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Influence of yield stress on free convective boundary-layer flow of a non-Newtonian nanofluid past a vertical plate in a porous medium †

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Abstract

The effect of yield stress on the free convective heat transfer of dilute liquid suspensions of nanofluids flowing on a vertical plate saturated in porous medium under laminar conditions is investigated considering the nanofluid obeys the mathematical model of power-law. The model used for non-Newtonian nanofluid incorporates the effects of Brownian motion and thermophoresis. The governing boundary-layer equations are cast into dimensionless system which is solved numerically using a deferred correction technique and Newton iteration. This solution depends on yield stress parameter Ω, a power-law index *n*, Lewis number *Le*, a buoyancy-ratio number *Nr*, a Brownian motion number *Nb*, and a thermophoresis number *Nt*. Analyses of the results found that the reduced Nusselt and Sherwood numbers are decreasing functions of the higher yield stress parameter for each dimensionless numbers, *n* and *Le*, except the reduced Sherwood number is an increasing function of higher *Nb* for different values of yield stress parameter.

Keywords: Non-Newtonian; Free convection; Nanofluid; Porous media; Yield stress

1. Introduction

Nowadays after a century of struggling for enhancing industrial heat transfer by fluid mechanics, the low thermal conductivity of conventional fluids such as water, oil, and Ethylene-Glycol (EG) for transferring the heat has been one of the great challenges on the heat transfer science. One of the ways to overcome this problem is to replace conventional fluids with some advanced fluids with higher thermal conductivities. Maxwell's study in 1873 [1] shows the possibility of increasing the thermal conductivity of a fluid–solid mixture by increasing volume fraction of solid particles. Thus, the particles with micrometer or even millimeter dimensions were used. Those particles caused several problems such as abrasion, clogging and pressure losses. During the past decade the technology to make particles in nanometer dimensions was improved and a new kind of solid–liquid mixture that is called nanofluid was appeared [2, 3]. The nanofluid is an advance kind of fluid containing small quantity of nanoparticles (usually less than 100 nm) that are uniformly and stably suspended in a liquid. The dispersion of a small amount of solid nanoparticles in conventional fluids such as water or Ethylene-Glycol changes their thermal conductivity remarkably (Masuda et al. [4]). This phenomenon suggests the possibility of using nanofluids in advanced nuclear systems (Buongiorno and Hu [5]). Buongiorno and Hu [5] noticed that several authors have suggested that convective heat transfer enhancement could be due to the dispersion of the suspended nanoparticles but he argues that this effect is too small to explain the observed enhancement. He also concludes that turbulence is not affected by the presence of the nanoparticles so this cannot explain the observed enhancement. In another paper, Buongiorno [6] has pointed out that the nanoparticle absolute velocity can be viewed as the sum of the base fluid velocity and a relative velocity (that he calls the slip velocity). He considered in turn seven slip mechanisms: inertia, Brownian diffusion, thermophoresis, diffusiophoresis, Magnus effect, fluid drainage, and gravity settling (Nield and Kuznetsov [7]). Buongiorno [6] proceeded to write down conservation equations based on these two effects. Numerous models and methods have been proposed by different authors to study convective flows of nanofluids and we mention here the papers by Khanafer et al. [8], Daungthongsuk and Wongwises [9], Tiwari and Das [10], Wang and Wei [11], Oztop and Abu-Nada [12], Nield and Kuznetsov [7], Kuznetsov and Nield [13-15], etc.

Non-Newtonian fluids in porous media exhibit a nonlinear behavior that is different from that of Newtonian fluids in porous media. The research on the heat and mass transfer for

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non-Newtonian fluids in porous media is very important due to its practical engineering applications, such as oil recovery, food processing, and materials processing. Chen and Chen [16] presented similarity solutions for natural convection of a non-Newtonian fluid over vertical surfaces in porous media. Chen and Chen [17] also studied the natural convection of a non-Newtonian fluid about a horizontal cylinder and sphere in porous media. Nakayama and Koyama [18] studied the natural convection of a non-Newtonian fluid over non-isothermal body of arbitrary shape in a porous medium. Yang and Wang [19] investigated the natural convection heat transfer of non-Newtonian power law fluids with yield stress over axisymmetric and two dimensional bodies of arbitrary shape embedded in a fluid-saturated porous medium. Kim and Hyun [20] studied the natural convection heat transfer of power law fluid in an enclosure filled with heat-generating porous media. Rastogi and Poulikakos studied [21] the problem of double diffusion from a plate in a porous medium saturated with a non-Newtonian power law fluid. Getachew et al. [22] performed a numerical and theoretical study of double-diffusive natural convection in a rectangular porous cavity saturated by a non-Newtonian power law fluid. Jumah and Mujumdar [23] studied the free convection heat and mass transfer of non-Newtonian power law fluids with yield stress over a vertical plate in saturated porous media subjected to constant wall temperature and concentration. Jumah and Mujumdar [24] also studied the natural convection heat and mass transfer of non-Newtonian power law fluids with yield stress over a vertical plate in saturated porous media subjected to variable wall temperature and concentration. Hady et al. [25] reported the problem of non-Darcian free convection of a non-Newtonian fluid from a vertical sinusoidal wavy plate embedded in porous medium. Hady and Ibrahim [26] studied the effect of the presence of an isotropic solid matrix on the forced convection heat transfer rate from a flat plate to power-law non-Newtonian fluid-saturated porous medium. Mahdy and Hady [27] considered the present contribution deals with the effects of thermophoretic particle deposition of the free convective flow over a flat plate embedded in non-Newtonian fluid-saturated porous medium in the presence of a magnetic field. The effect of chemical reaction on free convection heat and mass transfer for a non-Newtonian power law fluid over a vertical flat plate embedded in a fluid-saturated porous medium has been studied in the presence of the yield stress and the Soret effect by Ibrahim et al. [28].

The principal aim of this work is to study the effect of yield stress on free convection boundary-layer flow past a vertical flat plate embedded in a porous medium filled with a nanofluid, the basic fluid being a non-Newtonian fluid. Based on the literature survey only the papers by Nield and Kuznetsov [7] have extended the paper by Cheng and Minkowycz [29] on free convection boundary layer flow past a vertical flat plate embedded in a porous medium to the case when porous medium is filled with a nanofluid. Numerous shapes and methods have been proposed by different authors to study convective flows of nanofluids and we mention here the papers by Kuznetsov and Nield [13-15], Tzou [30], Nield and Kuznetsov [31], Khan and Pop [32], Bachok et al. [33], Ahmad and Pop [34]. In these papers the authors have used the nanofluid model proposed by Buongiorno [6].

2. Basic equations

We consider the steady two-dimensional boundary layer flow of a non-Newtonian nanofluid past a vertical surface with the threshold gradient $\alpha_0 = a\tau_0 / \sqrt{K}$, where a is a constant, τ_0 yield stress and K is the permeability for the porous medium. We select a coordinate frame in which the x -axis is aligned vertically upwards. We supposed a vertical plate at $y = 0$. At this boundary the temperature T and the nanoparticles fraction ϕ take constant values T_w and ϕ_w , respectively. The ambient values, attained as y tends to infinity, of T and ϕ are denoted by T_∞ and ϕ_∞ , respectively. We assumed that a porous medium whose porosity is denoted by ε . The Darcy velocity is denoted by **v**, where $\mathbf{v} \equiv (u, v)$. We assume that Oberbeck–Boussinesq approximation is employed, and an assumption that the nanoparticles concentration is dilute. The basic steady conservation of mass, momentum, thermal energy and nanoparticles equations for nanofluids can be written in Cartesian coordinates x and y as, (see Refs. [7, 13]).

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u^n = \frac{K}{\mu}\left[(1-\phi_\infty)\rho_{\mathrm{f}\infty}\beta g(T-T_\infty) - (\rho_{\mathrm{p}} - \rho_{\mathrm{f}\infty})g(\phi-\phi_\infty) - \alpha_0\right],$$
$$\text{if } (1-\phi_\infty)\rho_{\mathrm{f}\infty}\beta g(T-T_\infty) - (\rho_{\mathrm{p}} - \rho_{\mathrm{f}\infty})g(\phi-\phi_\infty) > \alpha_0$$
$$u = 0, \quad \text{if } (1-\phi_\infty)\rho_{\mathrm{f}\infty}\beta g(T-T_\infty) - (\rho_{\mathrm{p}} - \rho_{\mathrm{f}\infty})g(\phi-\phi_\infty) \le \alpha_0, \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_m \frac{\partial^2 T}{\partial y^2} + \tau\left[D_B \frac{\partial \phi}{\partial y}\frac{\partial T}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right], \tag{3}$$

$$\frac{1}{\varepsilon}\left(u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y}\right) = D_B \frac{\partial^2 \phi}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} \tag{4}$$

subject to the boundary conditions

$$\begin{aligned} &v = 0; T = T_w; \phi = \phi_w \text{ at } y = 0 \\ &u = v = 0; T \to T_\infty; \phi \to \phi_\infty \text{ as } y \to \infty. \end{aligned} \tag{5}$$

Here, u and v are the velocity components along the axes x and y , respectively, n is the viscosity index of the non-Newtonian power-law nanofluid. ρ_{f} , μ and β are the density, viscosity, and volumetric expansion coefficient of the fluid while ρ_p is the density of the particles. The gravitational acceleration is denoted by g . Thermal diffusivity $\alpha_m = k_m / (\rho c)_{\mathrm{f}}$ is the ratio between the effective thermal conductivity k_m and the heat capacity of non-Newtonian fluid and $\tau = \varepsilon(\rho c)_{\mathrm{p}} / (\rho c)_{\mathrm{f}}$ is the ratio between the effective heat capacity of the nanoparticles material and heat capacity of the fluid with ρ being the density, c is the specific heat capacity. D_B is the Brownian diffusion coefficient and D_T is the thermophoretic diffusion coefficient.

We look for a similarity solution of Eqs. (1)-(4) with the boundary conditions (5) of the following form:

$$\psi = \alpha Ra_x^{\frac{1}{2}} s(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad f(\eta) = \frac{\phi - \phi_\infty}{\phi_w - \phi_\infty}, \quad \eta = \frac{y}{x} Ra_x^{\frac{1}{2}} \tag{6}$$

where the stream function $\psi(x,y)$ is defined usually as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$; and Ra_x is the modified Rayleigh number defined by:

$$Ra_x = \left(\frac{x}{\alpha_m}\right)\left[\frac{(1-\phi_\infty)K\rho_{f\infty}\beta g(T_w - T_\infty)}{\mu}\right]^{\frac{1}{n}}. \tag{7}$$

Substituting variables (6) into Eqs. (1)-(5), we obtain the following system of ordinary differential equations:

$$s' = [\theta - \mathrm{Nr} f - \Omega]^{\frac{1}{n}}; \quad \text{if } \theta - \mathrm{Nr} f > \Omega \qquad s' = 0; \quad \text{if } \theta - \mathrm{Nr} f \le \Omega , \tag{8}$$

$$\theta'' + \frac{1}{2} s\theta' + \mathrm{Nb} f'\theta' + \mathrm{Nt}\theta'^2 = 0, \tag{9}$$

$$f'' + \frac{1}{2}\mathrm{Le} f' s + \frac{\mathrm{Nt}}{\mathrm{Nb}}\theta'' = 0 \tag{10}$$

along with the boundary conditions

$$s(0) = 0, \theta(0) = 1, f(0) = 1, \quad s'(\infty) = 0, \theta(\infty) = 0, f(\infty) = 0 \tag{11}$$

where primes denote differentiation with respect to η and the parameters are defined by:

$$Nr = \frac{(\rho_p - \rho_{f\infty})(\phi_w - \phi_\infty)}{\rho_{f\infty}\beta(1-\phi_\infty)(T_w - T_\infty)}, \tag{12}$$

$$Nb = \varepsilon \frac{(\rho c)_p D_B(\phi_w - \phi_\infty)}{(\rho c)_f \alpha_m}, \tag{13}$$

$$Nt = \varepsilon \frac{(\rho c)_p D_T (T_w - T_\infty)}{(\rho c)_f \alpha_m T_\infty}, \tag{14}$$

$$Le = \frac{\alpha_m}{\varepsilon D_B}, \tag{15}$$

$$\Omega = \frac{\alpha_0}{(1-\phi_\infty)\rho_{f\infty}\beta g(T_w - T_\infty)}. \tag{16}$$

Here, Le, Nr, Nb, Nt and Ω denote the Lewis number, the buoyancy ratio, the Brownian motion parameter, the thermophoresis parameter and the yield stress parameter, respectively. It is important to note that this boundary value problem reduces to the classical problem of heat and mass transfer of non-Newtonian fluid flow past a vertical surface when Nr, Nb, Nt and Ω are zero in Eqs. (8)-(10). The boundary value problem for f then becomes ill-posed and is of no physical significance. The quantities of practical interest, in this study are the Nusselt number Nu and the Sherwood number Sh which are defined as:

$$Nu = \frac{xq_w}{k_m(T_w - T_\infty)}, \quad Sh = \frac{xq_m}{D_B(\phi_w - \phi_\infty)} \tag{17}$$

where q_w and q_m are the wall heat and mass fluxes. Using variables (11), we obtain:

$$Nu / Ra_x^{\frac{1}{2}} = -\theta'(0) , \quad Sh / Ra_x^{\frac{1}{2}} = -f'(0) . \tag{18}$$

3. Results and discussion

The system of equations in Eqs. (8)-(10) subject to the boundary conditions (11) has been solved numerically using an implicit finite-difference scheme depending on a deferred correction technique and Newton iteration method for some values of the governing parameters n, Ω and Le. The problem for a regular (Newtonian) nanofluid involves just one independent parameter, namely the Lewis number Le in addition to the dimensionless numbers. The present extension involves two more independent parameters: n and Ω. Therefore, we need to be very selective in the choice of the values of the parameters. Since most nanofluids examined to date have large values of the Lewis number Le, we are interested mainly in the case $Le > 1$. It is also important that the computational time for each set of input parameter values should be short. Because the physical domain in this problem is unbounded, whereas the computational domain has to be finite, we apply the far field boundary conditions for the similarity variable η at a finite value denoted here by $\eta_{\max}$. We ran our bulk computations with the value $\eta_{\max} = 10$, which is sufficient to achieve the far field boundary conditions asymptotically for all values of the parameters considered.

The results for no yield stress ($\Omega = 0$) compared for the Newtonian case ($n = 1$) with those obtained by the study of Nield and Kuznetsov [7] and the present study for the reduced Nusselt number for different values of Le are given in Table 1. Here C_r, C_b and C_t are the coefficients in the linear regression estimate $Nu / Ra_x^{1/2} = 0.444 + C_r Nr + C_b Nb + C_t Nt$, applicable for Nr, Nb, Nt each in [0, 0.5]. Moreover the results of the case of regular non-Newtonian fluid when $Nr = Nb = Nt = 0$ with different values of Ω compared with those obtained by the study of Jumah and Mujumdar [24]. We notice that the comparison show good agreement for each value of Le. Therefore, we are sure that the present results are very accurate. Fig. 1 shows the effect of yield stress parameter Ω on dimensionless stream function, temperature

Table 1. Comparison between the linear regression coefficients for the study of Nield and Kuznetsov [7] and the present study.

	Nield and Kuznetsov [7]			Present study		
Le	C_r	C_b	C_t	C_r	C_b	C_t
1	-0.309	-0.060	-0.166	-0.30946	-0.05980	-0.16595
10	-0.111	-0.245	-0.150	-0.11108	-0.24456	-0.14980
50	-0.064	-0.288	-0.149	-0.06386	-0.28830	-0.14867
100	-0.053	-0.298	-0.148	-0.05306	-0.29795	-0.14856
500	-0.039	-0.310	-0.148	-0.03831	-0.31163	-0.14851
1000	-0.036	-0.313	-0.148	-0.03667	-0.31346	-0.14780

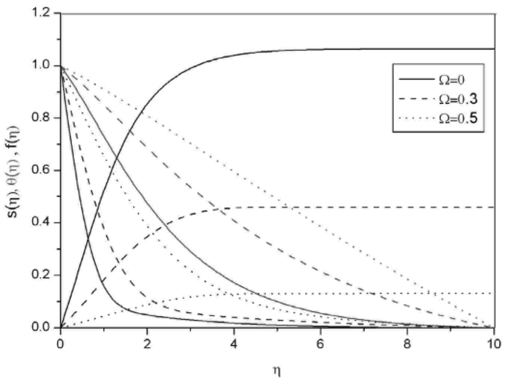

Fig. 1. Profiles of stream function $s(\eta)$, temperature function $\theta(\eta)$ and mass fraction function $f(\eta)$ for the case $\Omega = 0, 0.3, 0.5$ with $Nr = Nb = Nt = 0.3$, $n = 0.5$ and $Le = 10$.

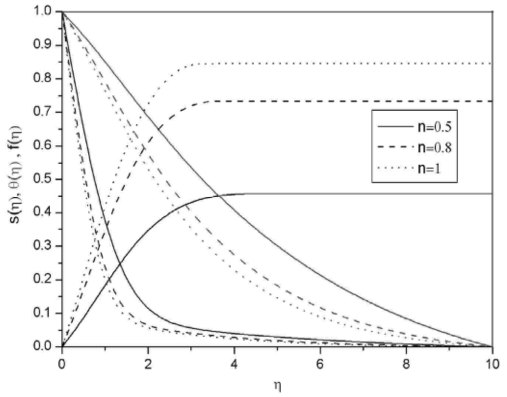

Fig. 2. Profiles of stream function $s(\eta)$, temperature function $\theta(\eta)$ and mass fraction function $f(\eta)$ for the case $n = 0.5, 0.8, 1$ with $Nr = Nb = Nt = 0.3$, $\Omega = 0.5$ and $Le = 10$.

function and mass fraction function (rescaled nanoparticles volume fraction) with $Nr = Nb = Nt = 0.3$, $n = 0.5$ and $Le = 10$. It is shown that the momentum boundary layer thickness decreases with Ω. On the other hand, the thermal and concentration boundary layer thicknesses increase as Ω increase. This means that higher values of heat and mass transfer rates are associated with small Ω. Figs. 2 and 3 depict the effect of n and Le respectively, on dimensionless stream function, temperature function and mass fraction function, when $Nr = Nb = Nt = 0.3$. Whereas Fig. 2 for different

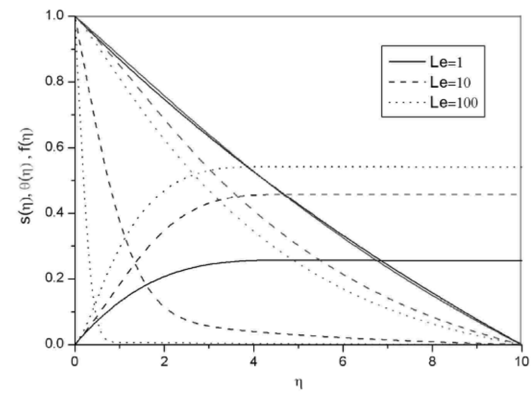

Fig. 3. Profiles of stream function $s(\eta)$, temperature function $\theta(\eta)$ and mass fraction function $f(\eta)$ for the case $Le = 1, 10, 100$ with $Nr = Nb = Nt = 0.3$, $n = 0.5$ and $\Omega = 0.3$.

values of n with $\Omega = 0.5$ and $Le = 10$ and Fig. 3 for different values Le with $\Omega = 0.3$ and $n = 0.5$. It is shown that $s(\eta)$ increase with n and Le, while $\theta(\eta)$ and $f(\eta)$ decrease with n and Le.

The effects of Ω, n and Le on the velocity profiles $s'(\eta)$ for the selected parameters are shown in Figs. 4-6. It is clear that the velocity decreases as the Ω increases, in the opposite to the case of variation with n and Le, we found that the velocity increase. Since our main objective is to explore the influence of the parameters Ω and n on the heat and mass fluxes characteristics, we will present here the results for the effect of these parameters considering various values of Ω and n. The variations of the reduced Nusselt number (heat transfer rate) $-\theta'(\eta)$ and the reduced Sherwood number (mass flux rate) $-f'(\eta)$ with $Nr = Nb = Nt = 0.5$ and $Le = 1$ and different values of Ω and n are presented in Figs. 7 and 8. It is seen that the heat and mass transfer rates are consistently higher for a nanofluid with smaller values of Ω, this mean that the Nusselt and Sherwood numbers are decreasing function of yield stress parameter, while the heat and mass transfer rates are higher for a nanofluid with higher values of n which as expected. Further, it is seen that explanation for the variation of Nusselt and Sherwood number with Ω and Le when $Nr = Nb = Nt = 0.5$ and $n = 0.5$, but the increase on Sherwood number is wide, which is depicted in Figs. 9 and 10.

The variation in dimensionless heat and mass transfer rates vs. Nr, Nb and Nt numbers are shown in Figs. 11 and 12. They show the effects of these dimensionless numbers Nr, Nb and Nt one by one on the dimensionless heat and mass transfer rates for the same Ω when $\Omega = 0, 0.2, 0.4$ with $n = 0.5$, $Le = 10$ and the other two $= 0.1$. It is clear that the dimensionless heat transfer rates decrease with the increase in Nr, Nb and Nt, but the dimensionless mass transfer rates decrease with Nr and Nt in the opposite in the case of Nb the Sherwood number increase. These results show that the Nusselt and Sherwood numbers are decreasing functions with Nr, Nb and Nt except the Sherwood number is increasing function with Nb for taking different values of Ω.

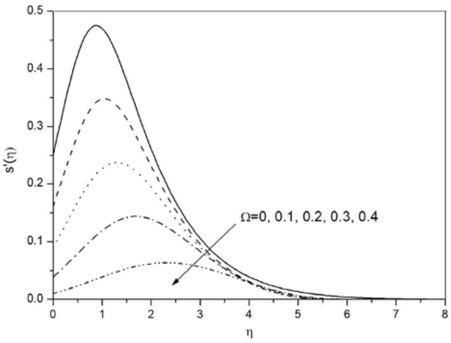

Fig. 4. Variation of the velocity distributions $s'(\eta)$ for various values of Ω with $Nr = Nb = Nt = 0.5$, $n = 0.5$ and $Le = 10$.

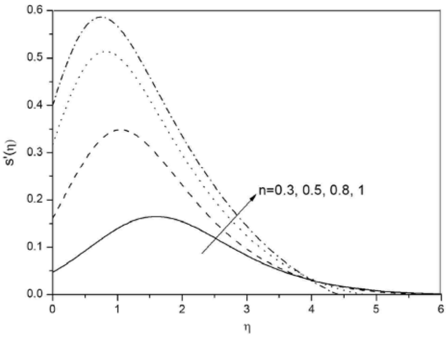

Fig. 5. Variation of the velocity distributions $s'(\eta)$ for various values of n with $Nr = Nb = Nt = 0.5$, $\Omega = 0.1$ and $Le = 10$.

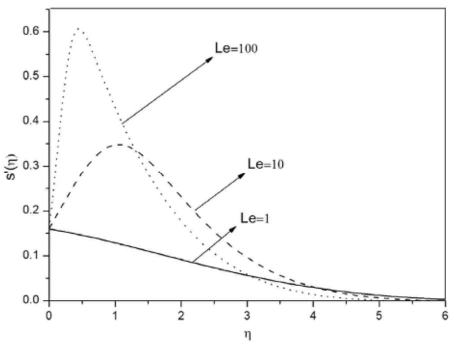

Fig. 6. Variation of the velocity distributions $s'(\eta)$ for various values of Le with $Nr = Nb = Nt = 0.5$, $\Omega = 0.1$ and $n = 0.5$.

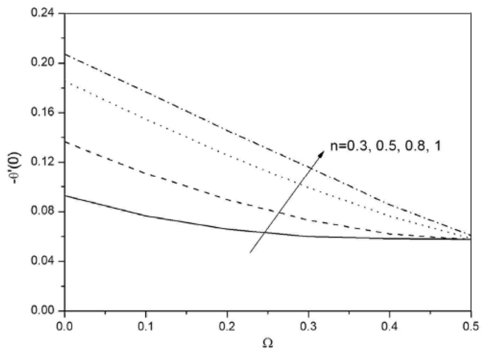

Fig. 7. Variation of the heat transfer rate $-\theta'(\eta)$ with Ω for various values of n with $Nr = Nb = Nt = 0.5$ and $Le = 1$.

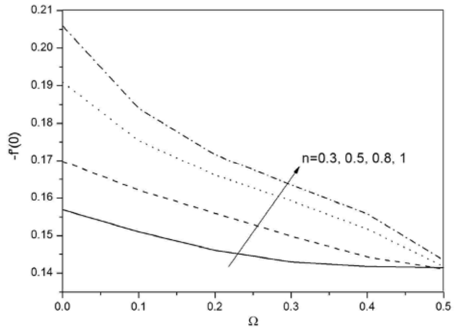

Fig. 8. Variation of the mass transfer rate $-f'(\eta)$ with Ω for various values of n with $Nr = Nb = Nt = 0.5$ and $Le = 1$.

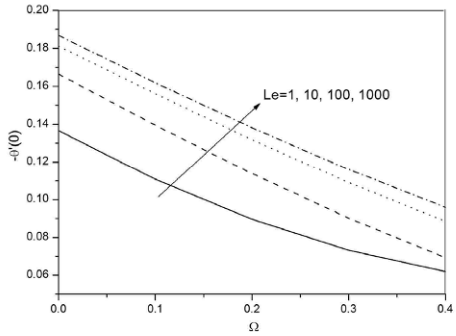

Fig. 9. Variation of the heat transfer rate $-\theta'(\eta)$ with Ω for various values of Le with $Nr = Nb = Nt = 0.5$ and $n = 0.5$.

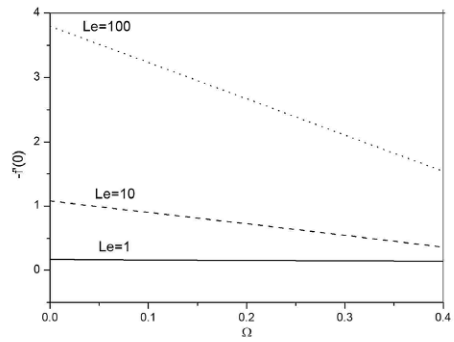

Fig. 10. Variation of the mass transfer rate $-f'(\eta)$ with Ω for various values of Le with $Nr = Nb = Nt = 0.5$ and $n = 0.5$.

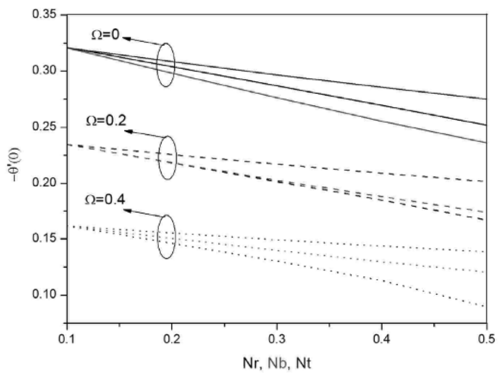

Fig. 11. Variation of the heat transfer rate $-\theta'(\eta)$ with Nr, Nb and Nt for various values of Ω with $n = 0.5$ and $Le = 10$.

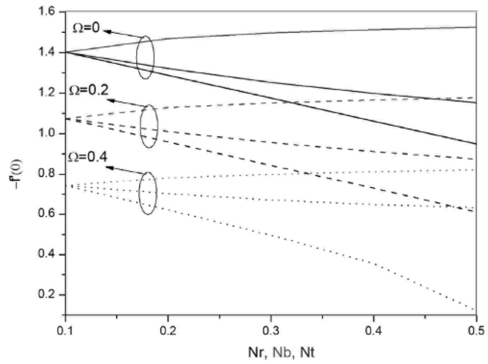

Fig. 12. Variation of the mass transfer rate $-f'(\eta)$ with Nr, Nb and Nt for various values of Ω with $n=0.5$ and $Le=10$.

The change in the dimensionless mass transfer rates is found to be higher than the change in dimensionless heat transfer rates for the various values of Nr, Nb and Nt.

4. Conclusions

In this article, we presented a boundary layer analysis for the natural convection past a vertical flat plate embedded in a porous medium saturated with a non-Newtonian nanofluid in the presence of yield stress effect. The governing partial differential equations are transformed into ordinary differential equations, a more convenient form for numerical computation, using a similarity transformation. These equations are solved numerically using the deferred correction technique and Newton iteration method. Numerical results for velocity distributions $s'(\eta)$, surface heat transfer rate $-\theta'(\eta)$, and mass transfer rate $-f'(\eta)$ have been presented for parametric variations of the dimensionless rheological parameter Ω, power index of non-Newtonian fluid n and Lewis number Le. The results indicate that as Ω increases, the velocity distribution, heat transfer rate (Nusselt number) and mass transfer rate (Sherwood number) increase, while the velocity, the heat transfer rate and mass transfer rate increase with n and Le increase. As Nr and Nt increase, the heat and mass transfer rates decrease, whereas the surface mass transfer rate increases with increase of Nb in the opposite heat transfer rate which decrease. From the results obtained in this paper the onset of free convection will occur provided that certain inequalities, which depend strongly on the yield stress, are satisfied.

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Nomenclature

a : Dimensionless constant
D_B : Brownian diffusion coefficient
D_T : Thermophoretic diffusion coefficient
f : Rescaled nanoparticle volume fraction, defined by Eq. (6)
g : Gravitational acceleration
k_m : Effective thermal conductivity
K : Permeability of the porous medium
Le : Lewis number, defined by Eq. (15)
n : Power index of non-Newtonian fluid, $n \geq 0$
Nr : Buoyancy ratio, defined by Eq. (12)
Nb : Brownian parameter, defined by Eq. (13)
Nt : Thermophoresis parameter, defined by Eq. (14)
Nu : Reduced Nusselt number, $Nu / Ra_x^{1/2}$
q_w : Wall heat flux
q_m : Wall mass flux
Ra_x : Modified Rayleigh number, defined by Eq. (7)
s : Dimensionless stream function, defined by Eq. (6)
T : Temperature
T_w : Temperature at the vertical plate
T_∞ : Ambient temperature attained as $y \to \infty$
u, v : Darcian velocity components in x - and y -directions
x, y : Cartesian coordinates

Greek symbols

α_0 : Threshold gradient
α_m : Thermal diffusivity, defined by Eq. (6)
β : Volumetric expansion coefficient of the non-Newtonian fluid
ε : Porosity
η : Similarity variable, defined by Eq. (6)
θ : Dimensionless temperature, defined by Eq. (6)
μ : Effective viscosity of non-Newtonian fluid
ρ_f : Density of the non-Newtonian fluid
ρ_p : Nanoparticles mass density
$(\rho c)_f$: Heat capacity of the fluid
$(\rho c)_p$: Effective heat capacity of the nanoparticles material
τ_0 : Yield stress
ϕ : Nanoparticles volume fraction
ϕ_w : Nanoparticles volume fraction at the vertical plate
ϕ_∞ : Ambient nanoparticles volume fraction attained as $y \to \infty$
ψ : Stream function, defined by Eq. (6)
Ω : Dimensionless rheological parameter, defined by Eq. (16)

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