of mechanical devices strongly affects their dynamics behaviors. Consequently, for various purposes, a variety of CNTs such as single-, double-, and multi-walled, as well as Y-, bamboo-, cone-shaped, horn-shaped CNTs [6-10] have been synthesized. Nonuniform CNTs possess varying cross-section and are one of the most attractive shapes of CNTs. On the other hand, mass sensors with varying cross-section have some advantages over those with uniform cross-section. Therefore, it is extremely necessary to study the vibration performance of mass sensors made of nonuniform

# CNTs.

<u> La componenta de la compo</u>

Generally, the theoretical analysis of CNTs is classified into two main categories, the discrete atomic modeling and the continuum modeling. In addition, experimental evidence [11] has showed pronounced size effects in CNTs. Atomic modeling such as molecular dynamics simulation are more suitable in accurately describing size-dependent mechanical properties. However, these discrete simulations are limited to systems with a small number of molecules and atoms and therefore restricted to small-scale modeling. Moreover, it is extraordinary hard to conduct experimental tests efficiently at nanoscale. To overcome these deficiencies, modified continuum mechanics approaches, the nonlocal elasticity theory [12] describing long-range interactions of the nanoscale effect, has been widely accepted to deal with size-dependent problems. In this way, a lot of researches have been reported on the vibration of CNTs with an attached mass based on the nonlocal Euler-Bernoulli beam theory (EBT) [13] and Timoshenko beam theory (TBT) [14, 15]. It is mentioned that all of these studies are only suitable for uniform CNTs with attached mass. For nonuniform CNTs, free vibration analyses without considering shear deformation and rotary inertia of the crosssection have been made by Lee and Chang [16], Murmu and Pradhan [17], respectively. For nonuniform nanocantilever with attached nanoparticle, Tang et al. [18] investigated the vibration of horn-shaped single wall CNT (SWCNTs) based on the nonlocal EBT. However, there are few studies on resonant frequency of vibration of a nonuniform CNT-based mass

<sup>\*</sup>Corresponding author. Tel.: +86 731 84573178, Fax.: +86 731 84512301

E-mail address: zb\_shen@yeah.net (Z.B.Shen); lidaokui@nudt.edu.cn (D.K.Li) † Recommended by Associate Editor Jin Weon Kim

<sup>©</sup> KSME & Springer 2014



Fig. 1. Nonuniform SWCNT-based mass sensor.

sensor using nonlocal TBT, to the best knowledge of the authors.

The present paper aims at analyzing the vibration response of nonuniform SWCNTs with an attached mass. The nonlocal TBT with the scale parameter is applied. Using the transfer function method (TFM) [19] incorporating with the perturbation method (PM), the natural frequencies of the SWCNTbased mass sensor are evaluated. A detailed investigation is carried out for the effects of the nonlocal parameter, attached mass and geometry parameters on the natural frequencies and frequency shifts.

### **2. Governing equations and boundary conditions**

# *2.1 Dynamic equation of nonuniform SWCNT-based mass sensors*

In this study, a nonuniform SWCNT-based mass sensor can be modeled as a nonuniform cantilever beam of length *L* and carrying a concentrated mass *m* at the free tip, as shown in Fig. 1. Its cross section is a circle with radius r varying linearly  $r_0$ to  $r<sub>L</sub>$ , and the thickness of the pipe retains constant value  $\delta$ .<br>Based on the nonlocal TBT, the governing equations of *th* to the critics of the nonlocal parameter, and<br>and  $\rho A_x \left(1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right) \frac{\partial^2 W}{\partial t^2}$ <br>y shifts.<br> $-2\rho(e_0 a)^2 \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x}$ <br>y shifts.<br> $\frac{\partial}{\partial x}$  and **boundary conditions**<br> $\frac{\partial}{\partial x}$ <br> $\frac{\partial}{\partial$ and for the evaluation. A denoted investigation is<br>
out for the effects of the nonlocal parameter, attached<br>
d geometry parameters on the natural frequencies and<br>  $\rho A_x \left(1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right) \frac{\partial^2}{\partial t}$ <br>
cy shifts.<br> Let the critics of the interaction parameter, attention<br>
of a geometry parameters on the natural frequencies and<br>  $\left[\rho A_x\right] \left(1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right) \frac{\partial^2 f}{\partial t^2}$ <br>
ors study, a nonuniform SWCNT-based mass  $\rho I_x\left(1 - (e$ ency shifts.<br> **overning equations and boundary conditions**<br> **overning equations of nonuniform SWCNT-based mass**<br> **o**  $\rho I_s \left(1 - (e_0 a)^2 \frac{\partial^2 t}{\partial x^2} - kT_s \frac{\partial^2 t}{\partial x^2} - kT_s \frac{\partial^2 t}{\partial x^2} + R_S \frac{\partial^2 t}{\partial x^2} + R_S \frac{\partial^2 t}{\partial x^2} + R_S$ *o x x n**x x* Example a serving equations and boundary conditions<br>  $\rho_2$  and  $\rho_3$  and  $\rho_4$ <br>  $\rho_5$  and  $\rho_6$ <br>  $\rho_7$  and  $\rho_8$ <br>  $\rho_7$  $-2\rho(e_0 a)^2 \frac{\partial A_z}{\partial x}$ <br>
straine equations and boundary conditions<br>
stanic equation of nonuniform SWCNT-based mass<br>  $\rho I_x \left(1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2} \right) \frac{\partial^2}{\partial t^2}$ <br>
standy, a nonuniform SWCNT-based mass sensor can<br>
g a co

transverse vibration for SWCNTs with varying cross-section can be expressed as

$$
\rho A_x \frac{\partial^2 w}{\partial t^2} - \frac{\partial Q}{\partial x} = 0, \quad 0 < x < L \tag{1}
$$

$$
\rho I_x \frac{\partial^2 \theta}{\partial t^2} - \frac{\partial M}{\partial x} + Q = 0, \quad 0 < x < L \tag{2}
$$

where  $w$  and  $\theta$  are the transverse displacement and the rotation of cross-section, both of which depend on the longitudinal coordinate *x* and time *t*,  $\rho$  the mass density,  $A_x$ the area of cross-section,  $I<sub>x</sub>$  the moment of inertia of crosssectional area, *M* the bending moment, and *Q* the shearing force. For nonuniform SWCNTs, the radius of cross section varies and is assumed to obey Fig. (a transform contained manner  $\alpha = \frac{1}{2} \sum_{i=1}^{M} \frac{1}{i} \sum_{j=1}^{n} \frac{1}{j} \frac{1}{j} \frac{1}{j} \frac{1}{j} \frac{1}{j} \frac{1}{j} \frac{1}{j} \frac{1}{k} \frac{1}{k} \sum_{k=1}^{n} \frac{1}{k} \frac{1}{$ *x x* **a** *x* **c** *x x x x x x x x x x x x x x x x x x z z z z z z z z z z z z z z z PA*,  $\frac{\partial^2 w}{\partial t^2} - \frac{\partial Q}{\partial x} = 0, 0 < x < L$ <br> *A*  $t, \frac{\partial^2 w}{\partial t^2} - \frac{\partial M}{\partial x} + Q = 0, 0 < x < L$ <br> **CO**<br> *A*  $t, \frac{\partial^2 w}{\partial t^2} - \frac{\partial M}{\partial x} + Q = 0, 0 < x < L$ *<br> CO<br> CO<br>* 

$$
r_x = -(r_0 - r_L)x / L + r_0 = \varepsilon x + r_0
$$
\n(3)

then we have

$$
A_x = \gamma x + n, \quad (\gamma = 2\varepsilon \delta \pi, \quad n = 2\pi r_0 \delta)
$$
 (4)

$$
I_x = 0.25\pi \left[ (r_x + 0.5\delta)^4 - (r_x - 0.5\delta)^4 \right].
$$
 (5)

Furthermore, the bending moment *M* and the shearing force *Q* based on the nonlocal TBT can be obtained below, respectively ology 28 (9) (2014) 3741~3747<br>
more, the bending moment M and the shearing force<br>
on the nonlocal TBT can be obtained below, respec-<br>  $(e_0 a)^2 \frac{\partial (I_x \theta)}{\partial x} \frac{\partial^2 \theta}{\partial t^2} + \rho (e_0 a)^2 A_x \frac{\partial^2 w}{\partial t^2} + EI_x \frac{\partial \theta}{\partial x}$  (6)<br>  $e_0$ *x x a I cehnology 28 (9) (2014) 3741-3747*<br> *x* **Eurthermore, the bending moment** *M* **and the shearing force based on the nonlocal TBT can be obtained below, respectly<br>** *M* **= \rho(e\_0 a)^2 \frac{\partial (I\_x \theta)}{\partial x} \frac{\partial^2 \theta}{\partial t^2}** (2014) 3741-3747<br>
bending moment *M* and the shearing force<br>
alocal TBT can be obtained below, respec-<br>  $\frac{x\theta}{x} \frac{\partial^2 \theta}{\partial t^2} + \rho(e_0 a)^2 A_x \frac{\partial^2 w}{\partial t^2} + EI_x \frac{\partial \theta}{\partial x}$  (6)<br>  $\frac{x^2}{x} \frac{\partial^2 w}{\partial t^2} + \kappa GA_x \left( \theta + \frac{\partial w}{\partial x} \right$ echnology 28 (9) (2014) 3741-3747<br>
thermore, the bending moment M and the shearing force<br>
ed on the nonlocal TBT can be obtained below, respec-<br>  $= \rho(e_0 a)^2 \frac{\partial (I_x \theta)}{\partial x} \frac{\partial^2 \theta}{\partial t^2} + \rho(e_0 a)^2 A_x \frac{\partial^2 w}{\partial t^2} + EI_x \frac{\partial \theta}{\partial$ 9) (2014) 3741-3747<br>
e bending moment *M* and the shearing force<br>
onlocal TBT can be obtained below, respec-<br>  $\frac{I_x \theta}{\partial x} \frac{\partial^2 \theta}{\partial t^2} + \rho(e_0 a)^2 A_x \frac{\partial^2 w}{\partial t^2} + EI_x \frac{\partial \theta}{\partial x}$  (6)<br>  $\frac{A_x w}{\partial t} \frac{\partial^2 w}{\partial t^2} + \kappa G A_x \left( \$ *x* tively

$$
M = \rho(e_0 a)^2 \frac{\partial (I_x \theta)}{\partial x} \frac{\partial^2 \theta}{\partial t^2} + \rho(e_0 a)^2 A_x \frac{\partial^2 w}{\partial t^2} + EI_x \frac{\partial \theta}{\partial x}
$$
 (6)

$$
Q = \rho(e_0 a)^2 \frac{\partial (A_x w)}{\partial x} \frac{\partial^2 w}{\partial t^2} + \kappa G A_x \left(\theta + \frac{\partial w}{\partial x}\right)
$$
(7)

(9) (2014) 3741-3747<br>
ne bending moment M and the shearing force<br>
onlocal TBT can be obtained below, respec-<br>  $\frac{(I_x \theta)}{\partial x} \frac{\partial^2 \theta}{\partial t^2} + \rho(e_0 a)^2 A_x \frac{\partial^2 w}{\partial t^2} + EI_x \frac{\partial \theta}{\partial x}$  (6)<br>  $\frac{A_x w}{\partial x} \frac{\partial^2 w}{\partial t^2} + \kappa G A_x \left( \$ (9) (2014) 3741-3747<br>
the bending moment M and the shearing force<br>
nonlocal TBT can be obtained below, respec-<br>  $\frac{\partial (I_x \theta)}{\partial x} \frac{\partial^2 \theta}{\partial t^2} + \rho(e_0 a)^2 A_x \frac{\partial^2 w}{\partial t^2} + EI_x \frac{\partial \theta}{\partial x}$  (6)<br>  $\frac{(A_x w)}{\partial x} \frac{\partial^2 w}{\partial t^2} + \kappa G A_x$ nology 28 (9) (2014) 3741-3747<br>
rmore, the bending moment M and the shearing force<br>
on the nonlocal TBT can be obtained below, respec-<br>  $\phi(e_0 a)^2 \frac{\partial (I_x \theta)}{\partial x} \frac{\partial^2 \theta}{\partial t^2} + \rho(e_0 a)^2 A_x \frac{\partial^2 w}{\partial t^2} + EI_x \frac{\partial \theta}{\partial x}$  (6)<br> *xd Technology 28 (9) (2014) 3741-3747*<br> *X A W* **Eurthermore, the bending moment** *M* **and the shearing force based on the nonlocal TBT can be obtained below, respec-<br>** *M* **= \rho(e\_0 a)^2 \frac{\partial (I\_x \theta)}{\partial x} \frac{\partial^2 \theta}{\partial t^2} + \rho(e\_0** *xhnology 28 (9) (2014) 3741-3747*<br> *x* remmore, the bending moment *M* and the shearing force<br> *x* do n the nonlocal TBT can be obtained below, respec-<br>  $\rho(e_0 a)^2 \frac{\partial (I, \theta)}{\partial x} \frac{\partial^2 \theta}{\partial t^2} + \rho(e_0 a)^2 A_x \frac{\partial^2 w}{\partial t^2} + EI_x \$ Fechnology 28 (9) (2014) 3741-3747<br>
Thermore, the bending moment M and the shearing force<br>
sed on the nonlocal TBT can be obtained below, respec-<br>  $\rho(e_0 a)^2 \frac{\partial (I_x \theta)}{\partial x} \frac{\partial^2 \theta}{\partial t^2} + \rho(e_0 a)^2 A_x \frac{\partial^2 w}{\partial t^2} + EI_x \frac{\partial \theta}{\$ where *E* is Young's modulus, *G* the shear modulus,  $e_0a$  the nonlocal parameter with length unit which can be used to modify the classical TBT,  $\kappa$  the shear correction coefficient depending on the shape of the cross section. By substituting Eqs. (6) and (7) into Eqs. (1) and (2), then coupled governing  $Q = \rho(e_0 a)^2 \frac{\partial (A_x w)}{\partial x} \frac{\partial^2 w}{\partial t^2} + \kappa G A_x \left( \theta + \frac{\partial w}{\partial x} \right)$  (7)<br>where *E* is Young's modulus, *G* the shear modulus, *e<sub>9</sub>a* the<br>nonlocal parameter with length unit which can be used to<br>modify the classical TBT, *k*  $\partial x$   $\partial t^2$   $\partial t^2$   $\partial x$   $\partial y$   $\partial y$   $\partial z$   $\partial y$   $\partial z$   $\partial z$   $\partial y$   $\partial z$   $\partial z$   $\partial z$   $\partial y$   $\partial y$   $\partial z$   $\partial y$   $\partial y$ 

Furthermore, the bending moment M and the shearing force  
\n
$$
Q
$$
 based on the nonlocal TBT can be obtained below, respectively  
\n $M = \rho(e_{\alpha}a)^2 \frac{\partial (I_{,\theta})}{\partial a} \frac{\partial^2 \theta}{\partial t^2} + \rho(e_{\alpha}a)^2 A_{,\theta} \frac{\partial^3 w}{\partial t^3} + E I_{,\theta} \frac{\partial \theta}{\partial \theta}$   
\nor using nonlocal TBT, to the best knowledge of the au-  
\nis, we present paper aims at analyzing the vibration response  
\n $Q = \rho(e_{\alpha}a)^2 \frac{\partial (I_{,\theta})}{\partial x} \frac{\partial^2 w}{\partial t^2} + \rho(e_{\alpha}a)^2 A_{,\theta} \frac{\partial^3 w}{\partial t^3} + E I_{,\theta} \frac{\partial \theta}{\partial \theta}$   
\nor using nonlocal TBT, to the best knowledge of the au-  
\nis, we present paper aims at analyzing the vibration response  
\n $Q = \rho(e_{\alpha}a)^2 \frac{\partial (I_{,\theta})}{\partial x} \frac{\partial^2 w}{\partial t^2} + \kappa G A_{\theta} \left(\theta + \frac{\partial w}{\partial x}\right)$  (7)  
\n $Q = \rho(e_{\alpha}a)^2 \frac{\partial (I_{,\theta})}{\partial x} \frac{\partial^2 w}{\partial t^2} + \kappa G A_{\theta} \left(\theta + \frac{\partial w}{\partial x}\right)$  (9)  
\n $Q = \rho(e_{\alpha}a)^2 \frac{\partial (I_{,\theta})}{\partial x} \frac{\partial^2 w}{\partial t^2} + \kappa G A_{\theta} \left(\theta + \frac{\partial^2 w}{\partial x}\right)$  (1) and (2), then required growth length  
\nwith the scale parameter is applied. Using the transfer  
\n $Q$  means  
\nmethod (PM), the natural frequencies of the SWCN-1- equations of the nonlocal TBT can be obtained as follows  
\nd mass mesor are evaluated. A detailed investigation is  
\nd was seenor are evaluated. A detailed investigation is  
\nd was shown in Figure 2.12  
\n $Q$  (Eq. A)  $\frac{\partial^2 w}{\partial x^2} = \kappa G A_{\theta} \left(\frac{\partial^2 w}{\partial x^2} + \kappa G A_{\theta} \left(\theta + \frac{\partial^2 w}{\partial x}\right)\right)$   
\n $Q = \rho(e_{\alpha}a)^2 \left(\frac{\partial^2 w}{\partial x^2} - \kappa G A_{\$ 

#### *2.2 Boundary conditions*

For a cantilever beam, the corresponding boundary conditions read

$$
w(0,t) = \theta(0,t) = 0 , \quad M(L,t) = 0
$$
 (10)

$$
Q(L,t) + m \frac{\partial^2 w(L,t)}{\partial t^2} = 0.
$$
 (11)

Furthermore, the initial state of the sensor is assumed to be at rest, namely

$$
w(x,0) = \frac{\partial w(x,0)}{\partial t} = 0 , \quad \theta(x,0) = \frac{\partial \theta(x,0)}{\partial t} = 0 .
$$
 (12)

 $\rho I_x \left( 1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2} \right) \frac{\partial^2}{\partial t^2} - EI_x \frac{\partial^2 \theta}{\partial x^2} + \kappa G A_x \left( \theta + \frac{\partial \theta}{\partial x} \right)$ <br>  $-\rho(e_0 a)^2 \left( \frac{\partial^2 I_x}{\partial x^2} \frac{\partial^2 \theta}{\partial t^2} + 2 \frac{\partial I_x}{\partial x} \frac{\partial^2 \theta}{\partial x \partial t^2} \right) - E \frac{\partial I_x}{\partial x} \frac{\partial \theta}{\partial x} = 0$ .<br> **Boundary conditio** (eq. a)  $\left(\frac{\partial^2 I_x}{\partial x^2} \frac{\partial^2 \theta}{\partial t^2} + 2 \frac{\partial I_x}{\partial x} \frac{\partial^3 \theta}{\partial x \partial t^2}\right) - E \frac{\partial I_x}{\partial x} \frac{\partial \theta}{\partial x} = 0.$ <br>
(9)<br>  $\rho(e_0 a)^2 \left(\frac{\partial^2 I_x}{\partial x^2} \frac{\partial^2 \theta}{\partial t^2} + 2 \frac{\partial I_x}{\partial x} \frac{\partial^3 \theta}{\partial x \partial t^2}\right) - E \frac{\partial I_x}{\partial x} \frac{\partial \theta}{\partial x} = 0.$ <br>
(alt  $\frac{\partial^2 \theta}{\partial x^2} + E I_x \frac{\partial^2 \theta}{\partial x^2} + \kappa G A_x \left( \theta + \frac{\partial \theta}{\partial x} \right)$ <br>  $\frac{\partial^2 \theta}{\partial t^2} + 2 \frac{\partial I_x}{\partial x} \frac{\partial^2 \theta}{\partial x^2} - E \frac{\partial I_x}{\partial x} \frac{\partial \theta}{\partial x} = 0.$ <br>
(9)<br>  $\frac{\partial^2 \theta}{\partial t^2} + 2 \frac{\partial^2 I_x}{\partial x \partial x^2} - E \frac{\partial I_x}{\partial x \partial x} \frac{\partial \theta}{\partial x} = 0.$ <br>
(10)  $\frac{d_x}{dx} \frac{\partial^3 \theta}{\partial x \partial t^2}$  =  $E \frac{\partial I_x}{\partial x} \frac{\partial \theta}{\partial x} = 0$ .<br>
(9)<br>  $\frac{d_x}{\partial x \partial t^2}$  =  $E \frac{\partial I_x}{\partial x} \frac{\partial \theta}{\partial x} = 0$ .<br>
(10)<br>
(11)<br>
(11)<br>
(11)<br>
(12)<br>
the sensor is assumed to be<br>  $= \frac{\partial \theta(x,0)}{\partial t} = 0$ . (12)<br>
and rotary inert Neglecting shear deformation and rotary inertia of the crosssection, the governing equation of transverse vibration based on the nonlocal EBT for SWCNTs with varying cross-section is recovered [18].

#### **3. Solution method**

For the uniform Timoshenko beams with attached mass, it is easy to solve the govern equation using TFM [19]. For the present paper, since the governing Eqs. (8) and (9) are related to differential equations with variable coefficients, it is difficult to derive its exact solution directly. PM can be employed to obtain the approximate solution of the nonuniform structures, effectively. In this section, instead we invoke the TFM and the PM to determine the natural frequencies.

# *3.1 TFM for nonlocal Timoshenko beams*

With the aid of the initial conditions Eq. (12), after performing Laplace transform, the governing Eqs. (8) and (9) along with the corresponding boundary conditions (10) and (11) where  $F(X,s)$ ,  $M(s)$  and  $N(s)$  are all  $4 \times 4$  matrixes become

H.-L. Tang *et al.* /Journal of Mechanical Science and Technology 28 (9) (2014) 3741-3747  
\nres, effectively. In this section, instead we invoke the TFM and (14) together with Eqs. (15)-(17) can be rewritten  
\nd the PM to determine the natural frequencies.  
\n**1 TFM for nonlocal Timoshenko beams**  
\nWith the aid of the initial conditions Eq. (12), after perform-  
\ng Laplace transform, the governing Eqs. (8) and (9) along  
\nthe corresponding boundary conditions (10) and (11)  
\ncome  
\n
$$
\rho s^2 A_x \left(1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right) \tilde{w} - \kappa G A_x \left(\frac{\partial \tilde{\theta}}{\partial x} + \frac{\partial^2 \tilde{w}}{\partial x}\right) = 0
$$
\n
$$
-2\rho (e_0 a)^2 \gamma s^2 \frac{\partial \tilde{w}}{\partial x} - \kappa G \gamma \left(\tilde{\theta} + \frac{\partial \tilde{w}}{\partial x}\right) = 0
$$
\n
$$
\rho s^2 I_x \left(1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right) \tilde{\theta} - EI_x \frac{\partial^2 \tilde{\theta}}{\partial x^2} + \kappa G A_x \left(\tilde{\theta} + \frac{\partial \tilde{w}}{\partial x}\right) = 0
$$
\n
$$
- \rho (e_0 a)^2 s^2 \left(\tilde{\theta} + \frac{\partial^2 \tilde{\theta}}{\partial x} + 2 \frac{\partial^2 \tilde{\theta}}{\partial x}\right) \frac{\partial I_x}{\partial x} - E \frac{\partial I_x}{\partial x} \frac{\partial \tilde{\theta}}{\partial x} = 0
$$
\n(14) The problem under consideration is then reduce  
\n
$$
\rho s^2 I_x \left(1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right) \tilde{\theta} - EI_x \frac{\partial^2 \tilde{\theta}}{\partial x^2} + \kappa G A_x \left(\tilde{\theta} + \frac{\partial \tilde{w}}{\partial x}\right) = 0
$$
\n
$$
- \rho (e_0 a)^2 s^2 \left(\tilde{\theta} \frac{\partial}{\partial x} + 2 \frac{\partial \tilde{\theta}}{\partial x}\right) \frac{\partial I_x}{\partial x} - E \frac{\partial I_x}{\partial x} \frac{\partial \tilde{\theta}}{\partial
$$

H-L. Tang *e* at. *Journal of Mechanical Science and Technology* 28 (9) (2014) 3741-3747  
\nItures, effectively. In this section, instead we invoke the TFM and (14) together with Eqs. (15)-(17) can be rewritten in a  
\nand the PM to determine the natural frequencies.  
\n3.1 *TFM for nonlocal Timoshenko beams*  
\nWith the aid of the initial conditions Eq. (12), after perform-  
\ning Laplace transform, the governing Eqs. (8) and (9) along  
\nweib the corresponding boundary conditions (10) and (11)  
\nbecome  
\n
$$
\rho s^2 A_x \left(1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right) \vec{w} - \kappa G A_x \left(\frac{\partial \vec{\theta}}{\partial x} + \frac{\partial^2 \vec{w}}{\partial x^2}\right) = 0
$$
\n
$$
-2\rho (e_0 a)^2 s^2 \left(\frac{\partial^2}{\partial x} + 2 \frac{\partial^2}{\partial x}\right) \vec{\theta} - E I_x \frac{\partial^2 \vec{\theta}}{\partial x} + \kappa G A_x \left(\vec{\theta} + \frac{\partial^2 \vec{w}}{\partial x}\right) = 0
$$
\nand we have  
\n
$$
\rho (s_0 a)^2 s^2 \left(\frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial x}\right) \vec{\theta} - E I_x \frac{\partial^2 \vec{\theta}}{\partial x} + \kappa G A_x \left(\vec{\theta} + \frac{\partial^2 \vec{w}}{\partial x}\right) = 0
$$
\nand we have  
\n
$$
- \rho (e_0 a)^2 s^2 \left(\frac{\partial^2}{\partial x} + 2 \frac{\partial^2}{\partial x}\right) \frac{\partial^2}{\partial x} - E \frac{\partial^2 L}{\partial x} \frac{\partial \vec{\theta}}{\partial x} = 0
$$
\nand we have  
\nand  
\n
$$
s^2 = s_0 + s s_1
$$
\nand  
\n
$$
\vec{w}(0, s) = \vec{\theta}(0, s) = 0
$$
\n
$$
\vec{w}(0, s) = \vec{\theta}(0, s) = 0
$$
\n
$$
\vec{w}(0, s) = \vec{\theta} (0, s) = 0
$$
\n
$$
\vec{w}(0, s) = \vec{\theta} (0, s) = 0
$$
\n<

$$
\tilde{w}(0,s) = \theta(0,s) = 0 \tag{15}
$$

$$
\rho(e_0 a)^2 s^2 \left( \gamma \tilde{w} + A_L \frac{\partial \tilde{w}}{\partial x} \right) + \kappa G A_L \left( \tilde{\theta} + \frac{\partial \tilde{w}}{\partial x} \right) + m s^2 \tilde{w} = 0 \quad (16)
$$

$$
\rho(e_0 a)^2 s^2 \left( \tilde{\theta} \frac{\partial I_x}{\partial x} \big|_{x=L} + I_L \frac{\partial \tilde{\theta}}{\partial x} + A_L \tilde{w} \right) + EI_L \frac{\partial \tilde{\theta}}{\partial x} = 0 \tag{17}
$$

where *s* is the Laplace transform parameter.

To facilitate our treatment, we introduce the following dimensionless parameters

$$
\rho s^2 A_L \left[1 - (e_2 a)^2 \frac{v}{dx} - \frac{v}{dx} \right] \frac{v}{dx} - \kappa G A_L \left[ \frac{\partial v}{dx} + \frac{v}{dx} \right]
$$
\n
$$
-2\rho (e_2 a)^2 \frac{\partial v}{dx} - \kappa G A_L \left[ \frac{\partial v}{dx} + \frac{v}{dx} \right]
$$
\n
$$
-2\rho (e_2 a)^2 \frac{\partial v}{dx} - \kappa G \gamma \left( \hat{\theta} + \frac{\partial v}{dx} \right)
$$
\n
$$
-2\rho (e_2 a)^2 \frac{\partial^2 v}{dx^2} - \kappa G \gamma \left( \hat{\theta} + \frac{\partial v}{dx} \right)
$$
\n
$$
-2\rho (e_2 a)^2 \frac{\partial^2 v}{dx^2} - \frac{\partial^2 v}{dx^2} \frac{\partial^2 v}{dx} + \kappa G A_L \left( \hat{\theta} + \frac{\partial v}{dx} \right)
$$
\n
$$
- \rho (e_2 a)^2 s^2 \left( \hat{\theta} + \frac{\partial^2 v}{dx^2} \right) \frac{\partial l}{dx} - E \frac{\partial L}{dx} \frac{\partial \hat{\theta}}{\partial x} = 0
$$
\n(d) dimension 14 and we have  
\n
$$
- \rho (e_2 a)^2 s^2 \left( \hat{\theta} + \frac{\partial^2 v}{dx^2} \right) + \kappa G A_L \left( \hat{\theta} + \frac{\partial^2 v}{dx} \right) + m s^2 \hat{w} = 0
$$
\n
$$
y(13) = 16(1, 3, 5, 4, 5, 1)
$$
\n
$$
y(0, 3) = 0
$$
\n
$$
\rho (e_3 a)^2 s^2 \left( \hat{\rho} + \frac{\partial^2 v}{dx} \right) + \kappa G A_L \left( \hat{\theta} + \frac{\partial^2 v}{dx} \right) + m s^2 \hat{w} = 0
$$
\n
$$
\rho (e_4 a)^2 s^2 \left( \hat{\theta} + \frac{\partial^2 v}{dx} \right) + \kappa G A_L \left( \hat{\theta} + \frac{\partial^2 v}{dx} \right) + m s^2 \hat{w} = 0
$$
\n
$$
\rho (e_4 a)^2 s^2 \left( \
$$

where  $s_0$  is a reference value related to the Laplace transform parameter *s*.<br>Next, after substituting the dimensionless parameters above

into Eqs. (13)-(17) and introducing a state vector as

$$
\mathbf{\eta}(X,s) = \left[ \tilde{W}(X,s), \frac{\partial \tilde{W}(X,s)}{\partial X}, \tilde{\theta}(X,s), \frac{\partial \tilde{\theta}(X,s)}{\partial X} \right]^\mathrm{T}
$$
Considering t  
ferential equation

where the superscript T denotes matrix transpose, Eqs.  $(13)$ 

and (14) together with Eqs. (15)-(17) can be rewritten in a matrix form below, respectively

$$
\frac{\partial \mathbf{\eta}(X,s)}{\partial X} = \mathbf{F}(X,s)\mathbf{\eta}(X,s)
$$
\n(19)

$$
\mathbf{M}(s)\mathbf{\eta}(0,s) + \mathbf{N}(s)\mathbf{\eta}(1,s) = 0
$$
\n(20)

(*echnology 28 (9)* (2014) 3741-3747 3743<br>
14) together with Eqs. (15)-(17) can be rewritten in a<br>
x form below, respectively<br>  $\frac{(X,s)}{\partial X} = \mathbf{F}(X,s)\mathbf{\eta}(X,s)$  (19)<br>  $\frac{(S)\mathbf{\eta}(0,s) + \mathbf{N}(s)\mathbf{\eta}(1,s) = 0}{}(20)$ <br>  $\mathbf{F}(X,s)$ , *Technology 28 (9) (2014) 3741-3747* 3743<br>
(14) together with Eqs. (15)-(17) can be rewritten in a<br>
ix form below, respectively<br>  $\frac{\mathbf{n}(X,s)}{\partial X} = \mathbf{F}(X,s)\mathbf{n}(X,s)$  (19)<br>  $\mathbf{I}(s)\mathbf{n}(0,s) + \mathbf{N}(s)\mathbf{n}(1,s) = 0$  (20)<br> **re \mathbf{** and given in Appendix **A**. Thus the problem under consideration is then reduced to

 $\int_{x} \left(1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2} \right) \tilde{w} - \kappa G A_x \left(\frac{\partial \theta}{\partial x} + \frac{\partial^2 \tilde{w}}{\partial x^2}\right)$  solving ordinary differential Eq. (19) of first order subject to  $(13)$  the condition Eq.  $(20)$ . **M η N η** ( ) (0, ) ( ) (1, ) 0 *s s s s* + = (20) e and Technology 28 (9) (2014) 3741-3747 3743<br>
and (14) together with Eqs. (15)-(17) can be rewritten in a<br>
matrix form below, respectively<br>  $\frac{\partial \eta(X,s)}{\partial X} = \mathbf{F}(X,s)\eta(X,s)$  (19)<br>  $\mathbf{M}(s)\eta(0,s) + \mathbf{N}(s)\eta(1,s) = 0$  (20)<br>
where

#### *3.2 Perturbation solution for nonlocal Timoshenko beam*

 $\frac{\partial^2}{\partial z^2} + \kappa G A_x \left( \tilde{\theta} + \frac{\partial \tilde{w}}{\partial x} \right)$  To obtain approximate solution<br>with varying coefficients, we involve the varying coefficients, we involve the varying coefficients, we involve the varying coefficients of **x** and requencies. The state of the multiple of the state of the state of the state of the matrix form below, respectively<br> **x** and  $\cos \theta$  and  $\tilde{\theta} = EI \frac{\partial^2 \theta}{\partial x^2} + \kappa G A \left( \tilde{\theta} + \frac{\partial w}{\partial y} \right)$  To obtain approximate solutions of differential Eq. (19) with varying coefficients, we invoke the PM. To this end, a *e* and *Technology* 28 (9) (2014) 3741-3747<br>
3743<br>
and (14) together with Eqs. (15)-(17) can be rewritten in a<br>
matrix form below, respectively<br>  $\frac{\partial \eta(X,s)}{\partial X} = \mathbf{F}(X,s)\eta(X,s)$  (19)<br>  $\mathbf{M}(s)\eta(0,s) + \mathbf{N}(s)\eta(1,s) = 0$  (20)<br> and (14) together with Eqs. (15)-(17) can be rewritten in a<br>matrix form below, respectively<br> $\frac{\partial \eta(X,s)}{\partial X} = \mathbf{F}(X,s)\eta(X,s)$  (19)<br> $\mathbf{M}(s)\eta(0,s) + \mathbf{N}(s)\eta(1,s) = 0$  (20)<br>where  $\mathbf{F}(X,s)$ ,  $\mathbf{M}(s)$  and  $\mathbf{N}(s)$  are all  $4 \$  $\frac{\partial \eta(X,s)}{\partial X} = \mathbf{F}(X,s)\eta(X,s)$  (19)<br>  $M(s)\eta(0,s) + N(s)\eta(1,s) = 0$  (20)<br>
ere  $\mathbf{F}(X,s)$ ,  $M(s)$  and  $N(s)$  are all  $4 \times 4$  matrixes<br>
elgiven in Appendix **A**.<br>
Fhus the problem under consideration is then reduced to<br>
ving ordinary

$$
s^2 = s_0 + \varepsilon s_1 \tag{21}
$$

$$
\mathbf{\eta}(X,\mathbf{s}) = \mathbf{\eta}_0(X,\mathbf{s}_0) + \varepsilon \mathbf{\eta}_1(X,\mathbf{s}_0,\mathbf{s}_1) \tag{22}
$$

$$
\mathbf{F}(X,s) = \mathbf{F}_0(s_0) + \varepsilon \mathbf{F}_1(X, s_0, s_1) \tag{23}
$$

$$
\mathbf{N}(s) = \mathbf{N}_0(s_0) + \varepsilon \mathbf{N}_1(s_0, s_1) \tag{24}
$$

where  $s_i$  is a parameter related to the Laplace transform parameter *s* . For simplicity, only first order perturbation solution is used in the present study.

Substituting Eqs.  $(21)-(24)$  into Eqs.  $(19)$  and  $(20)$  and

$$
\rho s^2 A_z \left[1 - (e_0 a)^2 \frac{v}{dx} - kG A_z \left[ \frac{\partial v}{dx} + \frac{b}{dx} \right] \right]
$$
\n
$$
- 2\rho (e_0 a)^2 r s^2 \frac{\partial \tilde{v}}{\partial x} - \kappa G A_z \left[ \frac{\partial v}{dx} + \frac{b}{dx} \right]
$$
\n
$$
- 2\rho (e_0 a)^2 r s^2 \frac{\partial \tilde{v}}{\partial x} - \kappa G \sqrt{t} \left( \tilde{\theta} + \frac{\partial \tilde{w}}{\partial x} \right)
$$
\n
$$
- 2\rho (e_0 a)^2 r s^2 \left[ \frac{\partial}{\partial x} - kG \sqrt{t} \left( \tilde{\theta} + \frac{\partial \tilde{w}}{\partial x} \right) \right]
$$
\n
$$
- \rho (e_0 a)^2 s^2 \left[ \frac{\partial}{\partial x} - kG \frac{\partial}{\partial x} \right] \frac{\partial l}{\partial x} - E \frac{\partial l}{dx} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \right]
$$
\n
$$
- \rho (e_0 a)^2 s^2 \left[ \frac{\partial}{\partial x} + 2 \frac{\partial \tilde{\theta}}{\partial x} \right] \frac{\partial l}{\partial x} - E \frac{\partial l}{dx} \frac{\partial}{\partial x} \frac{\partial}{\partial x} = 0
$$
\n
$$
- \rho (e_0 a)^2 s^2 \left[ \frac{\partial}{\partial x} + 2 \frac{\partial \tilde{\theta}}{\partial x} \right] + \kappa G A_z \left[ \tilde{\theta} + \frac{\partial \tilde{w}}{\partial x} \right]
$$
\n
$$
+ \frac{\partial^2 \tilde{\theta}}{\partial x} - \frac{\partial^2 \tilde{\theta}}{\partial x} \right]
$$
\n
$$
= \rho (e_0 a)^2 s^2 \left[ \frac{\partial l}{\partial x} + l, \frac{\partial \tilde{\theta}}{\partial x} \right] + \kappa G A_z \left[ \tilde{\theta} + \frac{\partial \tilde{w}}{\partial x} \right]
$$
\n
$$
= \rho (3s) \frac{\partial l}{\partial x} - l, \frac{\partial l}{\partial x} \frac{\partial}{\partial x} \right]
$$
\n
$$
= \rho (3s) \frac{\partial l}{\partial x
$$

$$
\mathbf{F}_1(X, s_0, s_1) = \mathbf{F}_{10} + \mathbf{F}_{11} s_1 , \quad \mathbf{N}_1(s_0, s_1) = \mathbf{N}_{10} + \mathbf{N}_{11} s_1 .
$$

$$
f = \frac{\omega}{2\pi} = \frac{\sqrt{-s_0 - \varepsilon s_1}}{2\pi} \,. \tag{27}
$$

 $\left[\tilde{V}(X,s) \right]_{\tilde{\rho}(X,s)} \tilde{\rho}(\tilde{\theta}(X,s))$ <sup>T</sup> Considering the fact that Eqs. (25) and (26) are ordinary dif- $\chi$   $\chi$   $\chi$   $\chi$   $\chi$   $\chi$   $\chi$  ferential equations with constant coefficients, rather than varying coefficients, we can easily determine the solutions of Eqs. (25) and (26), respectively. That is, the solution to Eq. (25) is

$$
\det \left[ \mathbf{M} + \mathbf{N}_0(s_0) e^{\mathbf{F}_0(s_0)} \right] = 0 \,. \tag{28}
$$

$$
\mathbf{\eta}_0(X, s_0) = \mathbf{e}^{\mathbf{F}_0(s_0)X} \mathbf{V}_0 \tag{29}
$$

$$
\mathbf{\eta}_{1}(X, s_{0}, s_{1}) = e^{\mathbf{F}_{0}(s_{0})X} \left[ \int_{0}^{X} e^{-\mathbf{F}_{0}(s_{0})\xi} \mathbf{F}_{1}(\xi, s_{0}, s_{1}) \mathbf{\eta}_{0}(\xi, s_{0}) d\xi + \mathbf{A}_{0} \right].
$$
\n(30)

$$
\mathbf{\eta}_{1}(X, s_{0}, s_{1}) = e^{\mathbf{F}_{0}(s_{0})X}\mathbf{V}_{10} + s_{1}e^{\mathbf{F}_{0}(s_{0})X}\mathbf{V}_{11} + e^{\mathbf{F}_{0}(s_{0})X}\mathbf{A}_{0}
$$
(31)

(30)  
\nInserting 
$$
\mathbf{F}_1(X, s_0, s_1) = \mathbf{F}_{10} + \mathbf{F}_{11} s_1
$$
 together with Eq. (29)  
\ninto Eq. (30) leads to  
\n
$$
\mathbf{\eta}_1(X, s_0, s_1) = e^{\mathbf{F}_0(s_0)X} \mathbf{V}_{10} + s_1 e^{\mathbf{F}_0(s_0)X} \mathbf{V}_{11} + e^{\mathbf{F}_0(s_0)X} \mathbf{A}_0
$$
\n(31)  
\nwhere  
\n
$$
\mathbf{V}_{10} = \int_0^X e^{-\mathbf{F}_0(s_0) \xi} \mathbf{F}_{10} e^{\mathbf{F}_0(s_0) \xi} \mathbf{V}_0 d\xi \mathbf{V}_{11} = \int_0^X e^{-\mathbf{F}_0(s_0) \xi} \mathbf{F}_{11} e^{\mathbf{F}_0(s_0) \xi} \mathbf{V}_0 d\xi
$$
\n(32) of a

By rewriting 
$$
\mathbf{M} + \mathbf{N}_0(s_0)e^{F_0(s_0)}\mathbf{e}^{F_0(s_0)}\mathbf{V}_{11} + \mathbf{N}_{11}e^{F_0(s_0)}\mathbf{V}_0 = 0.
$$
\nBy rewriting 
$$
\mathbf{M} + \mathbf{N}_0(s_0)e^{F_0(s_0)}\mathbf{e}^{F_0(s_0)}\mathbf{V}_{10} + \mathbf{N}_{10}e^{F_0(s_0)}\mathbf{V}_0 = 0.
$$
\n
$$
\begin{aligned}\n\mathbf{B} & \mathbf{y} &= 0.34 \text{ nm}, \text{ the radius at the mass density } \rho = 2.24g/c \\
&= 1-r_1/r_0, \text{ which varies from (d) } \mathbf{A}.\mathbf{F} &= 0.\n\end{aligned}
$$
\nBy rewriting 
$$
\mathbf{M} + \mathbf{N}_0(s_0)e^{F_0(s_0)} = \left[\mathbf{P}\right]\lambda_0 \mathbf{F}^{\mathrm{T}}^{\mathrm{T}} \text{ and left null}
$$
\n
$$
\begin{aligned}\n\mathbf{B} &= 0.34 \text{ nm}, \text{ the radius at the mass density } \rho = 2.24g/c \\
&= 1-r_1/r_0, \text{ which varies from (e) } \mathbf{A}.\mathbf{F} &= 1-r_1/r_0, \text{ which varies from (f) } \mathbf{A}.\mathbf{F} &= 1-r_1/r_0, \text{ which varies from (g) } \mathbf{A}.\mathbf{F} &= 1-r_1/r_0, \text{ which varies from (h) } \mathbf{A}.\mathbf{F} &= 1-r_1/r_0, \text{ which varies from (i) } \mathbf{A}.\mathbf{F} &= 1-r_1/r_0, \text{ which varies from (i) } \mathbf{A}.\mathbf{F} &= 1-r_1/r_0, \text{ which varies from (ii) } \mathbf{A}.\mathbf{F} &= 1-r_1/r_0, \text{ which varies from (ii) } \mathbf{A}.\mathbf{F} &= 1-r_1/r_0, \text{ which varies from (iii) } \mathbf{A}.\mathbf{F} &= 1-r_1/r_0, \text{ which varies from (iv) } \mathbf{A}.\mathbf{F} &
$$

+ 
$$
\mathbf{N}_0(s_0) e^{\mathbf{F}_0(s_0)} \mathbf{V}_{10}
$$
 +  $\mathbf{N}_{10} e^{\mathbf{F}_0(s_0)} \mathbf{V}_0 = 0$ .  $\delta = 0.34$  nm,  
\n(33) otherwise state  
\n $c = 1 - r_1/r_0$ , whi  
\nBy rewriting  $\mathbf{M} + \mathbf{N}_0(s_0) e^{\mathbf{F}_0(s_0)} = [\mathbf{P} \mathbf{I} \lambda_0 \mathbf{I} \mathbf{P}]^{\text{T}}$  and left mul-  
\nplying Eq. (33) by  $[\mathbf{P}]^{\text{T}}$ , one gets  
\n
$$
[\lambda_0 \mathbf{I} \mathbf{P}]^{\text{T}} \mathbf{A}_0 + s_1 [\mathbf{P}]^{\text{T}} [\mathbf{N}_0(s_0) e^{\mathbf{F}_0(s_0)} \mathbf{V}_{11} + \mathbf{N}_{11} e^{\mathbf{F}_0(s_0)} \mathbf{V}_0]
$$
\n
$$
+ [\mathbf{P}]^{\text{T}} [\mathbf{N}_0(s_0) e^{\mathbf{F}_0(s_0)} \mathbf{V}_{10} + \mathbf{N}_{10} e^{\mathbf{F}_0(s_0)} \mathbf{V}_0] = 0
$$
\n(34) by comparing  
\nFEM software

eigenvalues and eigenvectors, respectively. Obviously, there then obtained easily by solving Eq. (34), and the first-order perturbation solution is then determined.

Table 1. Comparison of the natural frequencies (GHz) obtained from TFM & PM with those using FEM software for a SWCNT-based mass sensor with  $L = 22$  nm and  $r_0 = 0.8$  nm.

3744 H.-L. Tang et al. / Journal of Mechanical Science and Technology 28 (9) (2014) 3741~3747								
readily obtained to be		Table 1. Comparison of the natural frequencies (GHz) obtained from TFM & PM with those using FEM software for a SWCNT-based mass						
det $\left[{\bf M} + {\bf N}_0(s_0) e^{{\bf F}_0(s_0)}\right] = 0$ .	(28)	sensor with $L = 22$ nm and $r_0 = 0.8$ nm.						
		$\mathbf c$	m/g			3	5	
Hence, the zeroth-order perturbation of the circular fre-				TFM&PM	14.4238	199.0966	546.4015	
quency $\omega_0 = \sqrt{-s_0}$ , and the corresponding modal shape can be			$\overline{0}$	<b>FEM</b>	14.5396	201.1667	558.1628	
evaluated by		1/4		%Error	0.7964	1.0290	2.1071	
$\mathbf{\eta}_0(X, s_0) = e^{\mathbf{F}_0(s_0)X} \mathbf{V}_0$	(29)			TFM&PM	13.9968	193.8567	534.5559	
			$10^{-21}$	<b>FEM</b>	14.1113	195.7882	545.7362	
where $V_0$ is the eigenvector corresponding to the matrix		1/8		%Error	0.8114	0.9865	2.0487	
$\mathbf{M} + \mathbf{N}_0(s_0) e^{\mathbf{F}_0(s_0)}$ with zero as its eigenvalue.			$\overline{0}$	TFM&PM <b>FEM</b>	14.2111 14.2528	208.1089 211.1869	566.7282 582.3247	
To obtain its first-order perturbation solution, we express the solution to Eq. (26) according to TFM				%Error	0.2926	1.4575	2.6783	
				TFM&PM	13.8460	203.6253	556.4650	
			$10^{-21}$	<b>FEM</b>	13.8864	206.4349	571.3588	
$\mathbf{\eta}_{1}(X, s_{0}, s_{1}) = e^{\mathbf{F}_{0}(s_{0})X} \left  \int_{0}^{X} e^{-\mathbf{F}_{0}(s_{0})\xi} \mathbf{F}_{1}(\xi, s_{0}, s_{1}) \mathbf{\eta}_{0}(\xi, s_{0}) d\xi + \mathbf{A}_{0} \right .$				%Error	0.2909	1.3610	2.6067	
				TFM&PM	13.9952	216.7468	586.3507	
	(30)		$\overline{0}$	<b>FEM</b>	14.0169	220.6942	604.3092	
Inserting $\mathbf{F}_1(X, s_0, s_1) = \mathbf{F}_{10} + \mathbf{F}_{11} s_1$ together with Eq. (29)				%Error	0.1548	1.7886	2.9717	
into Eq. (30) leads to		$\bf{0}$		TFM&PM	13.6776	212.7577	577.7908	
			$10^{-21}$	<b>FEM</b>	13.6974	216.4300	594.5050	
$\mathbf{\eta}_{1}(X, s_{0}, s_{1}) = e^{\mathbf{F}_{0}(s_{0})X}\mathbf{V}_{10} + s_{1}e^{\mathbf{F}_{0}(s_{0})X}\mathbf{V}_{11} + e^{\mathbf{F}_{0}(s_{0})X}\mathbf{A}_{0}$	(31)			%Error	0.1446	1.6968	2.8114	
where								
		4. Numerical results and discussion In this section, to show the influences of some parameters						
$\mathbf{V}_{10} = \int_{0}^{A} e^{-\mathbf{F}_0(s_0)\xi} \mathbf{F}_{10} e^{\mathbf{F}_0(s_0)\xi} \mathbf{V}_0 d\xi \quad \mathbf{V}_{11} = \int_{0}^{A} e^{-\mathbf{F}_0(s_0)\xi} \mathbf{F}_{11} e^{\mathbf{F}_0(s_0)\xi} \mathbf{V}_0 d\xi$		on the resonance frequencies or frequencies shift, an example						
	(32)	of a nonuniform SWCNT with an attached mass is presented						
Consequently, substituting $N_1(s_0, s_1) = N_{10} + N_{11} s_1$ and Eq. $(31)$ into the second equation in Eq. $(26)$ yields			and numerical results of the resonance frequencies and fre- quency shifts are calculated. In the following computations, the material properties and geometry of a SWCNT are chosen as follows: Young's modulus $E = 1TPa$ , Poisson's ratio					
$\left({\bf M}+{\bf N}_0(s_0){\rm e}^{{\bf F}_0(s_0)}\right){\bf A}_0+s_1\Big({\bf N}_0(s_0){\rm e}^{{\bf F}_0(s_0)}{\bf V}_{11}+{\bf N}_{11}{\rm e}^{{\bf F}_0(s_0)}{\bf V}_0\Big)$		$v = 0.3$ , the shear correction coefficient $\kappa = 2(1 + v)/(4 + 3v)$ ,						
$+ \mathbf{N}_0(s_0) e^{\mathbf{F}_0(s_0)} \mathbf{V}_{10} + \mathbf{N}_{10} e^{\mathbf{F}_0(s_0)} \mathbf{V}_0 = 0$ .		the mass density $\rho = 2.24g/cm^3$ , the effective tube thickness $\delta$ = 0.34 nm, the radius at the clamped end 0.8nm unless						
	(33)	otherwise stated. For convenience, we denote the taper ratio as $c = 1 - r_1/r_0$ , which varies from 0 to 0.5 for the present study.						
By rewriting $\mathbf{M} + \mathbf{N}_0(s_0) e^{\mathbf{F}_0(s_0)} = [\mathbf{P} \mathbf{I} \lambda_0 \mathbf{I} \mathbf{P}]^{\text{-}1}$ and left multiplying Eq. (33) by $[\mathbf{P}]^{\text{-}1}$ , one gets		4.1 Result validation						
				Prior to the presentation of numerical results, let us examine				
$\left[\lambda_0\right]\mathbf{P}\right]^{-1}\mathbf{A}_0 + s_1\left[\mathbf{P}\right]^{-1}\left(\mathbf{N}_0(s_0)\mathbf{e}^{\mathbf{F}_0(s_0)}\mathbf{V}_{11} + \mathbf{N}_{11}\mathbf{e}^{\mathbf{F}_0(s_0)}\mathbf{V}_0\right)$		the accuracy and validity of the present approach. This is done						
$\mathbf{E} \mathbf{D}^{-1} [\mathbf{N}(s) e^{\mathbf{F}_0(s_0)} \mathbf{V} + \mathbf{N} e^{\mathbf{F}_0(s_0)} \mathbf{V} \mathbf{V}] = 0$	(34)	by comparing our numerical results with those obtained from						

#### **4. Numerical results and discussion**

 $=\int e^{-F_0(s_0)\xi} F_{11} e^{F_0(s_0)\xi} V_0 d\xi$  In this section, to show the influences of some parameters (32) of a nonuniform SWCNT with an attached mass is presented The Consequently, substituting N<sub>1</sub>(S<sub>i</sub>, s<sub>i</sub>, s<sub>i</sub> N<sub>1</sub>, it is seed and the consequently, substituting N<sub>1</sub>(S<sub>i</sub>, s<sub>i</sub> N<sub>1</sub> + ε<sup>R</sup><sub>1</sub>(S<sub>i</sub>, s<sub>i</sub> + R<sub>1</sub>)<sup>5</sup><br>
(30)<br>
THEM 13.9952 216.7468 86.6350<br>
Inserting F<sub>1</sub>(X<sub>1</sub>, S<sub></sub>  $\begin{aligned}\n\mathbf{F}_{10}e^{\mathbf{F}_{0}(\lambda_{0})S}\mathbf{V}_{0}d\xi & \mathbf{V}_{11} = \int_{0}^{\infty} e^{-\mathbf{F}_{0}(\lambda_{0})S}\mathbf{F}_{11}e^{\mathbf{F}_{0}(\lambda_{0})S}\mathbf{V}_{0}d\xi & \text{in this section,} \\
&\text{on the resonance} \\
&\text{on$ (32) of a nonuniform S<br>
and numerical res<br>  $\mathbf{v}_1 = \mathbf{N}_{10} + \mathbf{N}_{11}\mathbf{s}_1$  and Eq.<br>
(6) yields<br>  $\mathbf{v}_0 = \mathbf{V}_{11} + \mathbf{N}_{11}e^{\mathbf{F}_0(\mathbf{s}_0)}\mathbf{V}_0$ <br>  $\mathbf{v}_0 = 0$ .<br>  $\mathbf{v}_0 = 0.3$ , the shear<br>
the mass density<br>  $\mathbf{v}_0$ **4. Numerical results and dis**<br>  $e^{-F_0(s_0)\xi}F_{10}e^{F_0(s_0)\xi}V_0d\xi$   $V_{11} = \int_0^X e^{-F_0(s_0)\xi}F_{11}e^{F_0(s_0)\xi}V_0d\xi$  In this section, to show the ir<br>
on the resonance frequencies or<br>
uently, substituting  $N_1(s_0, s_1) = N_{10}$ (50)<br>
(50)<br>
(60)<br>
(60)<br> Eq. (30) leads to<br>
Eq. (30) leads to<br>
Eq. (30) leads to<br>  $\mathbf{F}_{10} = \int_0^x e^{-\mathbf{F}_{10}(\mathbf{x})/E_{10}E_0(\mathbf{x})/E_{11}} + e^{\mathbf{F}_{10}(\mathbf{x})/E_0} \mathbf{Y}_{11} + e^{\mathbf{F}_{10}(\mathbf{x})/E_0} \mathbf{Y}_{12} = e^{\mathbf{F}_{10}(\mathbf{x})/E_0} \mathbf{Y}_{13} + \mathbf{F}_{20}(\mathbf{x})/E_$ **s**  $s_i(s_i, s_j) = F_{i0} + F_{i1} s_i$  together with Eq. (29)<br>
Jeads to<br>  $s_j) = e^{F_{i1}(s_j) x_i} V_{i0} + s_i e^{F_{i2}(s_j) x_i} V_{i1} + e^{F_{i3}(s_j) x_i} A_0$  (31)<br>  $s_j) = e^{F_{i2}(s_j) x_i} V_{i0} + s_i e^{F_{i3}(s_j) x_i} V_{i1} + e^{F_{i3}(s_j) x_i} A_0$  (31)<br>  $s_j) = e^{F_{i3}(s_j) x_i} V_{$  $s_0 \cdot \mathbf{F}_{11} s_1$  together with Eq. (29)<br>  $s_0 \cdot \mathbf{F}_{11} s_1$  together with Eq. (29)<br>  $s_1 e^{F_0(s_0)x} V_{11} + e^{F_0(s_1)x} A_0$ <br>  $s_1 e^{F_0(s_1)x} V_{11} + e^{F_0(s_1)x} A_0$ <br>
(31)<br>  $s_0 \cdot \mathbf{F}_{12}$ <br>  $s_1 e^{F_0(s_1)x} \cdot \mathbf{F}_{11} e^{F_0(s_1)x} V_$ **EXECUTE 20 FEM FIGUAL ASSESS**<br> **FEG. (30)** leads to<br> **FEG. (30)** leads to<br> **FEG. (30)** leads to<br> **FEG. (30)** leads to<br> **FEG. (40)**  $\mathbf{F}_{10} \mathbf{F}_{10} \mathbf{F}_{10} \mathbf{F}_{11} \mathbf{F}_{10} \mathbf{F}_{10} \mathbf{F}_{11} \mathbf{F}_{10} \mathbf{F}_{11} \mathbf$ + F<sub>1</sub>, s<sub>1</sub> together with Eq. (29)<br>
0<br>  $e^{F_0(x_0)X}V_{11} + e^{F_2(x_0)X}A_0$  (31)<br>
10<sup>21</sup> TEMAPM 13.6776 212.7577 577.7908<br>  $e^{F_0(x_0)X}V_{11} + e^{F_2(x_0)X}A_0$  (31)<br>
10<sup>21</sup> TEM 13.6974 216.4300 594.5050<br>
<sup>2</sup> V<sub>11</sub> =  $\int_0^x e$ (33) otherwise stated. For convenience, we denote the taper ratio as **1**<sub>(</sub>*X*, *s*<sub>0</sub>, *s*<sub>i</sub>) =  $e^{F_x(x_0)Y}V_{10} + s_1e^{F_x(x_0)Y}V_{11} + e^{F_x(x_0)Y}A_0$ <br>
Here<br>  $V_{10} = \int_0^x e^{-F_x(x_0)Y}F_{10}e^{F_x(x_0)Y}V_{0}d\xi$ <br>  $V_{11} = \int_0^x e^{-F_x(x_0)Y}F_{11}e^{F_x(x_0)Y}V_{0}d\xi$ <br>
(32) of a nonuniform SWCNT with an at  $\sum_{k=1}^{\infty} F_{k_0}e_{k_0}x_{k_0}y_{k_0} + S_0e_{k_0}x_{k_0}y_{k_0} + \sum_{k=1}^{\infty} F_{k_0}e_{k_0}y_{k_0}$ <br>  $\sum_{k=1}^{\infty} F_{k_0}e_{k_0}y_{k_0}y_{k_0} + \sum_{k=1}^{\infty} F_{k_0}e_{k_0}y_{k_0}y_{k_0}$ <br>  $\sum_{k=1}^{\infty} F_{k_0}e_{k_0}y_{k_0}y_{k_0} + \sum_{k=1}^{\infty} F$ Where<br>
Where<br>  $V_{10} = \int_{0}^{x} e^{-E_{\xi}(x_{0})t} F_{10}e^{E_{\xi}(x_{0})t} V_{01} d\xi$ <br>  $V_{11} = \int_{0}^{x} e^{-E_{\xi}(x_{0})t} F_{11}e^{E_{\xi}(x_{0})t} V_{01} d\xi$ <br>
(32) on the resonance frequencies or for the proton of the resonance frequencies of the reson 4. **Numerical results and discussion**<br>  $\mathbf{v}_{\text{0}}(s_{\text{0}}) \cdot \mathbf{v}_{\text{0}} d\xi$ <br>  $\mathbf{v}_{\text{1}} = \int_{0}^{X} e^{-\mathbf{F}_{0}(s_{\text{0}})} \mathbf{F}_{11} e^{\mathbf{F}_{0}(s_{\text{0}})} \mathbf{V}_{0} d\xi$ . In this section, to show the influences<br>
on the resonance freq  $V_{10} = \int_0^1 e^{-K_1(x_0)K} F_{10}e^{K_1(x_0)K}V_0 d\xi$ <br>  $V_{11} = \int_0^1 e^{-K_1(x_0)K} F_{10}e^{K_1(x_0)K}V_0 d\xi$ <br>
(32) and the resonance frequencies on forencomes the entergence is only the entergence of requencies of the resonance frequ **4. Numerical results and discussion**<br>  $\mathbf{F}_{10} \mathbf{e}^{\mathbf{F}_b(x_0)} \mathbf{V}_{01} = \int_0^x e^{-\mathbf{F}_b(x_0)\cdot\mathbf{F}} \mathbf{F}_{10} e^{\mathbf{F}_b(x_0)\cdot\mathbf{F}} \mathbf{V}_{00} d\xi$ . In this section, to show the influence<br>
on the resonance frequencies or fr 4. **Numerical results and disorcionally**<br>  $\mathbf{V}_{10} = \int_{0}^{\infty} e^{-\mathbf{F}_{6}(t_{0})/2} \mathbf{F}_{10} e^{\mathbf{F}_{6}(t_{0})/2} \mathbf{V}_{0} d\xi$   $\mathbf{V}_{11} = \int_{0}^{\infty} e^{-\mathbf{F}_{6}(t_{0})/2} \mathbf{F}_{11} e^{\mathbf{F}_{6}(t_{0})/2} \mathbf{V}_{0} d\xi$ . In this section, to show <sup>25</sup>F<sub>10</sub>e<sup>F<sub>1(49</sub>)E<sub>V</sub><sub>0</sub> ( $\mathbf{v}_0 = \mathbf{V}_{\text{in}}$ ,  $\mathbf{v}_0 = \mathbf{F}_{\text{in}}$  (34) of a nonuniform SWCNT with  $s$ , substituting  $N_1(s_0, s_1) = N_{10} + N_{11}s_1$  and Eq. (34) or a nonuniform SWCNT without in Eq. (26) yields as follo</sup> **F**  $\int_0^2 \int_0^2 F_1(x_0) \cdot F_{10} e^{\int_0^2 (x_0)^2 y_0} dy$ <br> **F**  $\int_0^2 e^{-F_1(x_0) \cdot x} \cdot V_0 d\xi$ <br> **F**  $\int_0^2 e^{-F_1(x_0) \cdot x} V_0 d\xi$ <br> **F**  $\int_0^2 e^{-F_1(x_0) \cdot x} V_0 d\xi$ <br> **F**  $\int_0^2 (x_0 - x_0) \cdot V_0 d\xi$ <br> **F**  $\int_0^2 (x_0 - x_0) \cdot V_0 d\xi$ <br> **F**  $\begin{pmatrix} \n\mathbf{B}^{\mathbf{F}_{\{i,j\}}\mathbf{V}_{\{j\}} \mathbf{V}_{\{j\}} = \n\mathbf{B}^{\mathbf{F}_{\{i,j\}}\mathbf{V}_{\{j\}}} + \mathbf{N}_{\{i,j\}}\mathbf{S}^{\mathbf{F}_{\{i,j\}}\mathbf{V}_{\{j\}}} + \mathbf{N}_{\{j\}}\mathbf{S}^{\mathbf{F}_{\{i,j\}}\mathbf{V}_{\{j\}}} + \mathbf{N}_{\{j\}}\mathbf{S}^{\mathbf{F}_{\{i,j\}}\mathbf{V}_{\{j\}}} + \mathbf{N}_{\{j\$ on the resonance frequencies or frequencies shift, an example and numerical results of the resonance frequencies and frequency shifts are calculated. In the following computations, the material properties and geometry of a SWCNT are chosen as follows: Young's modulus *E* =1TPa , Poisson's ratio 0.348 1.7886 2.9717<br>
TEM&PM 13.6776 212.7577 577.7908<br>
10<sup>21</sup> TEM 13.6974 216.4300 594.5050<br>
<sup>9</sup>/6Error 0.1446 1.6968 2.8114<br> **4. Numerical results and discussion**<br>
In this section, to show the influences of some paramete  $c = 1 - r_L/r_0$ , which varies from 0 to 0.5 for the present study.

# *4.1 Result validation*

 $(N_0(s_0) e^{F_0(s_0)} V_{11} + N_{11} e^{F_0(s_0)} V_0)$  the accuracy and validity of the present approach. This is done Consequently, substituting  $N_1(s_0, s_1) = N_{10} + N_{11}s_1$  and Eq. (26) yields<br>
(31) into the second equation in Eq. (26) yields<br>  $(M + N_0(s_0)e^{F_0(s_0)})A_0 + s_1[N_0(s_0)e^{F_0(s_0)}V_1 + N_{11}e^{F_0(s_0)}V_0$ <br>  $+ N_0(s_0)e^{F_0(s_0)}V_0 + N_0(e^{F_0(s_$ Consequently, substituting  $N_1(S_0, S_1) = N_{10} + N_{11}S_0$ .<br>
(A) into the second equation in Eq. (26) yields<br>
the material properties and geometry of a SWC<br>  $N_0(S_0)e^{k_1(z_0)}V_{11} + N_{11}e^{k_2(z_0)}V_0$ <br>  $V_0 = 0.3$ , the shear cor Prior to the presentation of numerical results, let us examine by comparing our numerical results with those obtained from FEM software MSC.Nastran for the case of  $e_0a = 0$ , which are tabulated in Table 1. It can be seen that our results and the FEM simulation results are in good agreement. The maximum relative error for the fundamental frequencies is 0.8114%. This means that the present approach is suitable for analyzing cantilevered nonuniform SWCNTs with a concentrated mass at the free end.



Fig. 2. Nonlocal effect on the natural frequency for different vibration modes of a nonuniform SWCNT with  $\mu$  = 0, *L* = 22 nm and *c* = 0.25.



Fig. 3. Effects of transverse shear deformation and rotary inertia on natural frequencies for a nonuniform SWCNT with  $e_0 a/L = 0.05$ ,  $\mu = 0$ ,  $L = 12$  nm and  $r_0 = 0.8$  nm.

#### *4.2 Study of natural frequencies*

In the absence of attached mass, Fig. 2 shows the effect of the nonlocal parameter on the natural frequencies for different vibration modes, where the frequencies with the subscripts NT and CT stand for those corresponding to the nonlocal and classical TBT, respectively. From Fig. 2, the nonlocal effect is found to be more apparent with an increase of **Example 12**<br> **Example 10**<br> **Example 12**<br> **Example 10**<br> **Example 12**<br> **Example 10**<br> **Example 12** The higher the vibration order is, the more apparent the nonlocal effect is.

In addition, the effects of transverse shear deformation and rotary inertia on vibration frequencies for nonuniform SWCNTs are investigated in Fig. 3. For comparison of the natural frequencies between nonlocal Timoshenko beams (NT) with nonlocal Euler-Bernoulli beams (NE), Fig. 3 shows the variation of the frequency ratio  $f_{NT}/f_{NE}$  against taper ratio *c*, and the length of a nonuniform SWCNT is taken as  $L = 12$  nm. From Fig. 3, we find that  $f_{NT}/f_{NE}$  is always less than unity for all modes. This implies that the frequencies based on the nonlocal EBT are still overestimated, in particular for higher-order modes.

Fig. 4 illustrates the frequency ratio  $f/f_0$  versus the taper ratio *c* for three different vibration modes using the nonlocal TBT with  $\mu = 0$  and  $e_0 a / L = 0.05$ , where  $f_0$  is the corresponding value when taking  $r_0 = r_L = 0.8$  nm which means  $c =$ 0. From Fig. 4, we can see that the frequency ratio is greater



Fig. 4. Taper ratio  $c$  effect on the natural frequency ratio  $f/f_0$  of a nonuniform SWCNT with  $e_0 a / L = 0.05$ ,  $\mu = 0$ ,  $L = 12$  nm and  $r_0 = 0.8$  nm.



Fig. 5. Taper ratio *c* effect on the fundamental frequency shift of a nonuniform SWCNT-based mass sensor with  $e_0 a / L = 0.1$ ,  $L = 22$  nm and  $r_0 = 0.8$  nm.



Fig. 6. Length-to-diameter ratio effect on the fundamental frequency shift of a nonuniform SWCNT-based mass sensor with  $e_0 a / L = 0.1$ , and  $d = r_0 + r_L = 1.2$  nm.

than unity and increases with taper ratio increasing for fundamental frequencies, while this trend is opposite for the higher vibration modes. This phenomenon indicates that the nonuniform characteristic can increase fundamental frequencies and reduce higher-order natural frequencies.

# *4.3 Study of frequency shift*

Of much interest is the change in the natural frequencies due to an attached mass. For this purpose, we denote the difference between the natural frequencies with and without attached mass as the frequency shift  $\Delta f$ . Using the nonlocal TBT, Figs. 5 and 6 show the variation of the frequency shift

for a nonuniform SWCNT with different taper ratios and length-to-diameter ratios, respectively. It is again viewed that the attached nanoparticle mass increases the frequency shift. From Fig. 5, the frequency shift becomes larger if the taper ratio increases. On the other hand, from Fig. 6 the influence of the length-to-diameter ratio on the frequency shift of shorter nonuniform SWCNTs is more sensitive.

# **5. Conclusions**

In this contribution, the vibration of a nonuniform SWCNTbased mass sensor was studied using the nonlocal TBT. The natural frequencies were determined by the TFM together with the PM. The accuracy and validity of numerical results were verified by a comparison with the corresponding FEM results. The conclusions are given as follows:

The nonlocal effect on the natural frequencies is more apparent with the vibration modes increasing. The taper ratio strongly affects natural frequencies. It increases fundamental frequency and decreases higher order natural frequencies.

Increasing the attached mass or decreasing the length-todiameter ratio increases the frequency shift. The SWCNTbased mass sensor is more sensitive to the frequency shift if the taper ratio becomes larger.

## **Acknowledgment**

We acknowledge the financial supports of the National Natural Science Foundation of China(No.11302254).

# **References**

- [1] S. Iijima, Helical microtubules of graphitic carbon, *Nature,* 354 (1991) 56-58.
- [2] E. W. Wong, P. E. Sheehan and C. M. Lieber, Nanobeam mechanics: elasticity, strength, and toughness of nanorods and nanotubes, *Science,* 277 (1997) 1971-1975.
- [3] Q. Zheng and Q. Jiang, Multiwalled carbon nanotubes as gigahertz oscillators, *Physical Review Letters,* 88 (2002) 045503.
- [4] C. Y. Li and T. W. Chou, Atomistic modeling of carbon nanotube-based mechanical sensors, *Journal of Intelligent Material Systems and Structures,* 17 (2006) 247-254.
- [5] R. Chowdhury, S. Adhikari and J. Mitchell, Vibrating 104-109.
- [6] X. Wu, Y. Tao, C. Mao, L. Wen and J. Zhu, Synthesis of nitrogen-doped horn-shaped carbon nanotubes by reduction of pentachloropyridine with metallic sodium, *Carbon,* 45 (2007) 2253-2259.
- [7] S. Y. Sawant, R. S. Somani and H. C. Bajaj, A solvothermalreduction method for the production of horn shaped multiwall carbon nanotubes, *Carbon,* 48 (2010) 668-672.
- [8] H. C. Su, C. M. Lin, S. J. Yen, Y. C. Chen, C. H. Chen, S. R. Yeh, W. Fang, H. Chen, D. J. Yao, Y. C. Chang and T. R.

Yew, A cone-shaped 3D carbon nanotube probe for neural recording, *Biosensors and Bioelectronics,* 26 (2010) 220-227.

- [9] P. Soltani, D. D. Ganji, I. Mehdipour and A. Farshidianfar, Nonlinear vibration and rippling instability for embedded carbon nanotubes, *Journal of Mechanical Science and Technology,* 26 (2012) 985-992.
- [10] M. H. Kim, S. Seo, W. K. Liu, B. S. Lim, J. B. Choi and M. K. Kim, A modal analysis of carbon nanotube using elastic network model, *Journal of Mechanical Science and Technology,* 26 (2012) 3433-3438.
- [11] P. Poncharal, Z. L. Wang, D. Ugarte and W. A. D. Heer, Electrostatic deflections and electro-mechanical resonances of carbon nanotubes, *Science,* 283 (1999) 1513-1516.
- [12] A. C. Eringen, *Nonlocal Continuum Field Theories*, Springer, New York (2002).
- [13] Z. B. Shen, G. J. Tang, L. Zhang and G. J. Tang, Vibration of double-walled carbon nanotube based nanomechanical sensor with initial axial stress, *Computational Materials Science,* 58 (2012) 51-58.
- [14] K. Yun, J. Choi, S. K. Kim and O. Song, Flow-induced vibration and stability analysis of multi-wall carbon nanotubes, *Journal of Mechanical Science and Technology,* 26 (2012) 3911-3920.
- [15] Z. B. Shen, D. K. Li, D. Li and G. J. Tang, Frequency shift of a nanomechanical sensor carrying a nanoparticle using nonlocal Timoshenko beam theory, *Journal of Mechanical Science and Technology,* 26 (2012) 1577-1583.
- [16] H. L. Lee and W. J. Chang, Surface and small-scale effects on vibration analysis of a nonuniform nanocantilever beam, *Physica E,* 43 (2010) 466-469.
- [17] T. Murmu and S. C. Pradhan, Small-scale effect on the vibration of nonuniform nanocantilever based on nonlocal elasticity theory, *Physica E,* 41 (2009) 1451-1456.
- [18] H. L. Tang, D. K. Li and S. M. Zhou, Vibration of hornshaped carbon nanotube with attached mass via nonlocal elasticity theory, *Physica E,* 56 (2014) 306-311.
- [19] B. Yang and C. A. Tan, Transfer function of one-dimension distributed parameter system, *ASME Journal of Applied Mechanics,* 59 (1992) 1009-1014.

# **Appendix**

# **A.1**

carbon nanotube based bio-sensors, *Physica E,* 42 (2009) In Eqs. (19) and (20), the elements of the matrices  $F(X,s)$ ,

of double-wailed carbon nanoube based nanomenanical  
sensor with initial axial stress, *Computational Materials*  
*Science*, 58 (2012) 51-58.  
[14] K. Yun, J. Choi, S. K. Kim and O. Song, Flow-induced  
vibration and stability analysis of multi-wall carbon  
manotubes, *Journal of Mechanical Science and Technology*,  
26 (2012) 3911-3920.  
[15] Z. B. Shen, D. K. Li, D. Li and G. J. Tang, Frequency shift  
of a nanomechanical sensor carrying a nanoparticle using  
nonlocal Timoshenko beam theory, *Journal of Mechanical*  
Storice and *Technology*, 26 (2012) 1577-1583.  
[16] H. L. Lee and W. J. Chang, Surface and small-scale effects  
on vibration analysis of a nonuniform anaocantilever beam,  
*Physical E*, 43 (2010) 466-469.  
[17] T. Mumu and S. C. Pradhan, Small-scale effect on the  
vibration of nonuniform anaocantilever based on nonlocal  
elasticity theory, *Physical E*, 54 (2014) 306-311.  
[18] H. L. Tang, D. K. Li and S. M. Zhou, Vibration of hom-  
shaped carbon nanotube with attached mass via nonlocal  
elasticity theory, *Physical E*, 56 (2014) 306-311.  
[19] B. Yang and C. A. Tan, Transfer function of one-dimensional  
distributot parameter system, *ASME Journal of Applied*  
*Mechanics*, 59 (1992) 1009-1014.  
**Appendix**  
**A.1**  
In Eqs. (19) and (20), the elements of the matrices 
$$
F(X,s)
$$
,  
 $F_{12} = F_{34} = 1$ ,  $F_{21} = \Gamma_s / (\lambda^2 \Gamma_s + \beta_s)$ ,  $F_{22} = -\beta_s / (\lambda^2 \Gamma_s + \beta_s)$ ,  
 $F_{23} = -(\frac{2\lambda^2 \Gamma_p + \beta_s}{\lambda^2 \Gamma_s + \beta_s})$ ,  $F_{24} = \beta_s / (\lambda^2 \Psi_s + 1)$ ,  
 $F_{44} = -I'_x L (2\lambda^2 \Psi_s + 1) / I_x / (\lambda^2 \Psi_s + 1)$ ,  
 $F_{44} = -I'_x L (2\lambda^2 \Psi_s +$ 

$$
H.-L. Tang et al. /Journal of Mechanical Science and Technology 28 (9) (2014) 3741~3
$$
\n
$$
\mathbf{N}_{32} = 1 + \lambda^2 \Gamma_{sL} / \beta_L, \quad \mathbf{N}_{33} = 1, \quad \mathbf{N}_{41} = \lambda^2 \Gamma_{sL},
$$
\n
$$
\mathbf{N}_{43} = \lambda^2 \rho I_x' L^3 s^2 / EI_L, \quad \mathbf{N}_{44} = \lambda^2 \rho L^2 s^2 / E + 1.
$$
\n
$$
\mathbf{N}_{10} + \mathbf{N}_{11} s_1 \quad \text{vanish unless those matrix } \mathbf{N}_{10} \text{ the corresponding}
$$
\n
$$
\mathbf{N}_{31} = 2\lambda^2 L^5 \rho \delta \pi s_0 / \beta_L EI_L, \quad \mathbf{N}_{11} \text{ the corresponding}
$$
\n
$$
\mathbf{N}_{12} = 2\lambda^2 L^5 \rho \delta \pi s_0 / \beta_L EI_L, \quad \mathbf{N}_{12} = 2\lambda^2 L^5 \rho \delta \pi s_0 / \beta_L EI_L, \quad \mathbf{N}_{13} = 2\lambda^2 L^5 \rho \delta \pi s_0 / \beta_L EI_L, \quad \mathbf{N}_{14} = 2\lambda^2 L^5 \rho \delta \pi s_0 / \beta_L EI_L.
$$

# **A.2**

$$
N_n = 1 + \lambda^2 \Gamma_{\alpha} / \beta_{\beta}
$$
,  $N_n = 1$ ,  $Y_n = \lambda^2 p'_i / \lambda^2 / \epsilon N_i$ ,  $N_n = \lambda^2 p'_i / \lambda^2 / \epsilon N_i$ ,  $N_n = \lambda^2 p'_i / \lambda^2 / \epsilon N_i$ ,  $N_n = \lambda^2 p'_i / \lambda^2 N_i$ ,  $N_n = \lambda^2 N_i / \lambda^2 N_i$ ,  $N_n = \lambda^2$ 

below. For matrix  $\mathbf{F}_{10}$  the corresponding elements are

$$
(X, s_0, s_1) = \mathbf{F}_{10} + \mathbf{F}_{11} s_1
$$
 vanish unless those that are displayed  
low. For matrix  $\mathbf{F}_{10}$  the corresponding elements are  

$$
\mathbf{F}_{22} = -\left(2\lambda^2 \chi_L \Gamma_0 + \beta_0 \chi_L\right) / \left(\lambda^2 \Gamma_0 + \beta_0\right),
$$

$$
\mathbf{F}_{23} = -\beta_0 \chi_L / \left(\lambda^2 \Gamma_0 + \beta_0 \chi_L\right) / \left(\lambda^2 \Gamma_0 + \beta_0\right),
$$

$$
\mathbf{F}_{31} = -\beta_0 \phi_x / \left(\lambda^2 \Gamma_0 + \beta_0\right),
$$

$$
\mathbf{F}_{42} = -\beta_0 \phi_x / \left(\lambda^2 \Gamma_0 \alpha_0 + 1\right),
$$

$$
\mathbf{F}_{43} = -\beta_0 \phi_x / \left(\lambda^2 \Gamma_0 \alpha_0 + 1\right),
$$

$$
\mathbf{F}_{44} = -4\Gamma_0 \left(2\lambda^2 \Gamma_0 \alpha_0 + 1\right) / \left(\lambda^2 \Gamma_0 \alpha_0 + 1\right).
$$
For matrix  $\mathbf{F}_{11}$  the corresponding elements are  
for matrix  $\mathbf{F}_{11}$  the corresponding elements are  

$$
\mathbf{F}_{21} = \Gamma_0 \beta_0 / \left(\Gamma_0 \lambda^2 + \beta_0\right)^2 / s_0,
$$

$$
\mathbf{F}_{24} = \lambda^2 \beta_0 \Gamma_0 / \left(\lambda^2 \Gamma_0 + \beta_0\right)^2 / s_0,
$$

$$
\mathbf{F}_{24} = \lambda^2 \beta_0 \Gamma_0 / \left(\lambda^2 \Gamma_0 + \beta_0\right)^2 / s_0,
$$

$$
\mathbf{F}_{34} = \left(1 - \lambda^2 \beta_0\right) \rho L^2 / \left(\lambda^2 \Gamma_0 \alpha_0 + 1\right)^2 / E,
$$

$$
\mathbf{F}_{43} = \left(1 - \lambda^2 \beta_0\right) \rho L^2 / \left(\lambda^2 \Gamma_0 \alpha_0 + 1\right)^2 / E.
$$

$$
\mathbf{F}_{43} = -\beta_0 \phi_x / (\lambda^2 \Gamma_0 \alpha_0 + 1),
$$
\n
$$
\mathbf{F}_{44} = -4 \Gamma_0 \left( 2\lambda^2 \Gamma_0 \alpha_0 + 1 \right) / (\lambda^2 \Gamma_0 \alpha_0 + 1).
$$
\nFor matrix  $\mathbf{F}_{11}$  the corresponding elements are\n
$$
\mathbf{F}_{21} = \Gamma_0 \beta_0 / (\Gamma_0 \lambda^2 + \beta_0)^2 / s_0,
$$
\n
$$
\mathbf{F}_{24} = \lambda^2 \beta_0 \Gamma_0 / (\lambda^2 \Gamma_0 + \beta_0)^2 / s_0,
$$
\n
$$
\mathbf{F}_{42} = -\lambda^2 \beta_0 \rho L^2 / (\lambda^2 \Gamma_0 \alpha_0 + 1)^2 / E,
$$
\n
$$
\mathbf{F}_{43} = (1 - \lambda^2 \beta_0) \rho L^2 / (\lambda^2 \Gamma_0 \alpha_0 + 1)^2 / E.
$$

*H.-L. Tang et al. / Journal of Mechanical Science and Technology 28 (9) (2014) 3741-3747*<br>  $\mathbf{N}_{32} = 1 + \lambda^2 \Gamma_{st} / \beta_L$ ,  $\mathbf{N}_{33} = 1$ ,  $\mathbf{N}_{41} = \lambda^2 \Gamma_{st}$ ,  $\mathbf{N}_{10} + \mathbf{N}_{15}$  (26), the elements of the matrices  $\math$ *H.-L. Tang et al. / Journal of Mechanical Science and Technology 28 (9) (2014) 3741-3747*<br>  $N_{32} = 1 + \lambda^2 \Gamma_{st} / \beta_L$ ,  $N_{33} = 1$ ,  $N_{41} = \lambda^2 \Gamma_{st}$ ,<br>  $N_{43} = \lambda^2 \rho I'_s L^3 s^2 / E I_L$ ,  $N_{44} = \lambda^2 \rho L^2 s^2 / E + 1$ .<br> **Appendix**<br> **A** *nd Technology 28 (9) (2014) 3741~3747* 3747<br>
In Eq. (26), the elements of the matrices  $N_1(s_0, s_1) =$ <br>  $N_1 s_1$  vanish unless those that are displayed below. For<br>
atrix  $N_{10}$  the corresponding elements are<br>  $N_{31} = 2\lambda$ and Technology 28 (9) (2014) 3741-3747 3747<br>
In Eq. (26), the elements of the matrices  $N_1(s_0, s_1) = N_{10} + N_{11} s_1$  vanish unless those that are displayed below. For<br>
matrix  $N_{10}$  the corresponding elements are<br>  $N_{31} =$ *e and Technology 28 (9) (2014) 3741~3747* 3<br>
In Eq. (26), the elements of the matrices  $N_1(s_0, s_1, \mathbf{N}_{10} + \mathbf{N}_{11} s_1$  vanish unless those that are displayed below.<br>
matrix  $N_{10}$  the corresponding elements are<br>  $N_{$ d Technology 28 (9) (2014) 3741~3747 3747<br>
in Eq. (26), the elements of the matrices  $N_1(s_0, s_1) =$ <br>  ${}_{0} + N_{11}s_1$  vanish unless those that are displayed below. For<br>
trix  $N_{10}$  the corresponding elements are<br>  $N_{31} = 2$ 3747<br>
of the matrices  $N_1(s_0, s_1) =$ <br>
e that are displayed below. For<br>
elements are<br>  $N_{43} = 4\lambda^2 L^2 \rho T_L s_0 / E$ .<br>
anding elements are<br>  $= \lambda^2 \rho A_L L^4 / \beta_L E I_L$ ,<br>  $\lambda^2 \rho L^2 / E$ . *d* Technology 28 (9) (2014) 3741-3747 3747<br>
in Eq. (26), the elements of the matrices  $N_1(S_0, S_1) =$ <br>  $\phi + N_{11}S_1$  vanish unless those that are displayed below. For<br>
trix  $N_{10}$  the corresponding elements are<br>  $N_{31} =$ *logy 28 (9) (2014) 3741-3747* 3747<br>
(26), the elements of the matrices  $N_1(s_0, s_1) =$ <br>
1 vanish unless those that are displayed below. For<br>
4<sup>2</sup> $L^5 \rho \delta \pi s_0 / \beta_L E I_L$ ,  $N_{43} = 4\lambda^2 L^2 \rho T_L s_0 / E$ .<br>
Trix  $N_{11}$  the corresp d Technology 28 (9) (2014) 3741-3747<br>
in Eq. (26), the elements of the matrices  $N_1(s_0, s_1) =$ <br>  ${}_{0} + N_{11} s_1$  vanish unless those that are displayed below. For<br>
trix  $N_{10}$  the corresponding elements are<br>  $N_{31} = 2\lambda^2$ 

$$
\mathbf{N}_{31} = 2\lambda^2 L^5 \rho \delta \pi s_0 / \beta_L E I_L , \ \mathbf{N}_{43} = 4\lambda^2 L^2 \rho T_L s_0 / E.
$$

For matrix  $N_{11}$  the corresponding elements are

$$
\mathbf{N}_{31} = \mu \rho A_L L^4 / \beta_L E I_L , \quad \mathbf{N}_{32} = \lambda^2 \rho A_L L^4 / \beta_L E I_L ,
$$
  

$$
\mathbf{N}_{41} = \lambda^2 \rho A_L L^4 / E I_L , \quad \mathbf{N}_{44} = \lambda^2 \rho L^2 / E .
$$



*H*. *L. Tang et al. / Journal of Mechanical Science and Technology 28 (9) (2014) 3741-3747<br>*  $\mathcal{B}_l$ *,*  $\mathbf{N}_{12} = 1$ *,*  $\mathbf{N}_{13} = \lambda^2 \Gamma_{24}$ *,*  $\mathbf{N}_{14} = \lambda^2 \rho L^2 s^2 / E + 1$ *.<br>*  $\mathbf{N}_{11} = \lambda^2 \rho L^2 s^2 / E + 1$ *.<br> \mathbf{N}\_{12} = \lambda* T.4, Eag et al. Zaurand of Moduntari Science and Technology 28 (b) (2613 214-1347 19<br>  $x^2/b$ ,  $N_1 = 1$ ,  $N_1 = \lambda^2 b^2 b^2 c^2 f = 1$ .<br>  $N_1 = 2\lambda^2 b^2 c^2 f = 1$ .<br>  $N_2 = 2\lambda^2 c^2 b^2 \cos t$  (b)  $B_x = N_1 = \lambda^2 b^2 c^2 f = 1$ .<br>  $N_1 = 2\lambda^2 b^2$ *d.d. Tange a d. / bound of Neckannia Science and Technology 25 (9) (2014) 574-7577*<br>  $\vec{r} = -\vec{k}$ ,  $\vec{k}$ , of the matrices  $\mathbf{F}_6(s_0)$  and<br>  $\mathbf{N}_{31} = \mu \rho A_L L^4 / \beta_L E I_L$ ,  $\mathbf{N}_{32} = \lambda^2 \rho A_L L^4 / \beta_L E I_L$ ,<br>  $\mathbf{N}_{41} = \lambda^2 \rho A_L L^4 / E I_L$ ,  $\mathbf{N}_{41} = \lambda^2 \rho L^4 / E L$ ,  $\mathbf{N}_{41} = \lambda^2 \rho L^2 / E$ .<br>  $\mathbf{F}_0 + \beta_0$ ,  $\mathbf{N}_{41} = \lambda^2 \rho A_L L^4 / E I_L$ *H*. *C*, T<sub>an</sub> *a L A*<sub>ma</sub> *a L A*<sub>ma</sub> *d* **L** *A*<sub>m</sub> *a L C*<sub>*n*</sub> **2** *A*<sub>m</sub> *a C*<sub>*n*</sub> *A*<sub>m</sub> *A C*<sub>*n*</sub> *A*<sub>m</sub> *A C*<sub>*n*</sub> *A*<sub>m</sub> (14) 3741-3747<br>
elements of the matrices  $N_1(s_0, s_1)$  =<br>
elless those that are displayed below. For<br>
sponding elements are<br>  $\beta_L EI_L$ ,  $N_{43} = 4\lambda^2 L^2 \rho T_L s_0 / E$ .<br>
e corresponding elements are<br>  $T_L$ ,  $N_{32} = \lambda^2 \rho A_L L^4 / \beta_L$ from the National University of Defense Technology (NUDT), China, in 2013. Mr. Li is currently an engineer in the College of Aerospace Science and Engineering, NUDT. His research interests include vibration analyses of carbon nanotubes and Graphene sheets.



**Daokui Li** received his Ph.D. in solid mechanics from the National University of Defense Technology (NUDT), China, in 2002. From 2011 to 2012, he worked as a visiting scholar at the School of Engineering, University of Glasgow in the U.K. Dr. Li is currently a professor at the College of Aerospace Science and

Engineering, NUDT. His research interests include composite structural mechanics and computational solid mechanics.