Springer

KSME

Journal of Mechanical Science and Technology 26 (9) (2012) 2879~2887
www.springerlink.com/content/1738-494x
DOI 10.1007/s12206-012-0730-y

Application of the differential transformation method and variational iteration method to large deformation of cantilever beams under point load †

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(Manuscript Received January 24, 2012; Revised March 13, 2012; Accepted April 25, 2012)

Abstract

Large deflection of a cantilever beam subjected to a tip-concentrated load is governed by a non-linear differential equation. Since it is hard to find exact or closed-form solutions for this non-linear problem, this paper investigates the aforementioned problem via the differential transformation method (DTM) and the variational iteration method (VIM), which are well-known approximate analytical solutions. The mathematical formulation is yielded to a non-linear two-point boundary value problem. In this study, we compare the DTM and VIM results, with those of Adomian decomposition method (ADM) and the established numerical solution obtained by the Richardson extrapolation in order to verify the accuracy of the proposed methods. As an important result, it is depicted from tabulated data that the DTM results are more accurate in comparison with those obtained by the VIM and ADM, which is one of the objectives of this article. Moreover, the effects of dimensionless end point load, α , on the slope of any point along the arc length and the dimensionless vertical and horizontal displacements are illustrated and explained. The results reveal that these methods are very effective and convenient in predicting the solution of such problems, and it is predicted that the DTM and VIM can find a wide application in new engineering problems.

Keywords: Cantilever beam; Differential transformation method; Large deformation; Variational iteration method

1. Introduction

The deflection of compliant mechanism which involves geometrical non-linearity in the wake of large deflection of members continues to be an interesting problem in mechanical systems. Therefore, many researches have been reported on the subject of the non-linear deformation of beams. Schmidt and Dadeppo [1] applied a modified Chebyshev's polynomial approximation method to analysis large deflections of cantilevered beams. Nageswara Rao and Venkateswara Rao studied the large deflection behavior of a cantilever beam subjected to a tip rotational concentrated load [2] and rotational distributed load [3]. Nageswara Rao et al. [4] analyzed the large deflection of a spring hinged cantilever beam subjected to a tip concentrated rational load. Nageswara Rao and Venkateswara Rao [5] investigated the large deflections of a nonuniform cantilever beam with end rotational load. Wang et al. [6] studied a class of large deflection beam problems where one end of the beam was being held while the other end portion was allowed to slide freely over a frictionless support fixed at a distance from that end. The elastic beam was subjected to a point load. This highly non-linear problem was solved using both the elliptic integral method and the shooting-optimization technique.

More recently, Lee [7] investigated large deflection of cantilever beams made of Ludwick type material subjected to a combined loading consisting of a uniformly distributed load and one vertical concentrated load at the free end. Chucheepsakul and Phungpaigram [8] investigated the exact closed-formed solutions using elliptic integrals for large deflection analysis of an elastic beam with variable arc-length subjected to an inclined follower force. The beam was hinged at end but slid freely over the support at the other end. Dado and Al-Sadder [9] studied the very large deflection behavior of prismatic and non-prismatic cantilever beams subjected to various types of loadings. The formulation was based on representing the angle of rotation of the beam by a polynomial on the position variable along the deflected beam axis. The coefficients of the polynomial were obtained by minimizing the integral of the residual error of the governing differential equation and by applying the beam's boundary conditions. Wang et al. [10] analyzed the large deformation of a cantilever beam under point load at the free tip by an analytical method, namely the homotopy analysis method (HAM). Tolou and Herder [11]

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investigated the feasibility of the Adomian decomposition method (ADM) in analyzing compliant mechanical systems. Mutyalarao et al. [12] examined large deflections of a uniform cantilever beam subjected to a tip-concentrated load whose inclination was normal to the deformed axis of the beam.

An analytical expression is more convenient for engineering calculations compare with experimental or numerical studies, and is also the obvious starting point for a better understanding of the relationship between the physical properties of the cantilever beam and the slope of any point along the arc length. Moreover, the pursuit of analytical solutions for the non-linear equation of large deformation in compliant beams under point load is of intrinsic scientific interest. To the best of the authors' knowledge, there is no paper that has solved this problem by the DTM and VIM. The primary purpose of present paper is to demonstrate the usefulness of the DTM and VIM to solve the aforementioned problem. Then, the ADM [11] and the numerical solution have been used for validity of these methods. Large deformation analysis of beams is a new application for the DTM and VIM which were used for other engineering applications [13-29].

2. Problem statement

Using the Bernoulli-Euler equation [30], the curvature of a prismatic beam, κ, can be written as

$$\kappa = \frac{d\theta}{ds} = \frac{M}{EI} \tag{1}$$

where M is the bending moment, E is Young's modulus, I is the area moment of inertia of the beam, while EI is called the bending stiffness of the beam. Furthermore, $\theta(s)$ is the slope of any point along the arc length with respect to the x-axis, and s is the arc-coordinate on the neutral axis of the beam from the fixed end to the base.

The moment at any point in the beam shown in Fig. 1 is given by

$$M = F\left(L - \delta_h - x\right) \tag{2}$$

where F is the point load at the free end. Thus, the bending equation of a uniform cross-section beam with large deflection is

$$\frac{d\theta}{ds} = \frac{F}{EI}\left(L - \delta_h - x\right), \qquad \theta(0) = 0, \qquad \theta'(L) = 0 \tag{3}$$

where the prime denotes the differential with respect to s, and where δ_h is the horizontal deflection of beam. The axial elongation of the beam is neglected, because it is much smaller than the lateral deflection at the free end point.

By differentiating Eq. (1) once with respect to s and rearranging it, we obtain

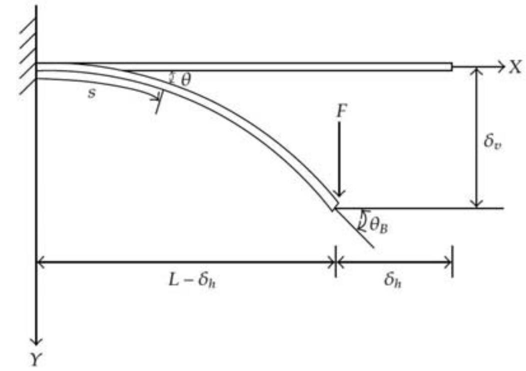

Fig. 1. Cantilever beam subjected to a free end point loading.

$$\frac{d^2\theta}{ds^2} = \frac{dM/ds}{EI}. \tag{4}$$

By introducing the dimensionless parameter $\zeta = s/L$, and differentiating Eq. (2) once with respect to s, taking into account the relation $\cos\theta = dx/ds$ and substituting in Eq. (4), we obtain the governing equation for large deformation of a cantilever beam under free end point vertical load shown in Fig. 1 in the following form:

$$\frac{d^2\theta}{d\zeta^2} + \alpha\cos\theta = 0 \tag{5a}$$

$$\theta(0) = 0 \tag{5b}$$

$$\theta'(1) = 0 \tag{5c}$$

where $\alpha = FL^2/EI$ is the dimensionless end point load. The rotation angle of the beam at free end point is denoted by $\theta_B = \theta(1)$. The dimensionless exact vertical and horizontal displacements of the free end point are given by [10, 30, 31]

$$\delta_v = L - 2\left[E(\mu) - E(\varphi,\mu)\right]\sqrt{\frac{EI}{F}} \tag{6}$$

where $E(\mu)$ is the complete elliptic integral of the second kind, $E(\varphi,\mu)$ is the elliptic integral of the second kind, and

$$\mu = \sqrt{\frac{1+\sin\theta_B}{2}}, \qquad \varphi = \arcsin\left(\frac{1}{\sqrt{2}\mu}\right). \tag{7}$$

Consequently, the dimensionless vertical displacement at free tip is given by

$$\frac{\delta_v}{L} = 1 - \frac{2}{\sqrt{\alpha}}\left[E(\mu) - E(\varphi,\mu)\right]. \tag{8}$$

Besides, we have

$$\frac{L-\delta_h}{L} = \sqrt{\left(\frac{2EI}{FL^2}\right)\sin\theta_B} = \sqrt{\frac{2\sin\theta_B}{\alpha}}. \tag{9}$$

Thus, the dimensionless horizontal displacement of the free tip is given by

$$\frac{\delta_h}{L} = 1 - \frac{1}{\sqrt{\alpha}}\sqrt{2\sin\theta_B} \,. \tag{10}$$

Clearly, the vertical and horizontal displacements δ_v and δ_h can be easily calculated as long as θ_B is known.

For infinitesimal deformation, it is enough to use the linear equation

$$\frac{d^2\theta}{d\zeta^2} + \alpha = 0, \qquad \theta(0) = 0, \qquad \theta'(1) = 0 \,. \tag{11}$$

The corresponding solution is

$$\theta(\zeta) = \frac{\alpha}{2}(2-\zeta)\,\zeta \tag{12}$$

which gives the linear result

$$\theta_B = \frac{\alpha}{2} \,. \tag{13}$$

If the large deformation is considered, one has to solve a non-linear algebraic equation [10]

$$\sqrt{\alpha} = K(\mu) - F(\varphi,\mu) \tag{14}$$

where μ and φ are defined by Eq. (7), and $K(\mu)$ is the complete elliptic integral of the first kind, and $F(\varphi,\mu)$ is the elliptic integral of the first kind, respectively.

3. Fundamental of differential transformation method [32]

Let $x(t)$ be analytic in a domain D and let $t = t_i$ represent any point in D. The function $x(t)$ is then represented by a power series whose center is located at t_i. A Taylor series expansion function of $x(t)$ about t_i takes the form:

$$x(t) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_i} \qquad \forall t \in D \,. \tag{15}$$

The particular case of Eq. (15) when $t_i = 0$ is the Maclaurin series of $x(t)$ and is expressed as:

$$x(t) = \sum_{k=0}^{\infty} \frac{(t)^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0} \,. \tag{16}$$

As explained in Ref. [33] the differential transformation of the function $x(t)$ is defined as follows:

Table 1. The fundamental operations of differential transform method.

Original function	Transformed function
$x(t) = \alpha f(t) \pm \beta g(t)$	$X(k) = \alpha F(k) \pm \beta G(k)$
$x(t) = \frac{df(t)}{dt}$	$X(k) = (k+1)F(k+1)$
$x(t) = \frac{d^2 f(t)}{dt^2}$	$X(k) = (k+1)(k+2)F(k+2)$
$x(t) = t^m$	$X(k) = \delta(k-m) = \begin{cases}1 & k=m\\ 0 & k \neq m\end{cases}$
$x(t) = \exp(\lambda t)$	$X(k) = \frac{\lambda^k}{k!}$
$x(t) = f(t)g(t)$	$X(k) = \sum_{l=0}^{k} F(l)G(k-l)$

$$X(k) = \frac{(H)^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0} \tag{17}$$

where $x(t)$ is the original function and $X(k)$ is the transformed function. The differential spectrum of $X(k)$ is confined within the interval $t \in [0, H]$, where H is a constant. The differential inverse transform of $X(k)$ is defined as follows:

$$x(t) = \sum_{k=0}^{\infty}\left(\frac{t}{H}\right)^k X(k) \,. \tag{18}$$

It is clear that the concept of differential transformation is based upon Taylor series expansion. Values of the function $X(k)$ are referred to as discretes, i.e. $X(0)$ is known as the zero discrete, $X(1)$ is the first discrete, and $X(k)$ is the kth discrete. The more discretes are available, the more precise it is possible to restore the unknown function. The function $x(t)$ consists of *T*-function $X(k)$, and its value is given by the sum of the *T*-function with $(t/H)^k$ as its coefficient. In real applications, with the right choice of the constant H, the larger values of argument k, the discrete of spectrum reduce rapidly. The function $x(t)$ is expressed by a finite series and Eq. (18) can be written as:

$$x(t) = \sum_{k=0}^{N}\left(\frac{t}{H}\right)^k X(k) \,. \tag{19}$$

Mathematical operations performed by the differential transform method are listed in Table 1.

3.1 Solution with DTM

Now we apply the differential transformation method to Eq. (5a). Taking the differential transform of Eq. (5a) with respect to ζ, and considering $H = 1$ according to Table 1, gives:

$$
\begin{aligned}
&(k+2)(k+1)\Theta(k+2)-\frac{1}{2}\alpha\left(\sum_{l=0}^{k}\Theta(l)\Theta(k-l)\right)\\
&+\frac{1}{24}\alpha\left(\sum_{m=0}^{k}\Theta(k-m)\left(\sum_{v=0}^{m}\Theta(m-v)\left(\sum_{w=0}^{v}\Theta(v-w)\Theta(w)\right)\right)\right)\\
&+\alpha\times\delta(k)=0\,.
\end{aligned}
\tag{20}
$$

From boundary condition in Eq. (5b), that we have it at point $\zeta = 0$, and exerting transformation

$$\Theta(0)=0\,. \tag{21}$$

The other boundary conditions are considered as follows:

$$\Theta(1)=C \tag{22}$$

where C is constant, and we will calculate it with considering another boundary condition in Eq. (5c) in point $\zeta = 1$.

Accordingly, from a process of inverse differential transformation, in this problem we calculated $\Theta(k+2)$ from Eq. (20) as follows:

$$\Theta(2)=-\frac{1}{2}\alpha \tag{23a}$$

$$\Theta(3)=0 \tag{23b}$$

$$\Theta(4)=\frac{1}{24}\alpha C^2 \tag{23c}$$

$$\Theta(5)=-\frac{1}{40}\alpha^2 C \tag{23d}$$

$$\Theta(6)=\frac{1}{240}\alpha^3-\frac{1}{720}\alpha\ C^4 \tag{23e}$$

$$\Theta(7)=\frac{1}{336}\alpha^2 C^3 \tag{23f}$$

$$\Theta(8)=-\frac{13}{6720}\alpha^3 C^2 \tag{23g}$$

$$\Theta(9)=\frac{1}{1920}\alpha^4 C-\frac{1}{8640}\alpha^2 C^5\,. \tag{23h}$$

$$\vdots$$

The above process may be continued further. Substituting Eq. (23) into the main equation based on the DTM, the closed form of the solutions is obtained as:

$$
\begin{aligned}
\theta(\zeta)=&\,C-\left(\frac{1}{2}\alpha\right)\zeta^2+\left(\frac{1}{24}\alpha C^2\right)\zeta^4-\left(\frac{1}{40}\alpha^2 C\right)\zeta^5+\left(\frac{1}{240}\alpha^3-\frac{1}{720}\alpha\ C^4\right)\zeta^6+\left(\frac{1}{336}\alpha^2 C^3\right)\zeta^7-\left(\frac{13}{6720}\alpha^3 C^2\right)\zeta^8\\
&+\left(\frac{1}{1920}\alpha^4 C-\frac{1}{8640}\alpha^2 C^5\right)\zeta^9+\left(\frac{257}{1209600}\alpha^3 C^4-\frac{1}{19200}\alpha^5\right)\zeta^{10}+\left(-\frac{67}{443520}\alpha^4 C^3+\frac{1}{475200}\alpha^2 C^7\right)\zeta^{11}\\
&+\left(\frac{157}{2956800}\alpha^5 C^2-\frac{13}{1182720}\alpha^3 C^6\right)\zeta^{12}+\left(\frac{529}{31449600}\alpha^4 C^5-\frac{7}{748800}\alpha^6 C\right)\zeta^{13}+\left(\frac{-120383}{9686476800}\alpha^5 C^4+\frac{1}{3548160}\alpha^3 C^8\right.\\
&\left.+\frac{1}{1497600}\alpha^7\right)\zeta^{14}+\left(\frac{547}{106444800}\alpha^6 C^3-\frac{391}{399168000}\alpha^4 C^7\right)\zeta^{15}+\left(\frac{25469}{18598035456}\alpha^5 C^6-\frac{15107}{12300288000}\alpha^7 C^2\right.\\
&\left.-\frac{19}{5474304000}\alpha^3 C^{10}\right)\zeta^{16}+\left(\frac{-13752547}{13173608448000}\alpha^6 C^5+\frac{4337}{130288435200}\alpha^4 C^9+\frac{163}{1018368000}\alpha^8 C\right)\zeta^{17}+\cdots.
\end{aligned}
\tag{24}
$$

To obtain the value of C, we substitute the boundary condition from Eq. (5c) into Eq. (24) giving

$$
\begin{aligned}
\theta'(1)=&-2\left(\frac{1}{2}\alpha\right)+4\left(\frac{1}{24}\alpha C^2\right)-5\left(\frac{1}{40}\alpha^2 C\right)+6\left(\frac{1}{240}\alpha^3-\frac{1}{720}\alpha\ C^4\right)+7\left(\frac{1}{336}\alpha^2 C^3\right)-8\left(\frac{13}{6720}\alpha^3 C^2\right)\\
&+9\left(\frac{1}{1920}\alpha^4 C-\frac{1}{8640}\alpha^2 C^5\right)+10\left(\frac{257}{1209600}\alpha^3 C^4-\frac{1}{19200}\alpha^5\right)+11\left(-\frac{67}{443520}\alpha^4 C^3+\frac{1}{475200}\alpha^2 C^7\right)\\
&+12\left(\frac{157}{2956800}\alpha^5 C^2-\frac{13}{1182720}\alpha^3 C^6\right)+13\left(\frac{529}{31449600}\alpha^4 C^5-\frac{7}{748800}\alpha^6 C\right)+14\left(\frac{-120383}{9686476800}\alpha^5 C^4+\frac{1}{3548160}\alpha^3 C^8\right.\\
&\left.+\frac{1}{1497600}\alpha^7\right)+15\left(\frac{547}{106444800}\alpha^6 C^3-\frac{391}{399168000}\alpha^4 C^7\right)+16\left(\frac{25469}{18598035456}\alpha^5 C^6-\frac{15107}{12300288000}\alpha^7 C^2\right.\\
&\left.-\frac{19}{5474304000}\alpha^3 C^{10}\right)+17\left(-\frac{13752547}{13173608448000}\alpha^6 C^5+\frac{4337}{130288435200}\alpha^4 C^9+\frac{163}{1018368000}\alpha^8 C\right)+\cdots=0\,.
\end{aligned}
\tag{25}
$$

The constant can be evaluated using numerical methods such as Newton-Raphson method. Substituting for C into Eq. (24), we determine $\theta(\zeta)$.

The calculations reported in this paper use $N = 24$, which was found to be sufficient to give an accurate solution. An implication of this is that Eq. (5) only requires the summation of a limited number of terms, and therefore the solution can be computed without excessive computational effort.

4. Fundamental of variational iteration method [34, 35]

To illustrate the basic concept of the technique, we consider the following general differential equation:

$$Lu+Nu=g(x) \tag{26}$$

where L is a linear operator, N a non-linear operator, and $g(x)$ is the forcing term. According to the variational iteration method, we can construct a correct functional as follows:

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda(Lu_n(t)+N\tilde{u}_n(t)-g(t))dt \tag{27}$$

where λ is a Lagrange multiplier [13-16], which can be identified optimally via the variational iteration method. The subscripts n denote the *nth* approximation, $\tilde{u}_n$ is considered as a restricted variation, that is, $\delta\tilde{u}_n = 0$; and Eq. (16) is called a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of the variational iteration method and its applicability to various kinds of differential equations are given in Refs. [13-16]. In this method, it is required first to optimally determine the Lagrange multiplier λ. The successive approximation u_{n+1}, $n \ge 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by

$$u=\lim_{n\to\infty} u_n\,. \tag{28}$$

The convergence and error of the VIM were estimated by Ramos [36].

4.1 Implementation of VIM

In order to solve Eq. (5a) using the VIM, we construct a correction functional as follows:

$$\theta_{n+1}(\zeta)=\theta_n(\zeta)+\int_0^\zeta \lambda\left\{\frac{d^2\theta_n(t)}{dt^2}+\alpha\cos\left(\tilde{\theta}_n(t)\right)\right\}dt . \quad (29)$$

Its stationary conditions can be obtained as follows:

$$\lambda''(t)\big|_{t=\zeta}=0 \quad (30a)$$

$$1-\lambda'(t)\big|_{t=\zeta}=0 \quad (30b)$$

$$\lambda(t)\big|_{t=\zeta}=0 . \quad (30c)$$

Thus the Lagrangian multiplier can therefore be identified as

$$\lambda=t-\zeta . \quad (31)$$

As a result, we obtain the following iteration formula:

$$\theta_{n+1}(\zeta)=\theta_n(\zeta)+\int_0^\zeta (t-\zeta)\left\{\frac{d^2\theta_n(t)}{dt^2}+\alpha\cos\left(\theta_n(t)\right)\right\}dt . \quad (32)$$

From the boundary condition in Eq. (5b), that we have it in point $\zeta=0$ an arbitrary initial approximation can be obtained

$$\theta_0(\zeta)=b\zeta \quad (33)$$

where b is constant, and we will calculate it with considering another boundary condition in Eq. (5c) in point $\zeta=1$.

Using the variational formula Eq. (32), we have

$$\theta_1(\zeta)=\theta_0(\zeta)+\int_0^\zeta (t-\zeta)\left\{\frac{d^2\theta_0(t)}{dt^2}+\alpha\cos\left(\theta_0(t)\right)\right\}dt . \quad (34)$$

Substituting Eq. (33) into Eq. (34) we have

$$\theta_1(\zeta)=b\zeta+\frac{\alpha\left(-1+\cos(b\zeta)\right)}{b^2} . \quad (35)$$

Accordingly, in the same manner the rest of the components of the iteration formula can be obtained. Here, in the wake of the large length of second and third iteration for the solution, the result of the first iteration is written; however the obtained results are calculated using three iterations. The variational iteration algorithm used in this paper is the variational iteration algorithm-I; there are also the variational iteration algorithm-II and variational iteration algorithm-III [37], which can also be used for such problems.

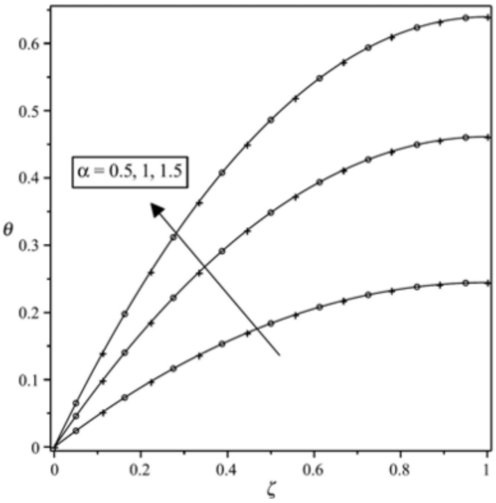

Fig. 2. The rotation of cross-section of the beam, θ, versus ζ obtained by DTM (circle), VIM (cross) and NS (solid line).

5. Numerical method

Eq. (5a) along with the boundary conditions (5b, c) were solved numerically using Maple 14.0. The software automatically detects the type of problem (boundary value problem or initial value) when the **dsolve** command is invoked and uses the appropriate algorithm accordingly. For the boundary value problems, the software uses a finite difference technique with Richardson extrapolation, whereas for the initial value problems, it uses a fourth-fifth order Runge-Kutta-Fehlberg method. The accuracy and robustness of Maple's algorithm for solving the boundary value problems has been repeatedly confirmed in various problems [38-40].

6. Results and discussion

Two analytical solutions named as differential transformation and variational iteration methods were applied to Eq. (5). To calculate a sufficient number of terms (N for the DTM and n for the VIM), four special cases for the rotation angle at the free end were studied.

Tables 2 and 3 show the convergence of the θ_B for different values of dimensionless end point loads. From Table 2 it is clearly visible that for the DTM more than 24 terms are needed to obtain the value of the θ_B, accurately to fourth significant digits. Also, it is seen from Table 3 that more than 3 iterations, i.e. $n=3$, are needed to obtain an accurate value of the θ_B by the VIM. The bold numbers in these tables are those beyond which the fourth digit does not change as N or n increases. Therefore, the numerical results from the DTM and VIM which are presented in this paper were obtained by taking sufficient terms $N=24$ and $n=3$, respectively.

The fact that the DTM and the VIM fall on the numerical solution (NS) confirms the validity and accuracy of analytical solutions, which is shown in Fig. 2. In this figure, we presented slope of any point of the beam versus the dimen-

Table 2. Convergence test of the DTM results for θ_B .

N	α			
	0.5	1	1.5	2
4	0.2470591834	0.4800471884	0.6967061088	0.9040967099
6	**0.2442493956**	0.4540331169	0.5989222138	0.6632405766
8	0.2445257363	**0.4615279094**	0.6436114887	0.8069077667
10	0.2445371302	0.4617104206	0.6442188633	0.8087292779
12	0.2445335658	0.4612911521	0.6377879226	0.7660452317
14	0.2445336605	0.4613486440	**0.6395081776**	0.7823797966
16	0.2445336876	0.4613588944	0.6397917906	0.7853664547
18	0.2445336854	0.4613548057	0.6395172023	0.7804035620
20	0.2445336853	0.4613549776	0.6395588028	0.7816919102
22	0.2445336857	0.4613551255	0.6395779582	0.7822444716
24	0.2445336857	0.4613550965	0.6395684280	**0.7817589123**
26	0.2445336855	0.4613550948	0.6395688638	0.7818299637
28	0.2445336855	0.4613550961	0.6395697594	0.7819026271
30	0.2445336855	0.4613550960	0.6395694829	0.7818605881
NS	0.2445337143	0.4613522662	0.6395398261	0.7817498319

Table 3. Convergence test of the VIM results for θ_B .

n	α			
	0.5	1	1.5	2
1	0.2357278931	0.4079438338	0.5095927414	0.5583459484
2	**0.2444918905**	0.4603303323	0.6340981838	0.7660791021
3	0.2445335606	**0.4613439981**	**0.6394513742**	**0.7812978835**
4	0.2445336472	0.4613519147	0.6395386164	0.7816435338
5	0.2445336474	0.4613519497	0.6395393686	0.7816374050
6	0.2445336475	0.4613519499	0.6395393707	0.7816367582
NS	0.2445337143	0.4613522662	0.6395398261	0.7817498319

sionless parameter ζ , for various dimensionless end point loads i.e. $\alpha = 0.5$,1 ,1.5 . As can be seen clearly in this figure, the value of the rotation angle of the beam is increased as the dimensionless displacement ζ and dimensionless applied force α increase. Also, both increasing ζ and α give rise to a more significant increase of rotating angle.

Accordingly, in order to investigate the effectiveness of the DTM and VIM solutions with a finite number of terms, the corresponding results are compared with the Adomian decomposition method (ADM) [11] and numerical solution using Maple which uses a finite difference method with Richardson extrapolation, which are tabulated in Table 4. This table represents the rotation angle at the free end, θ_B . The results of the comparison clearly show that the maximum difference between the ADM and numerical results for θ_B for the strongest non-linearity condition, i.e. $\alpha = 1.5$, is 0.78%. However, this value for the VIM and DTM solutions is 0.0138% and 0.0044%, respectively. It is apparent from these to sample calculations that these two methods are more accurate than the ADM. Although the VIM results are acceptable, but it is shown that with the DTM, a highly accurate analytical solution of the problem is achievable.

Additionally, Fig. 3 demonstrates the rotation angle of cross-section plane at tip versus α . This figure clearly illustrates that increasing in the values of α produce increase in values of θ_B .

Now, since analytical expressions by two methods for the rotation angle of any cross-section of the beam were obtained, the vertical and horizontal displacements can readily be calculated from Eqs. (8) and (10). Accordingly, the trends of vertical and horizontal displacements of the beam at the free end point versus α are shown in Figs. 4 and 5, respectively. As expected, an increase in the value of α causes an increase in the value of the vertical and horizontal displacements. Also, from Fig. 4, $\alpha = 2$ corresponds to the vertical displacement at the free end almost as half of the original length of the beam. In addition, from Fig. 5, $\alpha = 1.5$ corresponds to the horizontal displacement at the free end almost as one-tenth of the original length of the beam.

Table 4. The error of DTM, VIM and ADM [11].

α	$\theta_{B(DTM)}$	$\theta_{B(VIM)}$	$\theta_{B(NS)}$	$Error_{DTM}$	$Error_{VIM}$	$Error_{ADM}$ [11]
0.1	0.0499542557	0.0499542558	0.0499546770	4.2E-07	4.2E-07	4E-10
0.2	0.0996361632	0.0996361627	0.0996361680	4.8E-09	5.3E-09	5E-9
0.3	0.1487837281	0.1487837240	0.1487837430	1.5E-08	1.9E-08	2E-8
0.4	0.1971546322	0.1971546048	0.1971546604	2.8E-08	5.6E-08	3E-8
0.5	0.2445336856	0.2445335607	0.2445337142	2.9E-08	1.5E-07	3E-8
0.6	0.2907378954	0.2907374735	0.2907378738	2.2E-08	4.0E-07	2E-8
0.7	0.3356190220	0.3356178611	0.3356188285	1.9E-07	9.7E-07	2E-7
0.8	0.3790638165	0.3790610713	0.3790632154	6.0E-07	2.1E-06	6E-7
0.9	0.4209923703	0.4209865896	0.4209909600	1.4E-06	4.4E-06	2E-6
1	0.4613550965	0.4613439980	0.4613522661	2.8E-06	8.3E-06	1E-5
1.1	0.5001289124	0.5001091382	0.5001235667	5.3E-06	1.4E-05	5E-5
1.2	0.5373130531	0.5372799303	0.5373040831	9.0E-06	2.4E-05	2E-4
1.3	0.5729248872	0.5728722173	0.5729110066	1.4E-05	3.9E-05	7E-4
1.4	0.6069959702	0.6069158871	0.6069755444	2.0E-05	6.0E-05	2E-3
1.5	0.6395684291	0.6394513739	0.6395398260	2.9E-05	8.8E-05	5E-3

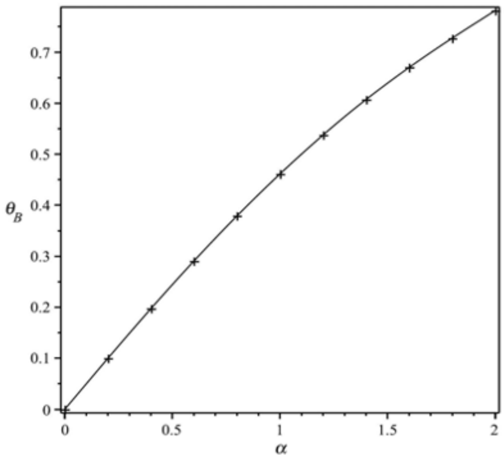

Fig. 3. The rotation angle at the free end, θ_B , versus α obtained by DTM (solid line), VIM (cross).

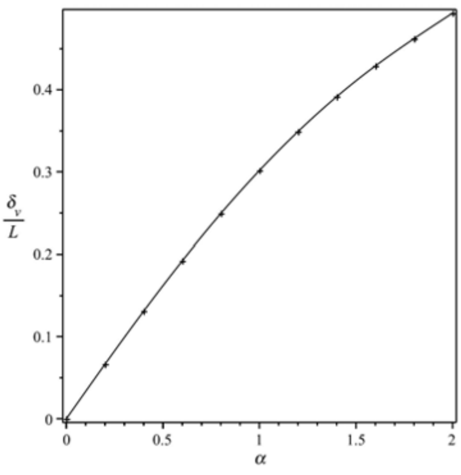

Fig. 4. The dimensionless vertical displacement, δ_v/L , versus α at the tip obtained by DTM (solid line), VIM (cross).

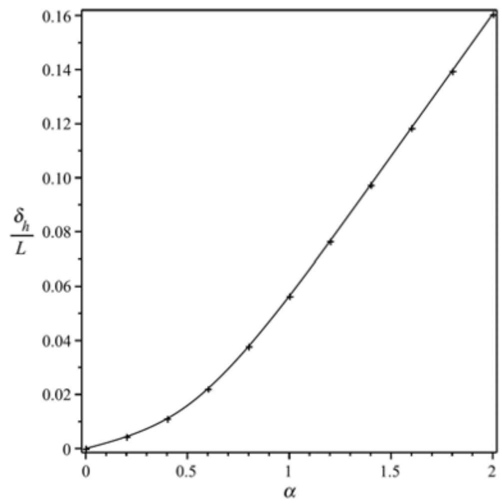

Fig. 5. The dimensionless horizontal displacement, δ_h/L , versus α at the tip obtained by DTM (solid line), VIM (cross).

Furthermore, for $\alpha > 1$ it can be concluded that the horizontal displacement varies linearly with α . As can be seen clearly in these figures the DTM and VIM results are in good agreement with each other. Needless to say that, since the DTM results are more accurate compare with the VIM, as cited in Table 2, the displacements which are achieved by the DTM are more reliable than the VIM.

It is worth mentioning that for infinitesimal deformation i.e. Eq. (11), these two analytical solutions lead into Eq. (12) which gives the exact solution.

7. Conclusions

Two extremely simple and elementary but rigorous the DTM and VIM have been utilized to derive approximate explicit analytical solutions for non-linear equation of large deformation in cantilever beams under point load. The results show that these schemes provide excellent approximations to the solution of this non-linear equation with high accuracy. However, the DTM solution is more accurate compare with the VIM. These methods accelerated the convergence to the solutions. Finally, it has been attempted to show the capabilities and wide-range applications of the mentioned methods in comparison with the numerical solution in solving deflection of beams problems.

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