Springer

KSME

Journal of Mechanical Science and Technology 26 (11) (2012) 3555~3563
www.springerlink.com/content/1738-494x
DOI 10.1007/s12206-012-0871-z

Static analysis of nanobeams including surface effects by nonlocal finite element †

F. F. Mahmoud[1,2], M. A. Eltaher[1,*], A. E. Alshorbagy[1] and E. I. Meletis[2]

[1]*Department of Mechanical Design and Production Engineering, Zagazig University, Zagazig, Egypt*
[2]*Department of Material Science and Engineering, University of Texas at Arlington, USA*

(Manuscript Received February 12, 2012; Revised May 13, 2012; Accepted June 15, 2012)

Abstract

In reality, there are two phenomena should be considered to describe behaviors of nanostructures adequately and accurately. The first one is the surface properties, especially for a relatively high ratio of the surface area to the volume of structural. The second phenomenon is the information about bulk material, which contains the forces between atoms and the internal length scale. Therefore, the objective of the current work is to study the coupled effects of surface properties and nonlocal elasticity on the static deflection of nanobeams. Surface elasticity is employed to describe the behavior of the surface layer and the Euler-Bernoulli beam hypothesis is used to state the bulk deformation kinematics. Both, the surface layer and bulk volume of the beam are assumed elastically isotropic. Information about the forces between atoms, and the internal length scale are proposed by the nonlocal Eringen model. Galerkin finite element technique is employed for the discretization of the nonlocal mathematical model with surface properties. The present results are compared favorably with those published results. The effects of nonlocal parameter and surface elastic constants are figured out and presented.

Keywords: Euler-Bernoulli nanobeam; Nonlocal elasticity; Nonlocal finite element; Static analysis; Surface effects

---

1. Introduction

Recently, there has been significant interest in developing of nanomechanical and nanoelectromechanical systems (NEMS), which can be contributed to industrial revolution. Extremely small size of nanostructures such as beams, sheets and plates in nanosize, which are commonly used as components in NEMS devices, presents a significant challenge to researchers in nanomechanics, Li and Chou [1].

Nowadays, it is still a challenge to study the mechanics of nanomaterials by means of experimental tests due to the difficulties encountered on the nanoscale. Therefore, the theoretical methods such as atomistic simulations, multiscale computational models, and continuum mechanics theories are often used to analyze the behaviors of nanostructures, Hsu et al. [2]. It is known that using of an atomistic simulation method is an extremely time-consuming task and computationally intensive for relatively large scale nanostructures, Lee and Chang [3]. Multiscale computational models, based on atomistic /continuum coupling, have been recently developed for studying properties of nanomaterials. However, these methods are incapable of capturing atomic-scale surface stress effects; furthermore, the inclusion of thermal effects in multiscale modeling of nanomaterials, Yun and Park [4].

However, the continuum models have been proven to be important and efficient tools in the study of the nanostructures. Classical continuum mechanics is explicitly designed to be size-independent, Truesdell and Noll [5], which may call the applicability of classical continuum models on nanostructures into question. Several physical reasons may be ascribed to the breakdown of classical continuum mechanics, Maranganti and Sharma [6]. Among those reasons are the surface effects at nanoscale size and the discrete nature of the matter, as will be discussed briefly in the following:

Surface effect: On the basis of surface elasticity, the effect of surface energies, strains and stresses on the size-dependent elastic behavior of structural elements have been investigated experimentally. The results have shown that surface effects become important and induce a significant size dependency, which pointed out the limitation of the applicability of classical continuum model in nanotechnology. The surface of a solid is a region with negligible thickness which has its own atom arrangement and property differing from the bulk. For a solid with a large size, the surface effects can be ignored because the volume ratio of the surface region to the bulk is very small. However, for minute solids with large surface-to-bulk

*Corresponding author. Tel.: +2 01001089561
E-mail address: mmeltaher@zu.edu.eg
† Recommended by Associate Editor Jun-Sik Kim

ratio, the significance of surface is important.

Recently, mechanical experiments of nanoscale beams and plates indicate that the effective strength properties of these minute structural elements strongly depend on their size, Miller and Shenoy [7]. Classical elasticity lacks an intrinsic length scale, and thus cannot be used to model the size effect and cannot account for the significant surface contribution of those minute structural elements.

Gurtin and Murdoch [8, 9] developed a surface elasticity theory for isotropic materials based on some rational principles of mechanics. In their model, the surface layer of a solid is treated as a membrane with negligible thickness and perfectly bonded to the underlying bulk material. Miller and Shenoy [7] developed a simple and effective model to predict the size dependency of the effective stiffness properties of nanosized as plates and beams structural element. This model based on the general continuum formulation including surface effects as developed by Gurtin and Murdoch. Shenoy [10] predicted the size dependent torsional rigidities of nanosized structural elements, caused by surface effects.

Lim and He [11] investigated the surface effects on the large deflection of ultra-thin films by incorporating surface elasticity into the Von Karman plate theory using the Hamilton's principle. The model is applied to analyze the bending, buckling and vibration of simply supported micro- and nano-films in plane strains. Chen et al. [12] experimentally measured the Young's modulus of ZnO nanowire and found that the surface effects on the elastic property of the nanowire are significant. Lu et al. [13] improved Gurtin model by introducing normal stress, inside and on the surface of bulk substrate, to satisfy the equilibrium balance relations at the surface. Yun and Park [4] presented a multiscale, finite deformation formulation for the thermoelastic analysis of nanomaterials including surface stress effects on the dynamic, thermoelastic behavior of 1D nanostructure. Fu et al. [14] investigated the effects of the surface energies on the critical buckling, post-buckling and linear free vibration of nanobeams with size dependency. Assadi and Farshi [15] studied the Size dependent vibration of curved nanobeams and rings including surface energies. Yun and Park [16] developed nonlinear multiscale finite element techniques which account for nanoscale surface stress and surface elastic effects to investigate the elastic properties of silicon nanowires as obtained through bending deformation.

Discrete nature of matter: For realistic designing of nanodevices and nanostructures one must incorporate the small-scale effects and the atomic forces in the analysis of nano-components. The discrete nature of matter is usually associated with the long-range character of inter-atomic forces and may induce a nonlocal behavior which is in contradiction to the postulated local character of classical elasticity, Truesdell and Noll [5]. One promising theory which contains information about the forces between atoms and the internal length scale is the nonlocal elasticity theory developed by Eringen.

Linear theory of nonlocal elasticity, which has been proposed independently by many authors, Kroner [17], Edelen et al. [18], and Eringen [19-22] incorporates many important features of the lattice dynamics. Therefore, it has been proved that nonlocal elasticity theory is consistent with the molecular dynamics. The theory led to the classical elasticity, at macro-size limit and, therefore, the theory is capable of addressing small as well as large scale ratio phenomena. In the nonlocal elasticity theory, the stress state at a given point is regarded as being determined by the strain state of all points in the body; while the constitutive equations of classical elasticity is an algebraic relation between the stress and strain tensors only at the current location. In addition, the internal length scale is introduced into the constitutive equations as a material parameter.

Applications of nonlocal continuum mechanics have been demonstrated in the areas of lattice dispersion of elastic waves, fracture mechanics, dislocation mechanics, wave propagation in composites, Peddieson et al. [23]. The application of nonlocal elasticity to investigate the mechanical behavior of Euler–Bernoulli beam in micro- and nano-size has received, recently, much attention among the nanotechnology community, McFarland and Colton [24] and Wang and Liew [25]. Reddy [26] reformulated the local elasticity beam theory in the context of Eringen's nonlocal elasticity model, where the material constitutive relations are given in a differential form, to study bending, vibration and buckling behavior of nanobeams. Aydogdu [27] proposed a generalized nonlocal beam theory to study bending, buckling, and free vibration of Euler-Bernoulli nanobeams. Kong et al. [28, 29] solved analytically the static and dynamic problems of Bernoulli–Euler nanobeams on the basis of strain gradient elasticity theory.

Pisano et al. [30, 31] developed a nonlocal finite element model to study the homogeneous and non-homogeneous two-dimensional nonlocal elasticity problems. Pradhan and Murmu [32] and Civalek and Demir [33] developed a nonlocal beam models and employ the differential quadrature method (DQM) as a solver, to investigate the bending–vibration characteristics of a nano-size cantilever. Phadikar and Pradhan [34] presented finite element formulations for nonlocal elastic Euler–Bernoulli beam and Kirchhoff plate, based on the differential constitutive relations of Eringen. Mahmoud and Meletis [35] developed a nonlocal finite element model for solving the elasto-static frictional contact problems of nanostructures and nano-size devices. Xia et al. [36] exploited DQM method to study the static bending, post-buckling, and free vibration of nonlocal micro-beams. Eltaher et al. [37] presented a free vibration analysis of functionally graded (FG) size-dependent nanobeams using finite element method. The size-dependent FG nanobeam is investigated on the basis of the nonlocal continuum model of Eringen.

Recently, much research has been focused on exploring the combined effect of long–range interactions and surface properties of a nanoscale structures. Lee and Change [38] investigated analytically the effect of the surface properties on natural frequencies of nonlocal Timoshenko beam. Lee and

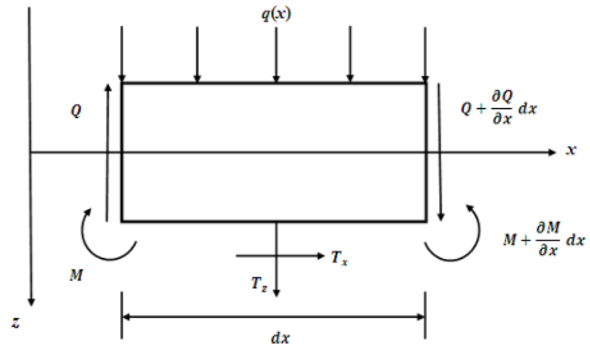


Fig. 1. Free-body diagram of nanobeam element.

Change [3] studied the natural frequency of a nonuniform nanocantilever beam with consideration of surface effects by using the nonlocal elastic theory. Lee and Change [39] used Rayleigh–Ritz method to analyze the influences of surface and nanolocal effects on the critical buckling load of the nonuniform nanowire. Lei et al. [40] used an analytical procedure based on the nonlocal Timoshenko beam mode to investigate the vibrational behavior of double-walled carbon nanotubes adhered by surface materials. These works are based on analytical solutions. Mahmoud et al. [41] presented a nonlocal continuum model of the nanoscale beams incorporated by surface effect to investigate the free vibration characteristics of nanobeams by using a finite element method.

In the present work a nonlocal finite element model is developed to study the static bending behavior of nanobeams, taking into account the surface effects. Euler-Bernoulli beam theory, incorporated with nonlocal differential constitutive relations of Eringen and surface constitutive relations of Gurtin and Murdoch, is used to derive the nonlocal equilibrium equations. The mathematical model is presented in detail in section 2. The details of nonlocal Galerkin finite element model for bending of the Euler-Bernoulli nanobeam, including surface effects, are presented in section 3. Section 4 contains numerical results and parametric studies. The results of the present model are addressed and compared with that previously published before. Concluding remarks are finally given in section 5.

2. Mathematical formulation

2.1 Local Euler-Bernoulli beam theory

The beam, based on Gurtin and Murdoch continuum model [8, 9], is considered to have an elastic surface of mathematically zero thickness and perfectly bonded to its bulk material. The elastic surface has different material properties and accounts for the surface energy effects. Assume we have a nanobeam with a rectangular cross section and consider the free-body diagram of an incremental beam element of length dx as shown in Fig. 1.

As a result of the interaction between the surface layer and bulk material, the contact tractions T_x and T_z exist on the contact surface between the bulk material and the surface layer. The bending moment and shear force of a bulk material cross section are denoted by M and Q, respectively.

The vertical force and bending moment equilibrium equations of the element $\boldsymbol{dx}$ can be expressed as,

$$\frac{\partial Q}{\partial x} + \int_S T_Z \, ds + q(x) = 0 \tag{1a}$$

$$\frac{\partial M}{\partial x} - Q + \int_S T_x z \, ds = 0 \tag{1b}$$

where S is the perimeter of the cross section and $q(x)$ denotes the magnitude of distributed vertical load on the beam. Differentiating Eq. (1b) and substituting it into Eq. (1a), to eliminate the shear force Q, we can get the following equation:

$$\frac{\partial^2 M^l}{\partial x^2} + \frac{\partial}{\partial x}\left(\int_S T_x z \, ds \right) + \int_S T_z \, ds + q(x) = 0 \,. \tag{2}$$

In the case of Euler-Bernoulli beam, the stress state of the bulk material is plane stress with stresses σ_{xx}, σ_{xz}, and σ_{zz}. The elastic surface of outward normal n has stresses τ_{xx} and τ_{zx}. The resultant bending moment M of a cross section is defined as,

$$M^l = -\int_A z \sigma_{xx} \, dA \tag{3}$$

where A is the area of the cross-section. The equilibrium relations of the surface layer can be expressed as given by Gurtin and Murdoch [8, 9],

$$\frac{\partial \tau_{xx}^l}{\partial x} - T_x = 0 \tag{4a}$$

$$\frac{\partial \tau_{zx}^l}{\partial x} - T_z = 0 \tag{4b}$$

where $T_x = \tau_{xj} n_j$; $T_z = \tau_{zj} n_j$; τ_{ij} are components of the bulk stresses; n_j is the component of the surface orientation vector $\boldsymbol{n}$. For the case of a rectangular cross-section the contact tractions are: $T_x = \tau_{xz}$ and $T_z = \tau_{zz}$. By substituting Eq. (4) into Eq. (2) results the following equilibrium equation:

$$\frac{\partial^2 M^l}{\partial x^2} + \frac{\partial}{\partial x}\left(\int_S \frac{\partial \tau_{xx}^l}{\partial x} z \, ds \right) + \int_S \frac{\partial \tau_{zx}^l}{\partial x} \, ds + q(x) = 0 \,. \tag{5}$$

We have to mention that the aforementioned equilibrium equations Eq. (5), is valid and applicable for both local and nonlocal elasticity fields. Assuming a homogenous isotropic material and neglecting any residual stress in the bulk material due to surface tension, the relevant bulk stress-strain relation of the beam, in the context of local elasticity, can be expressed as

$$\sigma_{xx} = E\varepsilon_{xx} + \upsilon \sigma_{zz} \tag{6}$$

where E and υ are the Young's modulus and Poisson's ratio, respectively. According to Gurtin and Murdoch model, the constitutive relations of the surface can be expressed as

$$\tau_{xx}^{l} = \tau_0 + (2\mu_0 + \lambda_0)\frac{\partial u_x}{\partial x} \qquad \text{at} \qquad z = \pm\frac{h}{2} \tag{7a}$$

$$\tau_{zx}^{l} = \tau_0 \frac{\partial u_z}{\partial x} \qquad \text{at} \qquad z = \pm\frac{h}{2} \tag{7b}$$

where τ_0 is the residual surface stress under unconstrained conditions; and μ_0 and λ_0 are surface Lame's constants, which can be determined from atomistic calculations, Miller and Shenoy [7]. In classical beam theory, the stress component σ_{zz} is simply neglected. However, σ_{zz} must be considered to satisfy the surface equilibrium Eq. (4). Following Lu et al. [13], the stress component σ_{zz} is assumed to vary linearly through the beam thickness such as,

$$\sigma_{zz} = \frac{1}{2}\left[\sigma_{zz}^{+} + \sigma_{zz}^{-}\right] + \frac{z}{h}\left[\sigma_{zz}^{+} - \sigma_{zz}^{-}\right] \tag{8a}$$

$$= \frac{1}{2}\left[\tau_{zx,x}^{+} - \tau_{zx,x}^{-}\right] + \frac{z}{h}\left[\tau_{zx,x}^{+} + \tau_{zx,x}^{-}\right] \tag{8b}$$

where σ_{zz}^{+} and σ_{zz}^{-} are stresses at the top and bottom fibers, respectively. It is noted that Eq. (8) is also suitable for anisotropic materials. Substitute Eqs. (4) and (7) into (8) yields,

$$\sigma_{zz} = \frac{1}{2}\left[\tau_0\left(\frac{\partial^2 u_z^{+}}{\partial x^2} - \frac{\partial^2 u_z^{-}}{\partial x^2}\right)\right] + \frac{z}{h}\left[\tau_0\left(\frac{\partial^2 u_z^{+}}{\partial x^2} + \frac{\partial^2 u_z^{-}}{\partial x^2}\right)\right] \tag{9}$$

where u_z^{+} and u_z^{-} are vertical displacements of the top and bottom fibers, respectively. Assuming the displacement is continuous with no slipping between the surface layer and the bulk. So, the displacement field at a point of the Euler-Bernoulli beam can be expressed as,

$$u_x = -z\frac{\partial w(x)}{\partial x} \tag{10a}$$

$$u_z = w(x) \tag{10b}$$

where $u(x)$ is the displacement component along the x-axis of the midplane. The geometrical fit condition in case of small deformation can be described by

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = -z\frac{\partial^2 w(x)}{\partial x^2}\ . \tag{11}$$

By substituting Eqs. (10) and (11) into Eqs. (7) and (9), then into Eqs. (6) and (3), the following equations can be obtained,

$$M^{l} = EI\frac{\partial^2 w(x)}{\partial x^2} - \frac{2\nu I}{h}\left(\tau_0\frac{\partial^2 w(x)}{\partial x^2}\right) \tag{12a}$$

$$\tau_{xx}^{l} = \tau_0 + (2\mu_0 + \lambda_0)\left(-z\frac{\partial^2 w(x)}{\partial x^2}\right) \tag{12b}$$

$$\tau_{zx}^{l} = \tau_0\frac{\partial w(x)}{\partial x} \tag{12c}$$

where $I = \int_A z^2\, dA$ is the moment of inertia of the beam cross section. Substituting Eq. (12) into Eq. (5) yields the following governing equilibrium equation of the local Euler-Bernoulli beam, including the surface effect,

$$\left[EI - I_0(2\mu_0 + \lambda_0) - \frac{2\nu I\tau_0}{h}\right]\frac{\partial^4 w}{\partial x^4} + \left[\tau_0\, S_0\right]\frac{\partial^2 w}{\partial x^2} = -q(x) \tag{13}$$

where $I_0 = \int_S z^2\, dS$ is the perimeter moment of inertia and $S_0 = \int_S dS$.

2.2 Nonlocal Euler-Bernoulli beam theory

According to Eringen's nonlocal elasticity theory [21], the stress at a point x in a body depends not only on the strain at point x but also on those at all other points of the body. Thus, the nonlocal stress tensor σ at point x is expressed as follows:

$$\sigma = \int_V \alpha(|x' - x|, \tau)\,\mathrm{t}(x')\,dx' \tag{14a}$$

$$\mathrm{t}(x) = C(x) : \varepsilon(x) \tag{14b}$$

where t (x) is the classical local stress tensor at point x; $\varepsilon(x)$ is the strain tensor; $C(x)$ is the fourth-order elasticity tensor; $\alpha(|x' - x|, \tau)$ is the nonlocal modulus or attenuation function which incorporating into the constitutive equations the nonlocal effects at the reference point x produced by the local strain at any source point x'; $|x' - x|$ is the Euclidean distance; and $\tau = e_0 a/l$ is considered as a scale factor, where e_0 is a material constant to be determined experimentally, a and l are the internal (e.g. lattice parameter, granular size, distance between C-C bonds) and external (e.g. crack length, wave length) characteristic length, respectively.

The properties of the nonlocal attenuation function $\alpha(|x' - x|, \tau)$ have been discussed in detail by Eringen [21]. When $\alpha(|x|)$ takes on a Green's function of a linear differential operator, L, such that $L\left[\alpha(|x' - x|)\right] = \delta(|x' - x|)$, the non-local integral constitutive relation (14a) is reduced to the differential equation $L[\sigma(x)] = \mathrm{t}(x)$.

Thus, Eringen proposed a nonlocal model with a linear differential operator defend by, $L = (1 - \tau^2 l^2 \nabla^2)$, and therefore the constitutive relation with this attenuation function may be simplified to

$$(1 - \tau^2 l^2 \nabla^2)\sigma = \mathrm{t}\ . \tag{15}$$

For nonlocal Euler-Bernoulli beam, Eq. (15) can be written as

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = \mathrm{t}_{xx} , \qquad (\mu = e_0^2 a^2) \tag{16}$$

where $t_{xx} = E\varepsilon_{xx}$. According to Eq. (16) the relation between nonlocal $\left(M^{nl}, \tau_{xx}^{nl}, \tau_{zx}^{nl}\right)$ and local $\left(M^{l}, \tau_{xx}^{l}, \tau_{zx}^{l}\right)$ moment resultant and surface stresses, respectively, can be expressed as,

$$\left(1-\mu_b \nabla^2\right) M^{nl} = M^l \tag{17a}$$

$$\left(1-\mu_s \nabla^2\right) \tau_{xx}^{nl} = \tau_{xx}^l \tag{17b}$$

$$\left(1-\mu_s \nabla^2\right) \tau_{xzx}^{nl} = \tau_{zx}^l \tag{17c}$$

where μ_b, and μ_s are the nonlocal parameters the bulk and surface materials, respectively, which should be evaluated experimentally. Since the atomic arrangement of the surface is different than that of the bulk material, therefore, the nonlocal parameter μ for each of them should be different. For sake of simplicity, we assume that the nonlocal parameter for both bulk and surface are identical, ($\mu_s = \mu_b = \mu$).

2.3 Nonlocal beam theory including surface effects

In case of nonlocal elasticity, the equilibrium equation is similar to that given by Eq. (5) and has the following form:

$$\frac{\partial^2 M^{nl}}{\partial x^2} + \frac{\partial}{\partial x}\left(\int_S \frac{\partial \tau_{xx}^{nl}}{\partial x} z\, ds\right) + \int_S \frac{\partial \tau_{zx}^{nl}}{\partial x} ds + q(x) = 0 . \tag{18}$$

Multiplying Eq. (18) by the nonlocal operator, $(1-\mu\nabla^2)$, the nonlocal governing equation of the nanobeam including surface effects are reduced to,

$$\frac{\partial^2 M^l}{\partial x^2} + \frac{\partial}{\partial x}\left(\int_S \frac{\partial \tau_{xx}^{l}}{\partial x} z\, ds\right) + \int_S \frac{\partial \tau_{zx}^{l}}{\partial x} ds + \left(1-\mu\nabla^2\right) q(x) = 0 . \tag{19}$$

Substituting Eq. (12) into Eq. (19) yields the following equilibrium equation in terms of transverse displacement:

$$\left[EI - I_0\left(2\mu_0 + \lambda_0\right) - \frac{2\nu I \tau_0}{h}\right]\frac{\partial^4 w}{\partial x^4} + \left[\tau_0 S_0\right]\frac{\partial^2 w}{\partial x^2} + \left(1-\mu\nabla^2\right) q(x) = 0. \tag{20}$$

We have to notice that the natural boundary conditions of the beam, which represent the moment and shear force at the boundary points, should be expressed in terms of nonlocal stress or nonlocal stress resultants, as given in the appendix.

It is obviously clear that if the surface effect is completely neglected, (μ_0, λ_0 and τ_0 are all set to zero), Eq. (20) is reduced to that of the nonlocal Euler-Bernoulli beam. In addition, if the nonlocal parameter μ is set to zero, the nonlocality would be eliminated and the equilibrium equation is transformed to the classical beam theory.

3. Numerical finite element formulation

While the minimum potential energy principle is only valid to apply on the level of the whole volume of nonlocal elastic continuum, it is not valid to apply on one element or subdomain of the continuum, Polizzotto [42]. Therefore, to develop a finite element model for a nonlocal elasticity problem, the weighted residual methods are more appropriate to derive the equivalent variational statement of the weak solution of the problem.

In the present work, a finite element model for nonlocal Euler-Bernoulli beam, including surface effects is developed. The conventional Galerkin technique is employed to derive the weighted residual variational functional of the equilibrium Eq. (20).

Firstly, the domain of the beam is discretized into a set of elements, each of them has a sub-domain $\Omega^e = (x_e, x_{e+1})$ and of length *L*. The variational statement of the weighted residual functional of the equilibrium Eq. (20) for a generic element, after performing the required partial integration, is expressed as,

$$\int_0^L \left\{ \begin{array}{l} \left[EI - I_0(2\mu_0 + \lambda_0) - \dfrac{2\nu I \tau_0}{h}\right]\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 \chi}{\partial x^2} + \\ \tau_0 S_0 \dfrac{\partial w}{\partial x}\dfrac{\partial \chi}{\partial x} + \left(q\chi - \mu q \dfrac{\partial^2 \chi}{\partial x^2}\right) \end{array} \right\} dx + \left[\bar{V}\chi + \bar{M}\frac{\partial \chi}{\partial x}\right]_0^L = 0 \tag{21}$$

where χ denotes the Galerkin's weight function and the over bar quantities denote the μ specified stress resultant at the two boundary points x_e and x_{e+1}. The deflection along the element, in a local coordinate system, is given in terms of the Hermite interpolation functions as

$$\bar{w}(\bar{x}) = \sum_{i=1}^{4} N_i U_i \tag{22}$$

where U_i denotes the nodal degrees of freedom, representing the deflection and rotation at each terminal node of the element; and N_i, i = 1, 2, 3, 4 are the Hermite interpolation functions which are given as follows:

$$N_1 = 1 - \frac{3\bar{x}^2}{l^2} + \frac{2\bar{x}^3}{l^3} \tag{23a}$$

$$N_2 = \bar{x} - \frac{2\bar{x}^2}{l} + \frac{\bar{x}^3}{l^2} \tag{23b}$$

$$N_3 = \frac{3\bar{x}^2}{l^2} - \frac{2\bar{x}^3}{l^3} \tag{23c}$$

$$N_4 = \frac{\bar{x}^3}{l^2} - \frac{\bar{x}^2}{l} . \tag{23d}$$

By substituting Eq. (23) into the modified weak form, Eq. (21), and performing the integration, we get the following element equilibrium equation,

$$\left([K_c]+[K_S]\right)\{\bar{U}\}=\{F\}+\{Q\} \tag{24}$$

where $[K_c]$, $[K_S]$, $\{F\}$ and $\{Q\}$ are the classical element stiffness matrix, surface stiffness matrix, distributed force vector, and concentrated force vector, respectively. The element equations can be assembled to form the system of global equilibrium equations. Finally, we have to focus the attention that the natural boundary conditions such as moments and shear forces should be expressed in terms of nonlocal stresses or nonlocal stress resultants.

4. Numerical results

This section is divided into three parts, the first one presents a comparison between the results of the proposed nonlocal finite element model, with and without surface effects, and the results that published before. In the second and third parts the effects of surface elastic constants and nonlocal parameter on the rigidity and deflection of nanobeams are presented and discussed.

All numerical computations have been implemented on simply supported nanobeams having two different materials constants as given in Gurtin and Murdoch [9],

Iron film on glass substrate (M1):
E = 5.625 × 10^{10} N/m^2; υ = 0.25; λ_0 = 7 × 10^3 N/m; μ_0 = 8 × 10^3 N/m; τ_0 = 110 N/m.

Iron free surface (M2):
E = 17.73 × 10^{10} N/m^2 ; υ = 0.27; λ_0 = -8 N/m; μ_0 = 2.5 N/m; τ_0 = 1.7 N/m.

4.1 Model verification

A study of the static bending behavior of nonlocal beam under uniform distributed load of unit amplitude (q_0 = 1) is carried out. Surface effect is taken into account for both materials, M1 and M2. A nonlocal beam of the following geometrical parameters have been considered for verifications, Reddy [26]: L = 10 m; L/h = 100; and b = h.

To verify the proposed model, non-dimensional central deflection $\bar{w}$, $\left(\bar{w}=100*\delta_{\max}*\frac{EI}{q_0 L^4}\right)$, is calculated and compared with previous published results. The comparison is presented in Table 1. As can be noted, the obtained results are identical for analytical solution and a good agreement with those of Reddy [26], Aydogdu [27], and Civalek [33] [Error = % 0.8448] for a local beam, $\mu = 0$. For the nonlocal behavior, the current work is very close to the previously published work as shown in Table 1.

Table 1. Comparison of non-dimensional central deflection ($\bar{w}$) with previous work for S-S beam.

	μ = 0	μ = 1	μ = 2	μ = 3	μ = 4
Analytical	1.302	-	-	-	-
Reddy [26]	1.313	1.4487	1.5844	1.7201	1.8558
Aydogdu [27]	1.313	1.4487	1.5844	1.7201	1.8558
Civalek [33]	1.313	1.4487	-	-	1.8558
Present (M1) no surface effect	1.30208	1.42708	1.55208	1.67708	1.80208
Present (M1) surface effect	1.30204	1.42703	1.55203	1.67703	1.80202
Present (M2) no surface effect	1.30208	1.42708	1.55208	1.67708	1.80208
Present (M2) surface effect	1.30208	1.42708	1.55208	1.67708	1.80208

Furthermore, it is noticed that as the nonlocal parameter increases the non-dimensional deflection increases slightly as a result of the size-dependency increased. Meanwhile, surface stress has no significant effect for the cases of macro-size scale since the ratio of the surface to the bulk volume is very small.

4.2 Surface effects

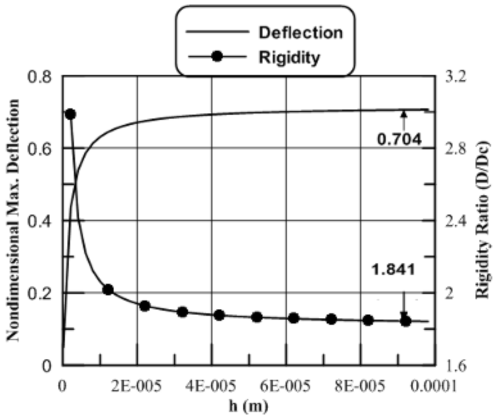
To investigate the surface effects on the rigidity and static deflection of a simple support beam irrespective of the nonlocality, (μ = 0), the proposed finite element model has been used to study the bending behavior of a beam having the following dimensions: L = 1 × 10^{-3} m; b = 1 × 10^{-6} m; and h = varied.

The variation of the non-dimensional maximum deflection and rigidity ratio versus the thickness h is shown in Fig. 2 for both materials M1 and M2. The rigidity ratio is defined as the ratio between the rigidity of beam with surface effect (D) to the rigidity of classical beam (D_c = EI).

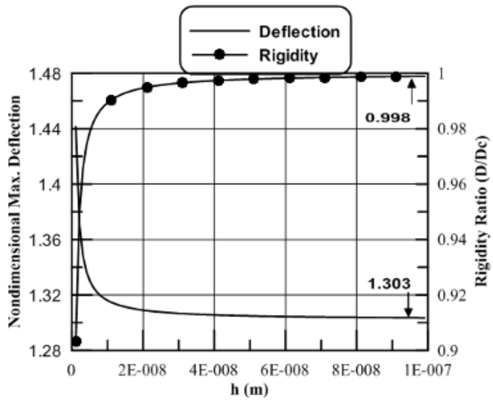
For Material M1, Fig. 2(a) shows that as the thickness increased the rigidity decreased and the non-dimensional central deflection, consequently, increased. Beyond the thickness value of 10 μm, there is no significant effect on both rigidity and deflection, and the behavior of beams is almost identical to that of the local classical theory. For the case of material M2, thickness has different effect on the rigidity and deflection as shown in Fig. 2(b) As the thickness increases, the rigidity will also increase and consequently the deflection decreases gradually until a thickness value of 1 nm.

Lim and He [11] concluded that in their work "*The size effect is obvious and the influence is determined by the intrinsic length scale: for Material M1 it is significant when the thickness of the film is smaller than 10 μm, while for Material M2 it is significant when the film thickness is of order of 1 nm*".

The current results are completely agreed with the conclusions given by Lim and He [11]. It is also shown that by the reduction of thickness, the bending rigidity increases noticeably for material M1, while decreases for material M2. The

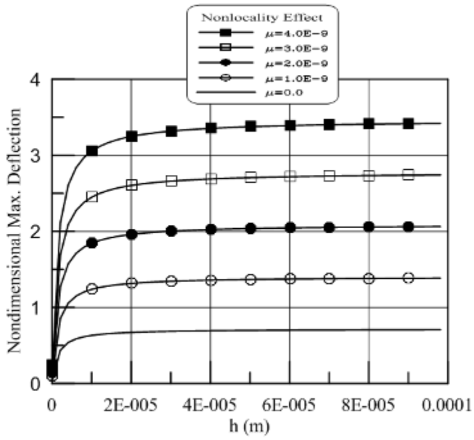


(a)

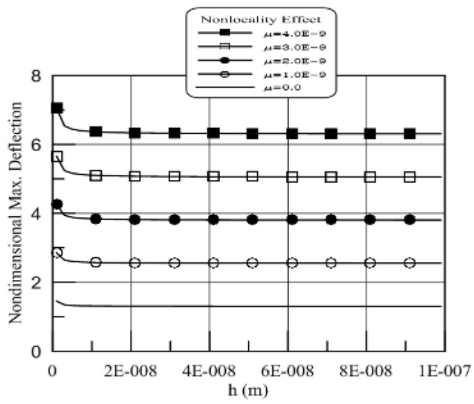


(b)

Fig. 2. (a) Effect of a beam thickness on beam rigidity and deflection including surface effects (M1); (b) Effect of a beam thickness on beam rigidity and deflection including surface effects (M2).



(a)



(b)

Fig. 3. (a) Nonlocality effect on the beam deflection for M1; (b) Nonlocality effect on the beam deflection for M2.

different effect of the thickness size, for the two materials, is directly correlated to their surface elastic constants.

4.3 Nonlocal parameter effect

The effect of nonlocal parameter μ on the bending behavior of nanobeam is investigated and shown in Fig. 3. The figure presents the variation of the non-dimensional maximum deflection, $w_{max,}$ for the aforementioned simply supported nanobeam, versus the thickness, *h,* and for different nonlocal parameter μ. It is obviously clear that w_{max} increases noticeably by increasing the nonlocal parameter, μ, for the both two materials.

Meanwhile, for any given value of μ, Fig. 3(a) shows that the non-dimensional deflection of the first material (M1) is gradually increasing by increasing the thickness value up to 10μm; beyond this value w_{max} is shown to be almost stationary, whatever the increasing of the thickness value. On the contrary, Fig. 3(b) shows that the non-dimensional deflection of the second material (M2) is gradually decreasing by increasing the thickness up to about 1nm, which means that the trend of w_{max} depends mainly on the elastic constants of the surface.

5. Conclusions

A nonlocal finite element model is developed for investigation of bending behavior of Euler-Bernoulli nanobeam, including surface effects. Natural boundary conditions such as the end moments and forces are expressed in terms of nonlocal stresses rather than the local one. Several computational experiments have been carried out to investigate the size dependent behavior due to the nature of nonlocal elasticity and surface effects.

Two materials of different bulk and surface elastic constants are used to study the effects of surface elastic constants on the rigidity and bending behavior of nanobeams. The stiffness, rigidity and bending behavior are found to be size-dependent and this dependency is more significant for slender nanobeams. The size effect for Material M1 is significant when the thickness of the film is smaller than 10 μm, while for Material M2 it is significant when the film thickness is of order of 1 nm. The results show that the size effects tends to be significant when the thickness of the beam decreases towards the intrinsic length scale of the material which is, generally, in agreement with the results from atomistic simulation.

The effect of nonlocal parameters on the deflection of Euler-Bernoulli nanobeam is investigated. The results show that the nonlocal effect on the deflection is significant, practically for a smaller thickness. Increasing the nonlocal parameter increased the deflection of nanobeams.

Acknowledgment

The first author wishes to express his thanks to the Department of Material Science and Engineering, University of Texas at Arlington, for the support of this work.

References

[1] C. Li and T. W. Chou, A structural mechanics approach for the analysis of carbon nanotubes, *Intentional journal of Solids and Structures*, 40 (2003) 2487-2499.

[2] J. C. Hsu, H. L. Lee and W. J Chang, Longitudinal vibration of cracked nanobeams using nonlocal elasticity theory, *Current Applied Physics*, 11 (2011) 1384-1388.

[3] H. L. Lee and W. J. Chang, Surface and small-scale effects on vibration analysis of a nonuniform nanocantilever beam, *Physica E*, 43 (2010) 466-469.

[4] G. Yun and H. S. Park, A multiscale, Finite deformation formulation for surface stress effects on the coupled thermomechanical behavior of nanomaterials, *Computer Methods in Applied Mechanics and Engineering*, 197 (2008) 3337-3350.

[5] C. Truesdell and W. Noll, *The nonlinear field theories of mechanics*, Springer (1992).

[6] R. Maranganti and P. Sharma, A novel atomistic approach to determine strain gradient elasticity constants, *Journal of the Mechanics and Physics of Solids*, 55 (2007) 1832-1852.

[7] R. E. Miller and V. B. Shenoy, Size-dependent elastic properties of nanosized structural elements, *Nanotechnology*, 11 (3) (2000) 139-147.

[8] M. E. Gurtin and A. I. Murdoch, A continuum theory of elastic material surfaces, *Archive for Rational Mechanics and Analysis*, 57 (1975) 291-323.

[9] M. E. Gurtin and A. I. Murdoch, Surface stress in solids, *International Journal of Solids and Structures*, 14 (1978) 431-440.

[10] V. B. Shenoy, Size-dependent rigidities of nanosized torsional elements, *International Journal of Solids and Structures*, 39 (15) (2002) 4039-4052.

[11] C. W. Lim and L. H. He, Size-dependent nonlinear response of thin elastic films with nano-scale thickness, *International Journal of Mechanical Sciences*, 46 (11) (2004) 1715-1726.

[12] T. Chen, M. S. Chiu and C. N. Weng, Derivation of the generalized Young-Laplace equation of curved interfaces in nanoscaled solids, *Journal of Applied Physics*, 100 (2006) 074308.

[13] P. Lu, L. H. He, H. P. Lee and C. Lu, Thin plate theory including surface effects, *International Journal of Solids and Structures* ,43 (16) (2006) 4631-4647.

[14] Y. Fu, J. Zhang and Y. Jiang, Influences of the surface energies on the nonlinear static and dynamic behaviors of nanobeams, *Physica E*, 42 (2010) 2268-2273.

[15] A. Assadi and B. Farshi, Size dependent vibration of curved nanobeams and rings including surface energies, *Physica E*, 43 (2011) 975-978.

[16] G. Yun and H. S. Park, Bridging the gap between experimental measurements and atomistic predictions of the elastic properties of silicon nanowires using multiscale modeling, *Finite Elements in Analysis and Design*, 49 (2012) 3-12.

[17] E. Kroner, Elasticity theory of materials with large range cohesive forces, *Intentional journal of Solids and Structures*, 3 (1967) 731-742.

[18] D. Edelen, A. E. Green and N. Laws, Nonlocal continuum mechanics, *Archive for Rational Mechanics and Analysis*, 43(1) (1971) 36-44.

[19] A. C. Eringen, "Nonlocal polar elastic continua", *International Journal of Engineering Science*, 10 (1972) 1-16.

[20] A. C. Eringen and D. Edelen, On nonlocal elasticity, *International Journal of Engineering Science*, 10 (3) (1972) 233-248.

[21] A. C. Eringen, On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves, Journal of Applied Physics, 54 (1983) 4703-4710.

[22] A. C. Eringen, *Nonlocal Continuum Field Theories*, Springer-Verlag, New York, (2002).

[23] J. Peddieson, G. R. Buchanan and R. P. McNitt, Application of nonlocal continuum models to nanotechnology, *International Journal of Engineering Science*, 41 (3-5) (2003) 305-312.

[24] A. W. McFarland and J. S. Colton, Role of material microstructure in plate stiffness with relevance to micro-cantilever sensors, *Journal of Micromechanics and Microengineering*, 15 (2005) 1060-1067.

[25] Q. Wang and K. M. Liew, Application of nonlocal continuum mechanics to static analysis of micro- and nanostructures, *Physics Letters A*, 363 (2007) 236-242.

[26] J. N. Reddy, Nonlocal theories for bending, buckling and vibration of beams, *International Journal of Engineering Science*, 45 (2007) 288-307.

[27] M. Aydogdu, A general nonlocal beam theory: Its application to nanobeam bending, buckling and vibration, *Physica E*, 41 (2009) 1651-1655.

[28] S. Kong, S. Zhou, Z. Nie and K. Wang, The size-dependent natural frequency of Bernoulli–Euler micro-beams, *International Journal of Engineering Science*, 46 (2008) 427-437.

[29] S. Kong, S. Zhou, Z. Nie and K. Wang, Static and dynamic analysis of micro beams based on strain gradient elasticity theory, *International Journal of Engineering Science*, 47 (2009) 487-498.

[30] A. Pisano, A. Sofi and P. Fuschi, Finite element solutions for nonhomogeneous nonlocal elastic problems, *Mechanics Research Communications*, 36 (2009) 755-761.

[31] A. Pisano, A. Sofi and P. Fuschi, Nonlocal integral elasticity: 2D finite element based solutions, *International Journal*

of Solids and Structures, 46 (2009) 3836-3849.

[32] S. C. Pradhan and T. Murmu, Application of nonlocal elasticity and DQM in the flap wise bending vibration of a rotating nano-cantilever , *Physica E*, 42 (2010) 1944-1949.

[33] O. Civalek and C. Demir, Bending analysis of microtubules using nonlocal Euler–Bernoulli beam theory, *Applied Mathematical Modeling*, 35 (5) (2011) 2053-2067.

[34] J. K. Phadikar and S. C. Pradhan, Variational formulation and finite element analysis for nonlocal elastic nanobeams and nanoplates, *Computational Materials Science*, 49 (3) (2010) 492-499.

[35] F. F. Mahmoud and E. I. Meletis, Nonlocal finite element modeling of the tribological behavior of nano-structured materials, *Interaction and Multiscale mechanics*, 3 (3) (2010) 267-276.

[36] W. Xia, L. Wang and L.Yin, Nonlinear non-classical microscale beams: Static bending, post-buckling and free vibration, *International Journal of Engineering Science*, 48 (12) (2010) 2044-2053.

[37] M. A. Eltaher, S. A. Emam and F. F. Mahmoud, Free vibration analysis of functionally graded size-dependent nanobeams, *Applied Mathematics and Computation*, 218 (2012) 7406-7420.

[38] H. L. Lee and W. J. Chang, Surface effects on frequency analysis of nanotubes using nonlocal Timoshenko beam theory, *Journal of Applied Physics*, 108 (2010) 093503.

[39] H. L. Lee and W. J. Chang, Surface effects on axial buckling of nonuniform nanowires using non-local elasticity theory, *Micro & Nano Letters*, 6 (1) (2011) 19-21.

[40] X. W. Lei, T. Natsuki, J. Shi and Q. Q. Ni, Surface effects on the vibrational frequency of double-walled carbon nanotubes using the nonlocal Timoshenko beam model, *Composites Part B*, 43 (2012) 64-69.

[41] F. F. Mahmoud, M. A. Eltaher, A. E. Alshorbagy and E. I. Meletis, Free vibration analysis of nonlocal nanobeams including surface effects, Applied Mathematical Modeling, (under review).

[42] C. Polizzotto, Nonlocal elasticity and related variational principles, *International Journal of Solids and Structures*, 38 (2001) 7359-7380.

Appendix

Derivation of the nonlocal bending moment and shear forces.

According to the differential constitutive relations proposed by Eringen, the nonlocal bending moment can be expressed as:

$$\left(1-\mu\frac{\partial^2}{\partial x^2}\right)M^{nl}=M^{l} \tag{A1}$$

where

$$M^{l}=\left[EI-\frac{2\nu I\tau_0}{h}\right]\frac{\partial^2 w}{\partial x^2}. \tag{A2}$$

By substituting M^{l} into Eq. (A1), M^{nl} can be written as:

$$M^{nl}=\mu\frac{\partial^2 M^{nl}}{\partial x^2}+\left[EI-\frac{2\nu I\tau_0}{h}\right]\frac{\partial^2 w}{\partial x^2} \tag{A3}$$

where $\frac{\partial^2 M^{nl}}{\partial x^2}$ is given by Eq. (18). Consequently, Eq. (A3) can be reduced to the following,

$$\begin{aligned}M^{nl}=&-\mu\left[\frac{\partial}{\partial x}\left(\int_S\frac{\partial\tau_{xx}^{nl}}{\partial x}z\,ds\right)+\int_S\frac{\partial\tau_{zx}^{nl}}{\partial x}ds+q(x)\right]\\&+\left[EI-\frac{2\nu I\tau_0}{h}\right]\frac{\partial^2 w}{\partial x^2}.\end{aligned} \tag{A4}$$

For the sake of simplicity, we can assume

$$\tau_{xx}^{nl}=\tau_{xx}^{l}=\tau_0-z\left(2\mu_0+\lambda_0\right)\frac{\partial^2 w}{\partial x^2} \tag{A5}$$

$$\tau_{zx}^{nl}=\tau_{zx}^{l}=\tau_0\frac{\partial w}{\partial x}. \tag{A6}$$

By substituting Eqs. (A5) and (A6) into Eq. (A4), the nonlocal moment resultant M^{nl} can be presented as

$$M^{nl}=\mu\left(2\mu_0+\lambda_0\right)I_0\frac{\partial^4 w}{\partial x^4}+\left[EI-2\mu\tau_0 S_0-\frac{2\nu I\tau_0}{h}\right]\frac{\partial^2 w}{\partial x^2}-\mu q. \tag{A7}$$

To obtain the expression of nonlocal shear force Q^{nl}, substitute Eq. (A7) into Eq. (1b), which produces

$$\begin{aligned}Q^{nl}=&\,\mu\left(2\mu_0+\lambda_0\right)I_0\frac{\partial^5 w}{\partial x^5}-\mu\frac{\partial q}{\partial x}\\&+\left[EI-2\mu\tau_0 S_0-\frac{2\nu I\tau_0}{h}-2\mu_0 I_0-\lambda_0 I_0\right]\frac{\partial^3 w}{\partial x^3}.\end{aligned} \tag{A8}$$

M. A. Eltaher received his B.Sc., M.Sc., and Ph.D in Mechanical Engineering from Zagazig University, Egypt in 1999, 2007, and 2012 respectively. Dr. Eltaher is currently a lecturer of the Department of Mechanical Design and Production Engineering of Zagazig University, Egypt. His research fields are composite material, functionally graded material, nanomechancis, MEMS, NEMS, nonlocal elasticity, viscoelasticity, dynamic contact, dynamic impact, vibrational analysis, plates and beams structures, and finite element analysis.