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Numerical integration of discrete mechanical systems with mixed holonomic and control constraints †

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Abstract

The present work aims to incorporate control (or servo) constraints into finite-dimensional mechanical systems subject to holonomic constraints. In particular, we focus on underactuated systems, defined as systems in which the number of degrees of freedom exceeds the number of inputs. The corresponding equations of motion can be written in the form of differential-algebraic equations (DAEs) with a mixed set of holonomic and control constraints. Apart from closed-loop multibody systems, the present formulation accommodates the so-called rotationless formulation of multibody dynamics. The rotationless formulation has proven to be especially well-suited for the design of energy and momentum conserving schemes, which typically exhibit superior numerical stability properties (see [4, 7, 10]). Subsequent to the incorporation of the servo constraints, we deal with a reformulation of the underlying DAEs, which is amenable to a direct numerical discretization. To this end, we apply a specific projection method to the DAEs in terms of redundant coordinates. A similar projection approach has been previously developed in the framework of generalized coordinates by Blajer & Kolodziejczyk [12]. A numerical example is presented, which deals with a 3D rotary crane.

Keywords: Differential algebraic equations; Redundant coordinates; Trajectory tracking; Underactuated mechanical systems

1. Introduction

1.1 Comments on the rotationless formulation of multibody dynamics

It is quite obvious that the choice of parametrization for the description of rigid bodies and multibody systems has a profound effect on the numerical discretization process. In this connection, some kind of rotational parameters, such as Euler angles, rotation vectors, or joint angles, are commonly applied. In contrast, the rotationless formulation circumvents the use of rotational parameters.

A rotationless formulation of rigid body dynamics can be achieved by considering a continuum body in which the kinematic assumption of rigidity is enforced by means of (scleronomic) holonomic constraints (cf. [6]). In the three-dimensional regime, the rotation of a rigid body can thus be characterized by nine redundant coordinates, which coincide with the elements of the rotation (or direction-cosine) matrix. The assumption of rigidity is then reflected in the orthogonality of the rotation matrix, which can be enforced by six independent constraints. At first glance this approach appears to be quite awkward because of its high level of redundancy. However, the rotationless formulation has several advantageous features:

- The equations of motion for an individual rigid

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body are governed by differential-algebraic equations (DAEs). In a multibody framework, the interconnection between bodies can be accounted for by additional 'external' constraints, which can be easily appended to the DAEs ([8]). Thus the underlying DAEs provide a uniform framework for the description of both open-loop and closed-loop multibody systems.

- The equations of motion in terms of rotational parameters are generally characterized by cumbersome and highly nonlinear expressions. In contrast, the structure of the DAEs associated with the advocated rotationless formulation is quite simple. For example, the mass matrix in the DAE formulation is generally constant. Moreover, the angular momentum map is quadratic in the coordinates/velocities. By applying a direct discretization to the DAEs, robust energy consistent integrators can be designed. In particular, if the multibody system at hand can be classified as Hamiltonian system with symmetry, previously developed energy-momentum conserving schemes ([14, 7]) can be employed.
- The number of unknowns in the resulting algebraic formulation (i.e., redundant coordinates and Lagrange multipliers) can be significantly reduced by applying the so-called discrete null space method ([3, 17])
- The aforementioned DAEs not only provide a uniform framework for classical multibody dynamics but also cover appropriate semi-discrete formulations of nonlinear structural dynamics (see [9, 17]). Accordingly, the present approach makes possible the straightforward extension to flexible multibody dynamics by incorporating well-established nonlinear finite element formulations. In this connection, energy-momentum conserving time-stepping schemes (or energy-decaying variants thereof) turn out to be of paramount importance for a stable time integration (see, for example, Simo & Tarnow [19], Gonzalez & Simo [15], Bauchau & Bottasso [2], Betsch & Steinmann [5] and the references cited therein).

1.2 Incorporation of servo constraints

In the present work we aim to incorporate servo (or control) constraints into the underlying rotationless formulation of multibody dynamics. Our long-term goal is the control of underactuated flexible multibody systems.

The servo constraints can be easily appended to the DAEs pertaining to the rotationless formulation of multibody dynamics. Thus we are confronted with a mixed set of standard (ideal) constraints and servo constraints. We aim to discretize the extended set of DAEs, which retains the advantageous numerical properties outlined in Section 1.1. To this end, we apply a Blajer-type projection method ([11, 12]) to the control-extension of the DAEs. Owing to the underlying rotationless formulation, the projection matrices turn out to be of Boolean (or binary) type. This greatly alleviates the discretization of the resulting modified DAEs. It is worth mentioning that, as a special case, our approach contains standard finite element displacement control procedures.

The remaining part of the paper is structured as follows. In Section 2, we incorporate servo constraints into a general description of the dynamics of discrete mechanical systems subject to holonomic constraints. Section 3 contains two alternative reformulations of the original DAEs.

A simple discretization of the resulting DAEs is outlined in Section 4. In Section 5, the numerical example of a 3D rotary crane is considered. Finally, conclusions are drawn in Section 6.

2. Discrete mechanical systems with mixed holonomic and servo constraints

In the present work we focus on finite-dimensional mechanical systems with mixed holonomic and servo constraints. The motion of the discrete mechanical systems under consideration is governed by DAEs of the form

$$\boxed{\begin{aligned} \dot{\mathbf{q}} - \mathbf{v} &= \mathbf{0} \\ \mathbf{M}\dot{\mathbf{v}} + \nabla V(\mathbf{q}) + \mathbf{G}^T\boldsymbol{\lambda} + \mathbf{B}^T\mathbf{u} &= \mathbf{0} \\ \mathbf{c}(\mathbf{q},t) &= \mathbf{0} \\ \mathbf{g}(\mathbf{q}) &= \mathbf{0} \end{aligned}} \tag{1}$$

Here, $\mathbf{q}(t) \in \Re^n$ is the vector of redundant coordinates that specifies the configuration of the mechanical system at time t. The corresponding vector of redundant velocities is given by $\mathbf{v} = \dot{\mathbf{q}}$, where a superposed dot denotes differentiation with respect to time. Together, ($\mathbf{q}$, $\mathbf{v}$) form the vector of state space coordinates (see, for example, Rosenberg [18]).

A characteristic feature of the rotationless formulation of multibody systems is a *constant* and symmetric mass matrix $\mathbf{M} \in \Re^{n\times n}$. The kinetic energy of the system can be written as

$$T(\mathbf{v}) = \frac{1}{2}\mathbf{v}\cdot\mathbf{M}\mathbf{v} \tag{2}$$

For the present purposes it suffices to deal with conservative systems with associated potential energy function $V(\mathbf{q}) \in \Re$. Moreover, $\mathbf{g}(\mathbf{q}) \in \Re^{m_h}$ is a vector of geometric (or holonomic) constraint functions, $\mathbf{G} = D\mathbf{g}(\mathbf{q}) \in \Re^{m_h \times n}$ is the constraint Jacobian, and $\boldsymbol{\lambda} \in \Re^{m_h}$ is a vector of multipliers that specify the relative magnitude of the constraint forces. We assume that the m_h constraints are independent. Note that, because of the presence of the holonomic constraints, the configuration space of the system is given by

$$\mathbf{Q} = \{\mathbf{q}(t) \in \Re^n \,|\, \mathbf{g}(\mathbf{q}) = \mathbf{0}\} \tag{3}$$

Consequently, the constrained mechanical system has $\tilde{n} = n - m_h$ degrees of freedom.

Furthermore, in $(1)_3$, $\mathbf{c}(\mathbf{q},t) \in \Re^{\tilde{m}}$ is a vector of servo (or control) constraint functions, which may be written in the form

$$\mathbf{c}(\mathbf{q},t) = \boldsymbol{\Phi}(\mathbf{q}) - \boldsymbol{\gamma}(t) \tag{4}$$

The servo constraints serve the purpose of partially specifying the motion of underactuated mechanical systems ($\tilde{m} < \tilde{n}$). Servo constraints may be used for trajectory-tracking problems where the vector-valued function $\boldsymbol{\gamma}(t) \in \Re^{\tilde{m}}$ and $\boldsymbol{\Phi}(\mathbf{q}) \in \Re^{\tilde{m}}$ specify the output. The corresponding actuator forces are determined by the control inputs $\mathbf{u} \in \Re^{\tilde{m}}$ in conjunction with the input transformation matrix $\mathbf{B} \in \Re^{\tilde{m}\times n}$.

3. Two alternative reformulations of the underlying DAEs

We next aim at reformulating the DAEs (1), which is amenable to a direct discretization. In particular, we present two alternative reformulations.

The first formulation relies on the introduction of generalized coordinates and the subsequent application of a projection method. This formulation coincides with that originally proposed by Blajer [11] (see also Lam [16]). The second formulation is based on a direct application of the Blajer-type projection method to the original DAEs in terms of redundant coordinates.

3.1 Projected formulation in terms of generalized coordinates

3.1.1 Introduction of generalized coordinates

Assume that it is feasible to find $\tilde{n}$ generalized coordinates $\mu \in U \subset \Re^{\tilde{n}}$ for the parametrization of the configuration manifold Q. Then there exits a mapping $\mathbf{F}: U \to Q$ such that

$$\mathbf{q} = \mathbf{F}(\boldsymbol{\mu}) \tag{5}$$

Admissible velocities $\mathbf{v} \in T\mathbf{q}Q = \ker(\mathbf{G}(\mathbf{q}))$ can be written in the form

$$\mathbf{v} = \mathbf{P}\boldsymbol{\upsilon} \tag{6}$$

with generalized velocities $\boldsymbol{\upsilon} = \dot{\boldsymbol{\mu}}$ and the null space matrix $\mathbf{P} = D\mathbf{F}(\boldsymbol{\mu})$. Note that the columns of $\mathbf{P} \in \Re^{n\times\tilde{n}}$ span the null space of $\mathbf{G} \in \Re^{m_h \times n}$. This implies that

$$\mathbf{G}\mathbf{P} = \mathbf{0} \tag{7}$$

Using (6), the reduced form of the kinetic energy $\tilde{T}$ is given by

$$\tilde{T} = \frac{1}{2}\boldsymbol{\upsilon}\cdot\tilde{\mathbf{M}}\boldsymbol{\upsilon} \tag{8}$$

with the reduced mass matrix

$$\tilde{\mathbf{M}} = \mathbf{P}^T\mathbf{M}\mathbf{P} \tag{9}$$

Note that $\tilde{\mathbf{M}}$ is generally configuration-dependent and assumed to be positive-definite. Pre-multiplying $(1)_2$ by $\mathbf{P}^T$ and making use of (7) and (6) yields the alternative reduced formulation

$$\begin{aligned} \dot{\boldsymbol{\mu}} - \boldsymbol{\upsilon} &= \mathbf{0} \\ \tilde{\mathbf{M}}\dot{\boldsymbol{\upsilon}} + \mathbf{P}^T\mathbf{M}\dot{\mathbf{P}}\boldsymbol{\upsilon} + \nabla\tilde{V}(\boldsymbol{\mu}) + \tilde{\mathbf{B}}^T\mathbf{u} &= \mathbf{0} \\ \tilde{\mathbf{c}}(\boldsymbol{\mu},t) &= \mathbf{0} \end{aligned} \tag{10}$$

where

$$\tilde{\mathbf{c}}(\boldsymbol{\mu},t) = \tilde{\boldsymbol{\Phi}}(\boldsymbol{\mu}) - \boldsymbol{\gamma}(t) \tag{11}$$

follows from inserting (5) into (4). Furthermore,

$$\nabla\tilde{V}(\boldsymbol{\mu}) = \mathbf{P}^T\nabla V(\mathbf{q}) \quad \text{and} \quad \tilde{\mathbf{B}}^T = \mathbf{P}^T\mathbf{B}^T \tag{12}$$

The DAEs (10) in terms of generalized coordinates can be regarded as the starting point of the formulation made by Blajer & Kolodziejczyk [12].

3.1.2 Application of the projection method

We next apply the projection method proposed by

Blajer & Kolodziejczyk [12] (see also Blajer [11]). Differentiating $(10)_3$ twice with respect to time yields the consistency condition

$$\frac{d^2}{dt^2}\tilde{\mathbf{c}}(\boldsymbol{\mu},t)=\tilde{\mathbf{C}}(\boldsymbol{\mu})\dot{\boldsymbol{\upsilon}}+\tilde{\boldsymbol{\xi}}=\mathbf{0} \tag{13}$$

with

$$\tilde{\mathbf{C}}(\boldsymbol{\mu})=D\tilde{\boldsymbol{\Phi}}(\boldsymbol{\mu}) \tag{14}$$

and

$$\tilde{\boldsymbol{\xi}}=\dot{\tilde{\mathbf{C}}}\boldsymbol{\upsilon}-\ddot{\boldsymbol{\gamma}} \tag{15}$$

Now, one has to devise a suitable projection matrix $\tilde{\mathbf{D}}\in\Re^{\tilde{n}\times(\tilde{n}-\tilde{m})}$, such that $\text{rank}(\tilde{\mathbf{D}})=\tilde{n}-\tilde{m}$, together with the relationship,

$$\tilde{\mathbf{C}}\tilde{\mathbf{D}}=\mathbf{0} \tag{16}$$

are satisfied.

Pre-multiplying $(10)_2$ by $\tilde{\mathbf{C}}\tilde{\mathbf{M}}^{-1}$ and subsequently taking into account (13) yields

$$-\tilde{\boldsymbol{\xi}}+\tilde{\mathbf{C}}\tilde{\mathbf{M}}^{-1}\left\{\mathbf{P}^T\mathbf{M}\dot{\mathbf{P}}\boldsymbol{\upsilon}+\nabla\tilde{V}(\boldsymbol{\mu})+\tilde{\mathbf{B}}^T\mathbf{u}\right\}=0 \tag{17}$$

Furthermore, pre-multiplying $(10)_2$ by $\tilde{\mathbf{D}}^T$ gives

$$\tilde{\mathbf{D}}^T\left\{\tilde{\mathbf{M}}\dot{\boldsymbol{\upsilon}}+\mathbf{P}^T\mathbf{M}\dot{\mathbf{P}}\boldsymbol{\upsilon}+\nabla\tilde{V}(\boldsymbol{\mu})+\tilde{\mathbf{B}}^T\mathbf{u}\right\}=0 \tag{18}$$

Together, the last two equations are used to replace $(10)_2$.

To summarize, the projection method yields

$$\begin{aligned}\dot{\boldsymbol{\mu}}-\boldsymbol{\upsilon}&=\mathbf{0}\\ \tilde{\mathbf{D}}^T\tilde{\mathbf{M}}\dot{\boldsymbol{\upsilon}}+\tilde{\mathbf{D}}^T\left\{\mathbf{P}^T\mathbf{M}\dot{\mathbf{P}}\boldsymbol{\upsilon}+\nabla\tilde{V}(\boldsymbol{\mu})+\tilde{\mathbf{B}}^T\mathbf{u}\right\}&=\mathbf{0}\\ \tilde{\mathbf{C}}\tilde{\mathbf{M}}^{-1}\left\{\mathbf{P}^T\mathbf{M}\dot{\mathbf{P}}\boldsymbol{\upsilon}+\nabla\tilde{V}(\boldsymbol{\mu})+\tilde{\mathbf{B}}^T\mathbf{u}\right\}-\tilde{\boldsymbol{\xi}}&=\mathbf{0}\\ \tilde{\mathbf{c}}(\boldsymbol{\mu},t)&=\mathbf{0}\end{aligned} \tag{19}$$

Similar to semi-explicit DAEs (see, for example, Ascher & Petzold [1]), the set of equations in (19) can be cast into the form

$$\begin{aligned}\mathbf{H}(\mathbf{x})\dot{\mathbf{x}}&=\mathbf{f}(\mathbf{x},\mathbf{z},t)\\ \mathbf{0}&=\mathbf{h}(\mathbf{x},\mathbf{z},t)\end{aligned} \tag{20}$$

where, in the present case,

$$\mathbf{x}=\begin{bmatrix}\boldsymbol{\mu}\\ \boldsymbol{\upsilon}\end{bmatrix}\quad\text{and}\quad\mathbf{z}=\mathbf{u} \tag{21}$$

Note that the application of the projection method yields a reduction of the number of differential equations from $\tilde{n}$ in (10) to $\tilde{n}-\tilde{m}$ in (19). The size-reduction of the differential part is accompanied by an increase of the algebraic equations from $\tilde{m}$ to $2\tilde{m}$.

3.2 Projected formulation in terms of redundant coordinates

In this section we do not introduce generalized coordinates but directly apply the projection method along the lines of Section 3.1.2 to the original DAEs (1) in terms of redundant coordinates.

Accordingly, calculating the servo constraints (4) on acceleration level yields

$$\frac{d^2}{dt^2}\tilde{\mathbf{c}}(\mathbf{q},t)=\mathbf{C}\dot{\mathbf{v}}+\left(\dot{\mathbf{C}}\mathbf{v}-\ddot{\boldsymbol{\gamma}}\right)=\mathbf{0} \tag{22}$$

with

$$\mathbf{C}=D\boldsymbol{\Phi}(\mathbf{q}) \tag{23}$$

Upon introduction of

$$\boldsymbol{\xi}=\dot{\mathbf{C}}\mathbf{v}-\ddot{\boldsymbol{\gamma}} \tag{24}$$

the servo constraints on acceleration level can be written as

$$\mathbf{C}\dot{\mathbf{v}}=-\boldsymbol{\xi} \tag{25}$$

To perform the projection, the matrix $\mathbf{D}\in\Re^{n\times(n-\tilde{m})}$ needs to be set up, such that the relationship

$$\text{range}(\mathbf{D})=\ker(\mathbf{C}) \tag{26}$$

is satisfied. Pre-multiplying $(1)_2$ by $\mathbf{CM}^{-1}$ and incorporating (25) yields

$$-\boldsymbol{\xi}+\mathbf{CM}^{-1}\left\{\nabla V(\mathbf{q})+\mathbf{G}^T\boldsymbol{\lambda}+\mathbf{B}^T\mathbf{u}\right\}=\mathbf{0} \tag{27}$$

Note that, for simplicity, it has been tacitly assumed that $\mathbf{M}$ is non-singular. Pre-multiplying $(1)_2$ by $\mathbf{D}^T$ yields

$$\mathbf{D}^T\left\{\mathbf{M}\dot{\mathbf{v}}+\nabla V(\mathbf{q})+\mathbf{G}^T\boldsymbol{\lambda}+\mathbf{B}^T\mathbf{u}\right\}=\mathbf{0} \tag{28}$$

The last two equations are used to replace $(1)_2$. Accordingly, (1) can be rewritten in the form

$$\begin{aligned}\dot{\mathbf{q}}-\mathbf{v}&=\mathbf{0}\\ \mathbf{D}^T\mathbf{M}\dot{\mathbf{v}}+\mathbf{D}^T\left\{\nabla V(\mathbf{q})+\mathbf{G}^T\boldsymbol{\lambda}+\mathbf{B}^T\mathbf{u}\right\}&=\mathbf{0}\\ \mathbf{CM}^{-1}\left\{\nabla V(\mathbf{q})+\mathbf{G}^T\boldsymbol{\lambda}+\mathbf{B}^T\mathbf{u}\right\}-\boldsymbol{\xi}&=\mathbf{0}\\ \mathbf{c}(\mathbf{q},t)&=\mathbf{0}\\ \mathbf{g}(\mathbf{q})&=\mathbf{0}\end{aligned} \tag{29}$$

It is worth mentioning that the description of trajectory tracking problems in the rotationless formulation of multibody dynamics typically yields projection matrices $\mathbf{C}$ and $\mathbf{D}$ of Boolean (or binary) type. This feature is highly beneficial to the time-discretization of the DAEs (29). The DAEs can again be written in the form (20), with

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \mathbf{v} \end{bmatrix} \text{ and } \mathbf{z} = \begin{bmatrix} \boldsymbol{\lambda} \\ \mathbf{u} \end{bmatrix} \tag{30}$$

It can be observed that the above incorporation of the servo constraints on acceleration level turns $\widetilde{m}$ of the original differential equations into algebraic equations.

4. Numerical discretization

As has been shown above, the projected formulation in terms of generalized coordinates (19) and the projected formulation in terms of redundant coordinates (29), yield DAEs of the form (20). In a first step towards the time discretization of the DAEs, we hoose to apply a backward Euler-type method. Accordingly, the time-stepping scheme is given by

$$\begin{aligned} \mathbf{H}(\mathbf{x}_{n+1})(\mathbf{x}_{n+1} - \mathbf{x}_n) &= \Delta t \mathbf{f}(\mathbf{x}_{n+1}, \mathbf{z}_{n+1}, t_{n+1}) \\ \mathbf{0} &= \mathbf{h}(\mathbf{x}_{n+1}, \mathbf{z}_{n+1}, t_{n+1}) \end{aligned} \tag{31}$$

where Δt is the step size. The corresponding discretization of the projected formulation in terms of generalized coordinates (19) leads to the scheme originally proposed by Blajer & Kolodziejczyk [12].

5. Rotary crane

In the numerical example, we consider a 3D rotary crane (Fig. 1). The description of the crane in terms of redundant coordinates can be found in [13]. The crane is composed of three rigid bodies and one point. The girder bridge (first body) is connected with the second body (trolley) via a prismatic joint, and the trolley and the third body (the winch) are coupled with revolute joints. By using augmented coordinates, the total number of redundant coordinates results in n=42. The interconnections between the bodies, along with the augmentation constraints, yield a total number of m=37 holonomic constraints. Accordingly, the present multibody system has five degrees of freedom. The equations of motion of the 3D rotary crane are presented in detail in [20].

5.1 Numerical results

We prescribe the rest-to-rest motion of the load mass (fourth body) along a straight line in space. In particular, the servo constraints are given by (4) with $\widetilde{m} = 3$. The load trajectory [13] is assumed to be of the form

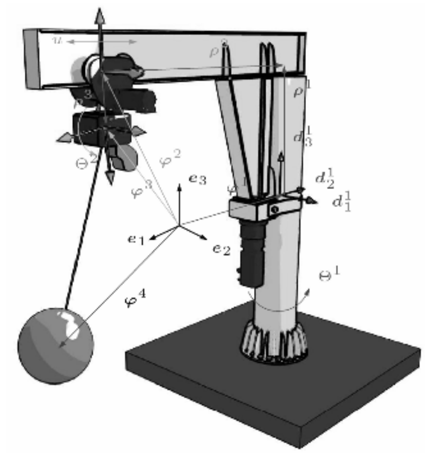

Fig. 1. Rotary crane.

$$\boldsymbol{\gamma}_d(t) = \boldsymbol{\gamma}_0 + (\boldsymbol{\gamma}_f - \boldsymbol{\gamma}_0)\, s(t) \tag{32}$$

The geometric properties [20] for this example are summarized in Tables 2 and 3.

Furthermore, the reference function *s(t)* and its acceleration are illustrated in Fig. 2. Fig. 3 represents the angle of the bridge, the trolley position and the hoisting rope length. Fig. 4 shows the torque M_b and the actuating force *F*. Both formulations under consideration, namely, in terms of generalized (GEN) coordinates and redundant (RED) coordinates, yield practically the same numerical results. Fig. 5 shows the initial and Fig. 6 the final posture of the 3D rotary crane. The numerical results have been calculated with $\Delta t = 0.1$.

6. Conclusions

comparison of the two alternative formulations of the rotary crane (Section 6) further supports the introductory comments (Section 1.1) on alternative (rotational) parametrizations. Typically, the introduction of local (generalized) coordinates for the parametrization of the configuration manifold yields a size-reduction of the equations of motion, while the structure of the equations gets increasingly involved. It can be observed that, in contrast to the generalized coordinates formulation, the alternative formulation in terms of redundant coordinates does not involve any transcendental functions. Similar conclusions can be drawn with regard to the incorporation of servo constraints by applying the projection method. In particular, the projection matrices pertaining to the rotationless formulation are typically of Boolean type.

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