Springer

KSME

Journal of Mechanical Science and Technology 26 (9) (2012) 2685~2694
www.springerlink.com/content/1738-494x
DOI 10.1007/s12206-012-0713-z

Free vibration analysis of orthotropic plates with variable thickness resting on non-uniform elastic foundation by element free Galerkin method †

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(Manuscript Received December 3, 2011; Revised April 20, 2012; Accepted May 10, 2012)

Abstract

This study intends to investigate the vibration behavior of a thin square orthotropic plate resting on non-uniform elastic foundation and its thickness varying in one or two directions. By using the classical plate theory and employing element free Galerkin method, it is shown that the fundamental frequency coefficients obtained are in good agreement with available results in the literature. The effects of thickness variation, foundation parameter and boundary conditions on frequency are investigated. The results show that the method converges very fast regardless of parameters involved.

Keywords: Element free Galerkin method; Non-uniform elastic foundation; Orthotropic thin plate; Plate of variable thickness; Vibration of plate

1. Introduction

Plates of non-uniform thickness are commonly used in ship and offshore structures. The variable thickness is used to change resonant frequency and to reduce the weight and size of the structures. Therefore, the vibration analysis of the plates with variable thickness is of great importance for researches. Analytical solutions for static analysis of plates with varying thickness have been obtained by various authors. In particular, Xu and Zhou [1, 2] studied three-dimensional elasticity solution for simply supported rectangular plates with variable thickness. Three-dimensional thermoelastic analysis of isotropic rectangular plates with variable thickness subjected to thermo-mechanical loads was investigated by Xu et al. [3]. Cheung and Zhou [4] used the Rayleigh-Ritz method for vibration analysis of a wide range of isotropic non-uniform rectangular plates in one or two directions. Using a new set of admissible functions in the Rayleigh-Ritz method, Cheung and Zhou [5] analyzed the free vibrations of isotropic tapered rectangular plates with an arbitrary number of intermediate line supports. The free vibration of isotropic point-supported rectangular plates with variable thickness using the Rayleigh-Ritz method was investigated by Cheung and Zhou [6]. Cheung and Zhou [7] studied vibrations of isotropic tapered Mindlin plates in terms of static Timoshenko beam functions.

Orthotropic rectangular plates with non-uniform thickness have been found to have great advantages, such as high strength to weight ratio, corrosion resistance and low cost. Comprehensive understanding of vibration behavior of orthotropic plates is important in many engineering fields. Malhotra et al. [8] employed the Rayleigh-Ritz method to study vibration of orthotropic square thin plates with parabolic variation thickness along one direction. They presented results for four types of boundary condition. Bert and Malik [9] studied the free vibration of isotropic and orthotropic rectangular thin plates of linearly varying thickness in one direction by the differential quadrature method.

Bambil et al. [10] used the Rayleigh-Ritz method and finite element method to study transverse vibration of orthotropic rectangular plate of linearly varying thickness in one direction. They presented the results of fundamental frequency for plates with a free edge. Ashur [11] investigated the flexural vibration of orthotropic plates of linearly varying thickness in one direction using the finite strip transition matrix technique. The results were obtained for plates with two opposite edges having the same boundary conditions and the same thickness. Huang et al. [12] applied a discrete method for analyzing the free vibration problem of orthotropic rectangular plates with variable thickness in one and two directions. They considered the effect of aspect ratio and different boundary condition on frequencies.

In vibration analysis of plates, the effect of elastic foundation must considered when plates are mounted on elastic springs such as pavement of roads, footing of buildings and bases of machines. The vibration analysis of plates resting on elastic foundation has been the subject of many researches. Gajendra [13] and Datta [14] used the one-term Galerkin

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method for solution of Berger's equations to analyze non-linear free vibration of simply and clamped supported thin isotropic circular plates resting on elastic foundation. Zhou et al. [15] studied three-dimensional free vibration of isotropic thick circular plates on Pasternak foundation. Bhaskar and Dumir [16] employed Von-Karman dynamic equation and orthogonal point collocation method to analyze non-linear vibration of two edges simply supported or clamped orthotropic thin plate resting on elastic foundation. Omurtag and Kadioglu [17] used mixed finite element formulation to study free vibration analysis of orthotropic Kirchhoff plate resting on elastic foundation. They presented the results of fundamental frequency for fully simply supported and clamped plates. Gupta and Bhardwaj [18] used the Rayleigh-Ritz method for vibration analysis of rectangular orthotropic elliptic plates with varying thickness resting on elastic foundation. Hsu [19] used differential quadrature method for vibration analysis of rectangular plate on elastic foundation. Liu et al. [20] used Galerkin's method to analyze the free vibration of orthotropic rectangular plates with tapered varying thickness in one or two directions and resting on Winkler type elastic foundation.

Element free Galerkin (EFG) method is considered as one of the meshless methods and is used for solution of many engineering problems. Yan et al. [21] used this method for vibration analysis of isotropic rectangular plate with interior elastic point support and elastically restrained edges. Chen et al. [22] and Dai et al. [23] used EFG method for vibration analysis of laminated composite plates.

The previous publications have concentrated on vibration of orthotropic plates resting on uniform elastic foundation. It is the main aim of present work to apply EFG method to study free vibration of orthotropic plates of variable thickness resting on non-uniform one-parameter elastic foundation with general boundary conditions. The plate is discretized by a set of regular nodes. On the basis of classical plate theory (CTP) a basic equation of vibration is derived. Moving least square (MLS) approximation is employed to produce displacement shape functions. Penalty method is used for imposing essential boundary condition. Different examples are solved to show accuracy, convergence and applicability of the EFG method.

2. Moving least square approximation (MLS)

Moving least square approximation is widely used for generation of shape function in mesh free methods. In this method, function $u(\mathbf{x})$ is approximated by $u^h(\mathrm{x})$, as follows:

$$u^h(\mathrm{x})=\sum_{j=1}^{m} p_j(\mathrm{x})a_j(\mathrm{x}) = \mathrm{p}^{\mathrm{T}}(\mathrm{x})\mathrm{a}(\mathrm{x}) \tag{1}$$

where m is the number of terms in the basis, $p_j(\mathrm{x})$ are the monomial basis functions, and $a_j(\mathrm{x})$ are unknown coefficients which depend on location x. Quadratic basis in two-dimensional domain has following form:

$$\mathrm{P}^{\mathrm{T}} = [1, x, y, x^2, xy, y^2]\,. \tag{2}$$

The unknown coefficients $a_j(\mathrm{x})$ in Eq. (1) can be determined by minimizing a functional of weighted residual

$$J = \sum_{I=1}^{n} w(\mathbf{x}-\mathbf{x}_I)[\mathbf{p}^{\mathrm{T}}(\mathbf{x}_I)\mathbf{a}(\mathbf{x}) - u_I]^2 \tag{3}$$

where u_I are nodal parameters of field variable at node I and n is the number of nodes in the neighborhood of $\boldsymbol{x}$ which called domain of influence. The unknown coefficient $\mathbf{a}(\mathbf{x})$ can be obtained by minimizing the functional of the weighted residual as follows:

$$\frac{\partial J}{\partial \mathbf{a}} = 0 \tag{4}$$

which yields following system of linear equations:

$$\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\mathbf{U}_s \tag{5}$$

where

$$\mathbf{A}(\mathbf{x}) = \sum_{I=1}^{n} w(\mathbf{x}-\mathbf{x}_I)\mathbf{p}(\mathbf{x}_I)\mathbf{p}^{\mathrm{T}}(\mathbf{x}_I)\,, \tag{6}$$

$$\mathbf{B}(\mathbf{x}) = [w(\mathbf{x}-\mathbf{x}_I)\mathbf{p}(\mathbf{x}_I),\ldots,w(\mathbf{x}-\mathbf{x}_n)\mathbf{p}(\mathbf{x}_n)]\,, \tag{7}$$

$$\mathbf{U}_s = \{u_1 \;\; u_2 \;\ldots\; u_n\}^{\mathrm{T}}\,. \tag{8}$$

By substituting Eq. (5) into Eq. (1), MLS approximant can be written as follows:

$$u^h(\mathbf{x}) = \sum_{I=1}^{n} \phi_I(\mathbf{x})u_I = \mathbf{\Phi}(\mathbf{x})\mathbf{U}_\mathbf{s} \tag{9}$$

where $\mathbf{\Phi}(\mathbf{x})$ is matrix of shape functions and defined as follows:

$$\mathbf{\Phi}(\mathbf{x}) = [\phi_1(\mathbf{x}) \;\; \phi_2(\mathbf{x}) \;\; \ldots \;\; \phi_n(\mathbf{x})] = \mathbf{p}^{\mathrm{T}}(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})\,. \tag{10}$$

To calculate partial derivatives of $\Phi(\mathrm{x})$, Eq. (10) is rewritten as follows [24]:

$$\mathbf{\Phi}(\mathbf{x}) = \gamma^{\mathrm{T}}(\mathbf{x})\mathbf{B}(\mathbf{x}) \tag{11}$$

where

$$\gamma(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x})\mathbf{p}(\mathbf{x}) \tag{12}$$

or

$$\mathbf{A}(\mathbf{x})\gamma(\mathbf{x}) = \mathbf{p}(\mathbf{x}) . \quad (13)$$

The partial derivatives of $\gamma(\mathbf{x})$ can be expressed as follows:

$$\mathbf{A}\gamma_{,x} = \mathbf{p}_{,x} - \mathbf{A}_{,x}\gamma , \quad (14)$$

$$\mathbf{A}\gamma_{,y} = \mathbf{p}_{,y} - \mathbf{A}_{,y}\gamma , \quad (15)$$

$$\mathbf{A}\gamma_{,xx} = \mathbf{p}_{xx} - (\mathbf{A}_{,xx}\gamma + 2\mathbf{A}_{,x}\gamma_{,x}) , \quad (16)$$

$$\mathbf{A}\gamma_{,xy} = \mathbf{p}_{,xy} - (\mathbf{A}_{,xy}\gamma + \mathbf{A}_{,x}\gamma_{,y} + \mathbf{A}_{,y}\gamma_{,x}) , \quad (17)$$

$$\mathbf{A}\gamma_{,yy} = \mathbf{p}_{,yy} - (\mathbf{A}_{,yy}\gamma + 2\mathbf{A}_{,y}\gamma_{,y}) . \quad (18)$$

The partial derivatives of Φ(x) would be as follows:

$$\mathbf{\Phi}_{I,x} = \gamma_{,x}^{\mathrm{T}}\mathbf{B}_I + \gamma^{\mathrm{T}}\mathbf{B}_{I,x} , \quad (19)$$

$$\mathbf{\Phi}_{I,y} = \gamma_{,y}^{\mathrm{T}}\mathbf{B}_I + \gamma^{\mathrm{T}}\mathbf{B}_{I,y} , \quad (20)$$

$$\mathbf{\Phi}_{I,xx} = \gamma_{,xx}^{\mathrm{T}}\mathbf{B}_I + 2\gamma_{,x}^{\mathrm{T}}\mathbf{B}_{I,x} + \gamma^{\mathrm{T}}\mathbf{B}_{I,xx} , \quad (21)$$

$$\mathbf{\Phi}_{I,xy} = \gamma_{,xy}^{\mathrm{T}}\mathbf{B}_I + \gamma_{,x}^{\mathrm{T}}\mathbf{B}_{I,y} + \gamma_{,y}^{\mathrm{T}}\mathbf{B}_{I,x} + \gamma^{\mathrm{T}}\mathbf{B}_{I,xy} , \quad (22)$$

$$\mathbf{\Phi}_{I,yy} = \gamma_{,yy}^{\mathrm{T}}\mathbf{B}_I + 2\gamma_{,y}^{\mathrm{T}}\mathbf{B}_{I,y} + \gamma^{\mathrm{T}}\mathbf{B}_{I,yy} . \quad (23)$$

Weight function plays an important role in the formulation of the MLS method. This function should be non-zero in domain of influence and zero outside of the domain. The precise character of this function seems to be unimportant although it is almost mandatory that it be positive and increase monotonically as $\|x\text{-}x_I\|$ decreases. Furthermore, it is desirable that weight function be smooth [25]. Several weight functions have been used in the EFG method. In this work quartic spline weight function is chosen which satisfies above mentioned conditions

$$w(\mathbf{x}-\mathbf{x}_I) \equiv w(r) = \begin{cases} 1-6r^2+8r^3-3r^4, & r \le 1 \\ 0, & r > 1 \end{cases} \quad (24)$$

where

$$r = \frac{\|\mathbf{x}-\mathbf{x}_I\|}{d_I} \quad (25)$$

where d_I determines the size of support domain at node *I*. The most commonly used supports are circles and rectangles (Fig. 1).

For circular domain, d_I is the radius of circle; for rectangular domain, d_I is equal to length of rectangle in *x*, and *y* direction. For the latter case, weight function can be written as follows:

$$w(\mathbf{x}-\mathbf{x}_I) = w(r_x).w(r_y) \quad (26)$$

where r_y and r_x *are* given by

$$r_x = \frac{\|x-x_I\|}{d_{Ix}} \quad (27)$$

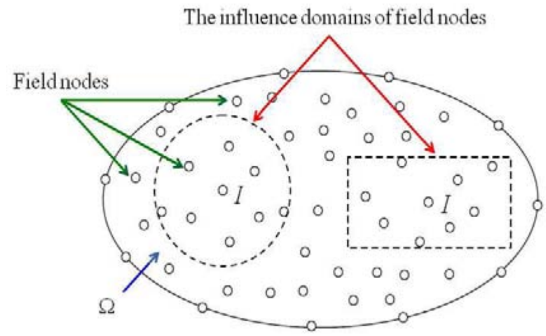

Fig. 1. Domain of influences.

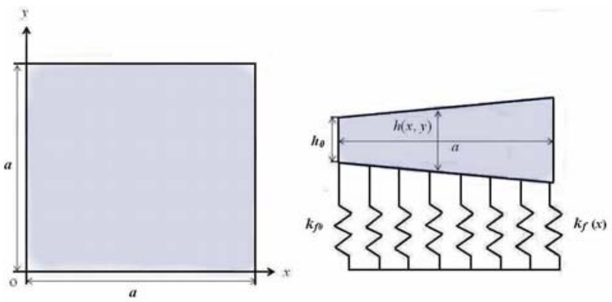

Fig. 2. A thin plate with variable thickness resting on non-uniform elastic foundation.

$$r_y = \frac{\|y-y_I\|}{d_{Iy}} \quad (28)$$

where

$$d_{Ix} = d_{\max}.c_{Ix} , \quad (29)$$

$$d_{Iy} = d_{\max}.c_{Iy} \quad (30)$$

where d_{max} is scaling parameter, c_{Ix} and c_{Iy} are determined by searching for enough nodes at the neighborhood for **A** in Eq. (13) to be invertible at every point in the domain [25].

3. Basic equations

3.1 Strain and kinetic energies of thin plate resting on elastic foundation

A thin square plate with thickness h(x,y), density ρ and resting on non-uniform Winkler-type elastic foundation is considered (Fig. 2).

Displacements in *x*, *y* and *z* directions are denoted by *u*, *v* and *w*, respectively. Based on CPT displacement field can be expressed as follows [26]:

$$\mathbf{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} -z\dfrac{\partial}{\partial x} & -z\dfrac{\partial}{\partial y} & 1 \end{Bmatrix}^{\mathrm{T}} w = \mathbf{L}_u w . \quad (31)$$

The pseudo-strains of the plate are denoted as

$$\boldsymbol{\varepsilon}_p = \left\{ -\frac{\partial^2}{\partial x^2} \quad -\frac{\partial^2}{\partial y^2} \quad -2\frac{\partial^2}{\partial x \partial y} \right\}^{\mathrm{T}} w = \mathbf{L}w. \tag{32}$$

The pseudo-stresses of the plate are given by

$$\boldsymbol{\sigma}_p = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}. \tag{33}$$

The relationship between the pseudo-strains and pseudo-stress is expressed as

$$\boldsymbol{\sigma}_p = \mathbf{D}\boldsymbol{\varepsilon}_p \tag{34}$$

where D is the stiffness matrix and defined as:

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}. \tag{35}$$

For an orthotropic plate, with the reference coordinate axes coinciding with the principal material directions, D would be as follows:

$$\mathbf{D} = \frac{h^3}{12}\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \tag{36}$$

where

$$Q_{11} = \frac{E_1}{(1-\nu_{12}\nu_{21})}, \tag{37}$$

$$Q_{22} = \frac{E_2}{(1-\nu_{12}\nu_{21})}, \tag{38}$$

$$Q_{12} = \frac{\nu_{12}E_2}{(1-\nu_{12}\nu_{21})}, \tag{39}$$

$$Q_{66} = G_{12} \tag{40}$$

where E_1 and E_2 are the Young moduli parallel to and perpendicular to the fibers, while ν_{12} and ν_{21} are the corresponding Poisson ratios.

The strain and kinetic energies of the plate resting on Winkler's type elastic foundation can be written as

$$U_p = \frac{1}{2}\int_S \boldsymbol{\varepsilon}_P^{\mathrm{T}} \boldsymbol{\sigma}_P \,\mathrm{d}S + \frac{1}{2}\int_S \mathbf{u}^{\mathrm{T}}\ \mathbf{q}\ \mathrm{d}S, \tag{41}$$

$$T_p = \frac{1}{2}\int_V \rho \dot{\mathbf{u}}^{\mathrm{T}}\ \dot{\mathbf{u}}\ \mathrm{d}V \tag{42}$$

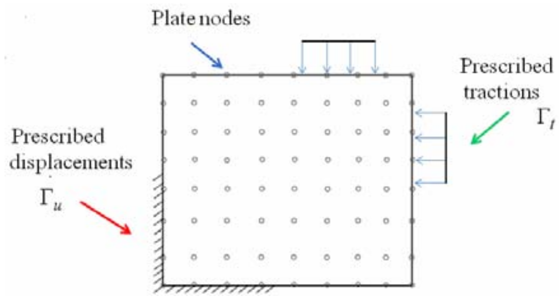

Fig. 3. Prescribed boundary conditions of the plate.

where $\mathbf{q}$ is the non-uniform surface pressure vector induce by foundation and is defined as follows:

$$\mathbf{q} = \left\{0 \quad 0 \quad k_f(x)w\right\}^{\mathrm{T}} \tag{43}$$

where k_f (x) is the non-uniform Winkler foundation stiffness which linearly varying along x-direction.

3.2 Approximation of field variables

Based on the CPT only the deflection w is independent and is approximated as follows:

$$w^h(\mathbf{x}) = \sum_{I=1}^{n} \phi_I(\mathbf{x}) w_I\ . \tag{44}$$

3.3 Imposing essential boundary conditions

As standard FEM, the EFG method uses the weak form of the problem to describe the equations of motion. Different variational principles can be used, depending upon the methods of enforcing the essential boundary conditions in the EFG formulation [27]. In this paper, a penalty method is applied to enforced essential boundary conditions. By assuming no body forces and prescribed tractions, Lagrangian of free vibration of a thin plate resting on elastic foundation can be written as follows:

$$L = T_P - U_p + \int_{\Gamma_u} \frac{1}{2}(\tilde{\mathbf{u}} - \bar{\mathbf{u}})^{\mathrm{T}} \boldsymbol{\alpha} (\tilde{\mathbf{u}} - \bar{\mathbf{u}}) \mathrm{d}\Gamma \tag{45}$$

where $\boldsymbol{\alpha}$ is a diagonal matrix of penalty coefficients, $\bar{\mathbf{u}}$ is prescribed displacement and $\tilde{\mathbf{u}}$ is defined as [26]

$$\tilde{\mathbf{u}} = \mathbf{L}_b w \tag{46}$$

where $\mathbf{L}_b$ is a vector of differential operators. Proper choice of penalty coefficients is very important on accuracy of solution. Usually large numbers of order $1 \times 10^{4\text{-}13} \times$ max (diagonals in stiffness matrix) can be chosen [26]. A plate with general boundary condition is considered (Fig. 3). For clamped boundary condition $\mathbf{L}_b$ is defined as,

$$\mathbf{L}_b = \left\{ 1 \quad \frac{\partial}{\partial n} \right\}^{\mathrm{T}} \tag{47}$$

where n denotes the outward normal direction on boundary of the plate. For simply supported boundary condition $\mathbf{L}_b$ is defined as

$$\mathbf{L}_b = \{1 \quad 0\}^{\mathrm{T}}. \tag{48}$$

3.4 Derivation of stiffness and mass matrices for free vibration analysis

The dynamical equation of plate resting on elastic foundation can be derived according to Hamilton's variational principle

$$\delta \int_{t_1}^{t_2} L\,\mathrm{d}t = 0 \tag{49}$$

where L is the Lagrangian function of system and t is time. By substituting Eqs. (41) and (42) into (45) and replacing in Eq. (49), following variational form is found:

$$\begin{aligned} &\int_S \delta \boldsymbol{\varepsilon}_P^{\mathrm{T}} \boldsymbol{\sigma}_P \,\mathrm{d}S + \int_S \delta \mathbf{u}^{\mathrm{T}} \mathbf{q}\,\mathrm{d}S + \int_V \rho \delta \mathbf{u}^{\mathrm{T}} \ddot{\mathbf{u}}\,\mathrm{d}V \\ &-\delta \int_{\Gamma_u} \frac{1}{2} (\tilde{\mathbf{u}} - \bar{\mathbf{u}})^{\mathrm{T}} \boldsymbol{\alpha} (\tilde{\mathbf{u}} - \bar{\mathbf{u}}) d\Gamma = 0\,. \end{aligned} \tag{50}$$

Substituting Eqs. (31), (32) and (46) into Eq. (50), it can be rewritten as:

$$\begin{aligned} &\int_S \delta (\mathbf{L}w)^{\mathrm{T}} \mathbf{D}\,(\mathbf{L}w) \mathrm{d}S \\ &+\int_S \delta (\mathbf{L}_u w)^{\mathrm{T}} \mathbf{q}\,\mathrm{d}S + \int_V \rho \delta (\mathbf{L}_u w)^{\mathrm{T}} \mathbf{L}_u \ddot{w}\,\mathrm{d}V \\ &-\frac{1}{2} \int_{\Gamma_u} \delta (\mathbf{L}_b w - \bar{\mathbf{u}})^{\mathrm{T}} \boldsymbol{\alpha} (\mathbf{L}_b w - \bar{\mathbf{u}}) \mathrm{d}\Gamma = 0\,. \end{aligned} \tag{51}$$

By substituting the approximated deflection function $w^h(\mathbf{x})$ (Eq. (44)) into Eq. (51), the discrete dynamical equations for free vibration analysis of plates resting on elastic foundation are deduced as:

$$\mathbf{M}\ddot{\mathbf{U}} + (\mathbf{K} + \tilde{\mathbf{K}})\mathbf{U} = 0 \tag{52}$$

where $\mathbf{U}$ is the vector of deflection of all nodes, and is defined by

$$\mathbf{U} = \{w_1, w_2, \ldots, w_{n_t}\}^{\mathrm{T}} \tag{53}$$

where n_t is the total number of nodes in the entire domain of the plate. The notations of $\mathbf{K}$, $\tilde{\mathbf{K}}$ and $\mathbf{M}$ denote the global stiffness matrix, the global penalty matrix, and global mass matrix which are given by:

$$\mathbf{K} = \int_S \mathbf{B}^{\mathrm{T}} \mathbf{D}\,\mathbf{B}\,\mathrm{d}S + \int_S \boldsymbol{\Phi}^{\mathrm{T}} k_f(x) \boldsymbol{\Phi}\,\mathrm{d}S, \tag{54}$$

$$\mathbf{M} = \int_V \rho \mathbf{N}^T \mathbf{N}\,\mathrm{d}V, \tag{55}$$

$$\tilde{\mathbf{K}} = \int_{\Gamma_u} \boldsymbol{\Psi}^T \boldsymbol{\alpha} \boldsymbol{\Psi}\,\mathrm{d}\Gamma \tag{56}$$

where

$$\mathbf{B} = \mathbf{L}\boldsymbol{\Phi} = \begin{Bmatrix} -\boldsymbol{\Phi}_{,xx} \\ -\boldsymbol{\Phi}_{,yy} \\ -2\boldsymbol{\Phi}_{,xy} \end{Bmatrix}, \tag{57}$$

$$\mathbf{N} = \mathbf{L}_u \boldsymbol{\Phi} = \begin{Bmatrix} -z\,\boldsymbol{\Phi}_{,x} \\ -z\,\boldsymbol{\Phi}_{,y} \\ \boldsymbol{\Phi} \end{Bmatrix}. \tag{58}$$

$\boldsymbol{\Psi}$ depends on boundary condition. For clamped boundary condition, $\boldsymbol{\Psi}$ is defined as follows:

$$\boldsymbol{\Psi} = \begin{Bmatrix} \boldsymbol{\Phi} \\ \boldsymbol{\Phi}_{,n} \end{Bmatrix} \tag{59}$$

and for simply support boundary condition is defined as follows:

$$\boldsymbol{\Psi} = \begin{Bmatrix} \boldsymbol{\Phi} \\ 0 \end{Bmatrix}. \tag{60}$$

Assuming a harmonic vibration form for plate, the deflection vector $\mathbf{U}$ can be expressed as:

$$\mathbf{U} = \bar{\mathbf{U}} e^{i\omega t} \tag{61}$$

where $\bar{\mathbf{U}}$ is the amplitude of the vibration and ω is the circular frequency.

Substituting Eq. (61) into (52), the following eigenvalue equation would be obtained:

$$(\mathbf{K} - \omega^2 \mathbf{M})\bar{\mathbf{U}} = 0 \tag{62}$$

where ω^2 is the eigenvalue and represent the square of circular frequency of transverse vibration and $\bar{\mathbf{U}}$ is the eigenvector that represent the vector of amplitude of transverse vibration .

4. Worked-out examples and discussion

In this paper, scaling parameter d_{max} is chosen as 3.9 [28]. To show the accuracy and convergence of the present method, a computer code in Matlab was developed and some examples have been solved.

The thickness variation function of plate is chosen as follows:

$$h(x,y) = h_0(1+\alpha x/a)(1+\beta y/a).$$

The linear variation function of stiffness of foundation is given as follows:

$$k_f(x) = k_{f0}(1+\gamma x/a).$$

E-glass/ epoxy material with the following properties is considered:

$E_1 = 60.7$ GPa, $E_2 = 24.8$ GPa, $G_{12} = 12.0$ GPa, $\nu_{12} = 0.23$.

To facilitate comparison of results, following dimensionless parameters are defined.

1-Natural frequency of isotropic is defined by

$$\bar{\omega} = \frac{\omega a^2}{\pi^2}\sqrt{\frac{\rho h}{D}} \tag{63}$$

where D is the bending stiffness of plate and defined as follows:

$$D = \frac{Eh^3}{12(1-\nu^2)}.$$

2-Foundation parameter for isotropic plate

$$K_{iso} = \frac{k_f a^4}{D}. \tag{64}$$

3- Natural frequency of orthotropic plate

$$\bar{\omega} = a\sqrt[4]{\frac{\omega^2 \rho h_0}{D_0(1-\nu_{12}\nu_{21})}} \tag{65}$$

where D_0 is defined as follows:

$$D_0 = \frac{E_2 h_0^3}{12(1-\nu_{12}\nu_{21})}. \tag{66}$$

4-Foundation parameter for orthotropic plate,

$$K_{orth} = \frac{12 k_{f0} a^4 (1-\nu_{12}\nu_{21})}{E_1 h^3}. \tag{67}$$

4.1 Free vibration of an isotropic square plate resting on elastic foundation

Vibration of an isotropic square plate resting on elastic foundation is analyzed. Different foundation parameters and thickness ratios are considered for two cases of boundary conditions: all edges simply supported (SSSS) and all edges clamped (CCCC). Dimensionless natural frequencies for the first 4th modes are calculated and compared with available results (Tables 1-2). The results are compared with those of Zhou et al. [29] (Ritz method), Ferreira et al. [30] (radial basis function method), Omurtag et al. [31] (mixed finite element formulation) and Xiang et al. [32] (Mindlin approach).

As can be seen from Table 1, for lower modes of vibration very good agreements are achieved for different thickness and foundation parameters even with very small number of nodes as 5 × 5. For higher modes, the present method yields good results with larger number of nodes as 9 × 9. When plate thickness increases, there is less agreement between present method and others for higher modes. This could be due to shear effect which is apparent in thicker plate. For orthotropic plate with CCCC boundary conditions, when the number of nodes is very small (5 × 5) the results are inaccurate (Table 2).

Table 1. Dimensionless parameter of natural frequency, $\bar{\omega}$ of an isotropic square plate with SSSS boundary conditions resting on Winkler's type elastic foundation ($\nu = 0.3$).

h/a	K_1	Method	Mode number			
			1st	2nd	3rd	4th
0.01	100	Pres. (5×5)	2.2514	6.9628	7.3435	15.701
		Pres. (9×9)	2.2431	5.1232	5.1286	8.1092
		Pres. (15×15)	2.2427	5.1107	5.1109	8.0735
		Ref. [29]	2.2413	5.0973	5.0973	8.0527
		Ref. [32]	2.2413	5.0971	5.0971	8.0523
		Ref. [30]	2.2414	5.0967	5.0967	8.0542
	500	Pres. (5×5)	3.0296	7.3458	7.7542	15.831
		Pres. (9×9)	3.0228	5.5093	5.5144	8.3587
		Pres. (15×15)	3.0225	5.4976	5.4978	8.3238
		Ref. [29]	3.0214	5.4850	5.4850	8.3035
		Ref. [32]	3.0215	5.4850	5.4850	8.3032
		Ref. [30]	3.0216	5.4846	5.4846	8.3051
0.1	200	Pres. (5×5)	2.4492	6.9107	7.2902	15.032
		Pres. (9×9)	2.4415	5.1192	5.1245	7.9186
		Pres. (15×15)	2.4411	5.1072	5.1074	7.8842
		Ref. [29]	2.3951	4.8262	4.8262	7.2338
		Ref. [32]	2.3989	4.8194	4.8194	7.2093
		Ref. [30]	2.3989	4.8194	4.8194	7.2093
	1000	Pres. (5×5)	3.7531	7.6327	8.0640	15.280
		Pres. (9×9)	3.7471	5.8392	5.8438	8.3916
		Pres. (15×15)	3.7468	5.8286	5.8287	8.3586
		Ref. [29]	3.7008	5.5661	5.5661	7.7335
		Ref. [32]	3.7212	5.5844	5.5844	7.7353
		Ref. [30]	3.7213	5.5844	5.5844	7.7353

* Numbers in parentheses refer to number of nodes in present method.

4.2 Free vibration of an orthotropic plate with varying thickness without elastic foundation

In the next step, vibration of an orthotropic plate with varying thickness is investigated. Tables 3 and 4 give the first 6th

Table 2. Dimensionless parameter of natural frequency $\bar{\omega}$ of an isotropic square plate with CCCC boundary conditions resting on Winkler's type elastic foundation ($\nu = 0.15$).

h/a	K_1	Method	Mode number			
			1st	2nd	3rd	4th
0.015	1390.2	Pres. (5×5)	8.0625	313.859	443.564	490.411
		Pres. (9×9)	5.2841	8.5491	8.5824	12.193
		Pres. (15×15)	5.2616	8.3914	8.3924	11.672
		Ref. [31]	5.2446	8.3156	8.3156	11.541
		Ref. [29]	5.2588	8.4322	8.4322	11.674
		Ref. [30]	5.2438	8.3129	8.3129	11.546
	2780.4	Pres. (5×5)	9.5780	362.739	512.641	566.773
		Pres. (9×9)	6.4953	9.3479	9.3789	12.771
		Pres. (15×15)	6.4769	9.2017	9.2027	12.268
		Ref. [31]	6.4629	9.1324	9.1324	12.142
		Ref. [29]	6.4601	9.2482	9.2482	12.263
		Ref. [30]	6.4625	9.1302	9.1302	12.147

* Numbers in parentheses refer to number of nodes in present method.

modes of natural frequencies for several boundary condition and thickness variation parameters. In all cases the ratio of thickness to length of plate is considered as 0.01. Examination of Tables 3 and 4 reveals that present method yields good results for vibration analysis of orthotropic plate with varying thickness.

4.3 Free vibration of an orthotropic plate with varying thickness resting on non-uniform elastic foundation

In the end, free vibration of an orthotropic plate with varying thickness resting on non-uniform elastic foundation has been studied. The effects of foundation parameter K_{orth}, variation parameter of stiffness of foundation γ, thickness variation parameters α and β, and boundary conditions have been investigated (Figs. 4 to 7).

Figs. 4 and 5 show the behavior of fundamental frequency parameter of an SSSS plate versus the variation parameter of stiffness of foundation γ for four different combinations of

Table 3. Dimensionless parameter of natural frequency $\bar{\omega}$ of an orthotropic square plate with variable thickness in one direction.

B.C	α	Method	Mode number					
			1st	*2nd*	*3rd*	*4th*	*5th*	*6th*
SSSS	0.0	Pres.(15×15)	4.9002	7.2562	8.3823	9.7997	10.1161	11.9501
		Ref. [12]	4.902	7.253	8.374	9.795	10.079	11.924
	0.4	Pres.(15×15)	5.3599	7.9314	9.1597	10.7146	11.0255	13.0771
		Ref. [12]	5.360	7.928	9.150	10.703	10.982	13.043
	0.8	Pres.(15×15)	5.7721	8.5298	9.8453	11.5289	11.7924	14.0924
		Ref. [12]	5.770	8.525	9.831	11.510	11.740	14.048
CCCC	0.0	Pres.(15×15)	6.7484	8.9048	10.2376	11.5664	11.6555	13.6193
		Ref. [12]	6.780	8.953	10.293	11.615	11.686	13.636
	0.4	Pres. (15×15)	7.3707	9.7225	11.1823	12.6420	12.7042	14.8974
		Ref. [12]	7.402	9.770	11.232	12.679	12.730	14.896
	0.8	Pres. (15×15)	7.9150	10.4322	12.0088	13.5895	13.5916	16.0405
		Ref. [12]	7.945	10.475	12.046	13.610	13.602	16.008
SSSC	0.0	Pres. (15×15)	5.2349	7.8344	8.4850	10.1042	10.7647	12.2377
		Ref. [12]	5.238	7.855	8.483	10.100	10.756	12.178
	0.4	Pres. (15×15)	5.7270	8.5589	9.2729	11.0507	11.7223	13.3650
		Ref. [12]	5.824	8.463	9.195	11.214	11.801	13.360
	0.8	Pres. (15×15)	6.1693	9.1948	9.9686	11.8965	12.5183	14.3490
		Ref. [12]	6.164	9.219	9.966	11.865	12.488	14.264
SCFC	0.0	Pres. (15×15)	4.8828	6.4843	7.9550	9.0959	9.6177	11.0582
		Ref. [12]	4.901	6.486	8.030	9.183	9.615	11.287
	0.4	Pres. (15×15)	5.5030	7.1047	8.9302	10.0324	10.4893	12.3231
		Ref. [12]	5.529	7.128	8.976	9.999	10.470	12.480
	0.8	Pres. (15×15)	6.0506	7.6760	9.7135	10.9257	11.2639	13.2116
		Ref. [12]	6.065	7.676	9.772	10.971	11.225	13.498
CSCS	0.0	Pres. (15×15)	6.3370	7.9288	10.0948	10.4357	11.0852	12.8111
		Ref. [12]	6.361	7.941	10.149	10.408	11.125	12.814
	0.4	Pres. (15×15)	6.9212	8.6599	11.0251	11.3855	12.1103	14.0039
		Ref. [12]	6.945	8.670	11.074	11.350	12.141	13.993
	0.8	Pres. (15×15)	7.4317	9.2989	11.8371	12.1999	13.0093	15.0597
		Ref. [12]	7.454	9.305	11.874	12.155	13.026	15.026
SSFS	0.0	Pres. (15×15)	3.5268	5.9534	6.5210	8.1283	9.4256	9.6292
		Ref. [12]	3.533	5.945	6.509	8.129	9.410	9.571
	0.4	Pres. (15×15)	3.9192	6.4899	7.3183	8.9307	10.2703	10.7809
		Ref. [12]	3.916	6.485	7.310	8.917	10.243	10.712
	0.8	Pres. (15×15)	4.2823	6.9783	8.0051	9.6649	11.0191	11.6635
		Ref. [12]	4.280	6.967	7.994	9.647	9.647	11.590

Table 4. Dimensionless parameter of natural frequency $\bar{\omega}$ of an orthotropic square plate with variable thickness in two directions.

B.C	α	β	Method	Mode number					
				1st	*2nd*	*3rd*	*4th*	*5th*	*6th*
SSSS	-0.5	-0.5	Pres. (15×15)	3.633	5.3460	6.0957	7.2323	7.3945	8.6625
			Ref. [12]	3.635	5.335	6.086	7.221	7.358	8.616
	-0.5	0.5	Pres. (15×15)	4.707	6.9422	7.9751	9.3885	9.5829	11.4636
			Ref. [12]	4.704	6.937	7.966	9.372	9.536	11.425
	0.5	-0.5	Pres. (15×15)	4.708	6.9420	7.9146	9.4104	9.6304	11.2629
			Ref. [12]	4.708	6.933	7.904	9.397	9.590	11.207
	0.5	0.5	Pres. (15×15)	6.0997	9.0128	10.359	12.1923	12.5082	14.8640
			Ref. [12]	6.086	9.022	10.350	12.136	12.439	14.858
CCCC	-0.5	-0.5	Pres. (15×15)	4.9361	6.5319	7.4167	8.4838	8.5834	10.0083
			Ref. [12]	4.955	6.548	7.440	8.502	8.533	9.989
	-0.5	0.5	Pres. (15×15)	6.4265	8.4814	9.7152	11.0179	11.1029	13.0914
			Ref. [12]	6.453	8.510	9.748	11.070	11.056	13.031
	0.5	-0.5	Pres. (15×15)	6.4203	8.5006	9.6385	11.0721	11.1413	12.9973
			Ref. [12]	6.447	8.525	9.671	11.103	11.108	12.993
	0.5	0.5	Pres. (15×15)	8.3598	11.0404	12.632	14.3719	14.4321	16.9525
			Ref. [12]	8.390	11.076	12.666	14.389	14.418	16.887
SSSC	-0.5	-0.5	Pres. (15×15)	3.8678	5.7167	6.2601	7.4914	7.7991	8.8361
			Ref. [12]	3.872	5.716	6.252	7.483	7.778	8.786
	-0.5	0.5	Pres. (15×15)	5.0309	7.5004	8.0312	9.6768	10.1684	11.4820
			Ref. [12]	5.038	7.499	8.015	9.678	10.148	11.433
	0.5	-0.5	Pres. (15×15)	5.0101	7.4316	8.1239	9.7270	10.1904	11.4859
			Ref. [12]	5.016	7.442	8.115	9.717	10.169	11.423
	0.5	0.5	Pres. (15×15)	6.5174	9.7528	10.4296	12.5434	13.3135	14.9329
			Ref. [12]	6.536	9.729	10.398	12.580	13.311	14.855
SCFC	-0.5	-0.5	Pres. (15×15)	3.4105	4.8053	5.4065	6.6735	7.0561	7.3755
			Ref. [12]	3.431	4.815	5.475	6.730	7.064	7.410
	-0.5	0.5	Pres. (15×15)	4.4339	6.2513	7.0334	8.6835	9.2230	9.5974
			Ref. [12]	4.467	6.303	6.989	8.628	9.206	9.717
	0.5	-0.5	Pres. (15×15)	4.8281	6.2126	7.8221	8.7692	9.1089	10.7546
			Ref. [12]	4.831	6.231	7.879	8.771	9.060	10.936
	0.5	0.5	Pres. (15×15)	6.2841	8.0762	10.1762	11.4172	11.8812	13.9901
			Ref. [12]	6.301	8.084	10.235	11.429	11.849	14.200
CCCS	-0.5	-0.5	Pres. (15×15)	4.7201	6.1800	7.2384	8.0610	8.3354	9.7410
			Ref. [12]	4.734	6.184	7.258	8.047	8.336	9.706
	-0.5	0.5	Pres. (15×15)	6.259	8.009	9.720	10.421	10.891	12.629
			Ref. [12]	6.2347	7.9912	9.6852	10.4441	10.8784	12.6795
	0.5	-0.5	Pres. (15×15)	6.158	8.052	9.444	10.522	10.832	12.613
			Ref. [12]	6.1398	8.0422	9.4116	10.5236	10.8227	12.6342
	0.5	0.5	Pres. (15×15)	8.137	10.417	12.630	13.605	14.132	16.382
			Ref. [12]	8.1110	10.3970	12.5950	13.6315	14.1263	16.4413
SSCC	-0.5	-0.5	Pres. (15×15)	4.2330	5.9495	6.7686	7.8845	8.0345	9.3860
			Ref. [12]	4.248	5.959	6.779	7.875	8.015	9.329
	-0.5	0.5	Pres. (15×15)	5.4687	7.7854	8.6994	10.1831	10.4270	12.2118
			Ref. [12]	5.481	7.795	8.715	10.187	10.412	12.198
	0.5	-0.5	Pres. (15×15)	5.6281	7.6423	8.9166	10.2252	10.3462	12.2241
			Ref. [12]	5.637	7.653	8.922	10.270	10.274	12.191
	0.5	0.5	Pres. (15×15)	7.2421	9.9894	11.4799	13.2585	13.3844	15.8434
			Ref. [12]	7.248	9.991	11.475	13.252	13.359	15.785

thickness variation parameters α and β and two values of foundation parameter K_{orth}. It is observed that the fundamental frequency parameter increases with the increasing values of variation parameter of stiffness of foundation γ and foundation parameter K_{orth} regardless of the value of thickness variation parameters. Fig. 6 shows the graph of fundamental frequency parameter of a CCCC plate versus the variation parameter of foundation stiffness γ for four different combinations of thickness variation parameters α and β when foundation parameter K_{orth} is 10. It can be seen that the fundamental frequency parameter has been increased as compared to SSSS. Fig. 7 shows the variation of fundamental frequency parameter against thickness variation parameter α in x direction, when thickness variation parameter β in y direction is equal to 0 and foundation parameter K_{orth} = 1000. As can be seen, the fundamental frequency parameter has been decreased very

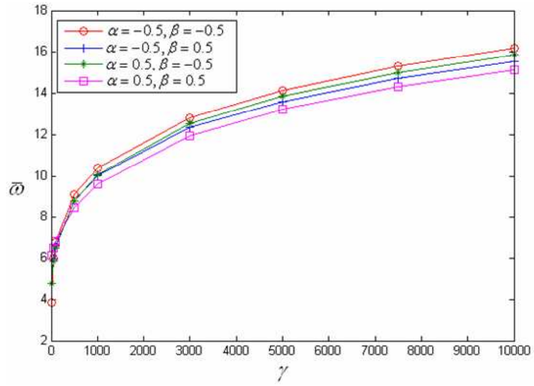

Fig. 4. Natural frequency parameter for a bilinear varying thickness resting on non-uniform elastic foundation (K_{orth} = 10, SSSS boundary condition).

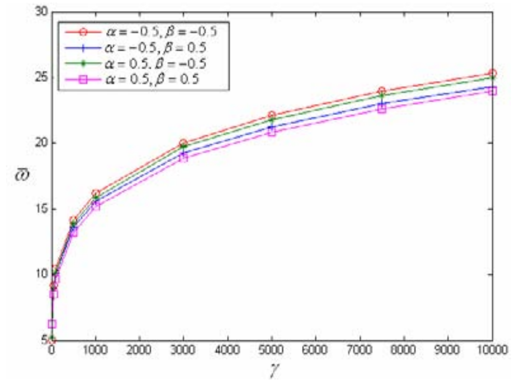

Fig. 5. Natural frequency parameter for a bilinear varying thickness resting on non-uniform foundation (K_{orth} = 100, SSSS boundary condition).

slightly by increasing in the thickness variation parameter α. Also, the frequency parameter increases with increasing values of variation parameter of foundation stiffness γ.

5. Conclusions

Element free Galerkin method has been used for free vibration analysis of thin orthotropic plates with variable thickness and resting on non-uniform elastic foundation. Accuracy and convergence of solution was examined by comparing the numerical results obtained by the present method with those previously published. It is observed that the fundamental frequency increases with increasing values of foundation parameter and variation parameter of foundation stiffness. It was found that by increasing values of thickness variation parameters, the fundamental frequency is decreased and the rate of decreasing is different.

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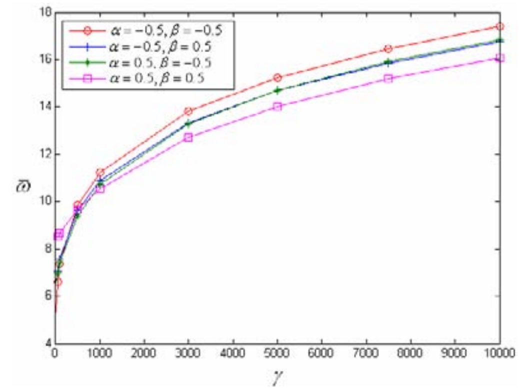

Fig. 6. Natural frequency parameter a bilinear varying thickness plate resting on non-uniform foundation (K_{orth} = 10, CCCC boundary condition).

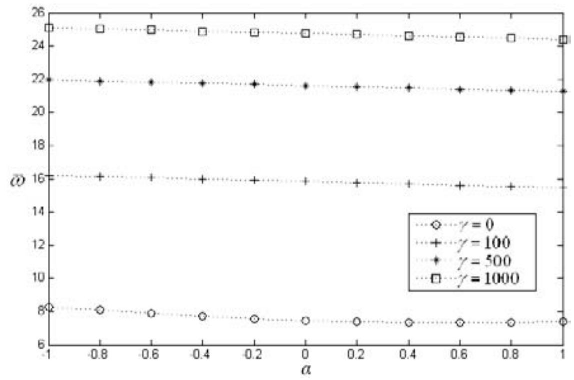

Fig. 7. Natural frequency parameter a bilinear varying thickness plate and resting on non-uniform foundation (K_{orth} = 1000, β = 0, SSSS boundary condition).

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