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Damage detection of damaged beam by constrained displacement curvature

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Abstract

Structural damage detection technique involves the problem of how to locate and detect damage that occurs in a structure by using the observed changes of its dynamic and static characteristics. The objective of this study is to present an analytical method for damage detection by utilizing displacement curvature and all static deflection data to be expanded from the measured deflection data. Utilizing the measured displacements as displacement constraints to describe a damaged beam, minimizing the change of the displacement vector between undamaged and damaged states, and neglecting the variation between the test data and the analytical results, the deflection shape and displacement curvature of the damaged beam can be estimated, and the damage can be detected by the curvature. The validity and effectiveness of the proposed method are illustrated through comparison with the experimental results of a simple cantilever beam test.

Keywords: Damage detection; Displacement curvature; Constraint; Data expansion; Incomplete data

1. Introduction

Regular inspection and condition assessment of civil structures are necessary to allow early detection of any damage and to enable maintenance and repair work at the initial damage phase, such that the structural safety and reliability are guaranteed with a minimum of costs. Structural damage detection technique involves the problem of locating and detecting damage that occurs in a structure by using the observed changes of its dynamic and static characteristics. Maintaining safe and reliable civil infrastructure for daily use is a topic that has received considerable attention in the literature in recent years.

Sheena et al. [1] presented an analytical method to assess the stiffness matrix by minimizing the difference between the actual and the analytical stiffness matrix subjected to the measured displacement constraints. Minimizing the difference between the applied and the internal forces, Sanayei and Scampoli [2] presented a finite element method for static parameter identification of structures by the systematic identification of plate-bending stiffness parameters for a one-third scale, reinforced-concrete pier-deck model. Sanayei and Onipede [3] provided an analytical method to identify the properties of structural elements from static test data such as a set of applied static forces and another set of measured displacements. Minimizing an index of discrepancy between the model and the measurements, Banan et al. [4, 5] proposed the mathematical formulations of two least-squares parameter estimators that estimate element constitutive parameters of a finite-element model that corresponds to a real structural system from measured static response to a given set of loads. And they investigated the performance of the force-error estimator and the displacement-error estimator.

Choi et al. [6] developed an elastic damage load theorem and an approach to the damage identification

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using static displacements. Chen et al. [7] presented a two-stage damage identification algorithm to use the change of measured static displacement curvature and grey system theory. Bakhtiari-Nejad et al. [8] presented a method to describe the change in the static displacement of certain degrees of freedom by minimizing the difference between the load vectors of damaged and undamaged structures. Wang et al. [9] proposed a two-stage identification algorithm for identifying the structural damages by employing the changes in natural frequencies and measured static displacements.

Damage detection by curvature or strain energy mode shapes has been widely discussed. Pandey et al. [10] detected the location of cracks by observing changes in curvature mode shapes from the displacement mode shapes. The paper showed that damage introduced into the structure led to local changes in the shape of curvature mode shapes, which could be observed by comparing the curvature mode shapes of the damaged structure to those of the undamaged structure. Lauwagie et al. [11] presented a finite element method procedure to identify the longitudinal stiffness profile of a beam specimen from the modal curvatures and resonance frequency of one single vibration mode. Sazonov and Klinkhachorn [12] provided both analytical and numerical justification for the selection of the proper sampling interval for performing damage detection on curvature and strain energy mode shapes with the most commonly used numerical methods.

The number of transducers will be far less than the number of degrees of freedom in the finite element model. It requires that the stiffness matrix must be reduced or the measured deflection data may be expanded to estimate the data at unmeasured locations. This paper discusses the application of a model-based approach to detect the flexural damage of a steel beam with all deflection data to be expanded from the measured deflection data. Starting from an analytical method to describe the static deflection of constrained structure provided by Eun, Lee and Chung [13], this study provides an analytical method to estimate all deflection data of the damaged beam, computes the displacement curvature from the displacement data, and detects the damaged location. Cantilever steel beams with a single damage and multiple damages were tested and the effectiveness of the proposed method was verified by a comparison with the experimental results.

2. Formulation

This section derives the damage detection method to detect the damaged location of the beam by estimating unmeasured displacement data from measured deflection data and introducing the displacement curvature from the deflection data. The deflection of the damaged beam has a tendency to increase due to the damage as if the intact structure is subjected to the additional unknown forces besides the applied forces. Figs. 1(a) and (b) exhibit the deflected shapes, $\mathbf{u}$ and $\mathbf{u}^*$, of the simply supported beam at undamaged and damaged states, respectively, subjected to the applied force P at x_p from the left support. It is apparent that the damaged beam represents larger deflection due to the deteriorated flexural rigidity. The beam is damaged at x_d from the left support and the displacements can be measured at other locations x_m. It can be assumed that the damaged deflection of Fig. 1(b) leads to the same displacements as the undamaged beam subjected to both the applied force P `of Fig. 1(a) and the additional unknown force F^c at the measuring point of Fig. 1(c). As the number of measurement points increases, the estimated shape of the damaged deflection can be more accurately obtained by the action of the additional

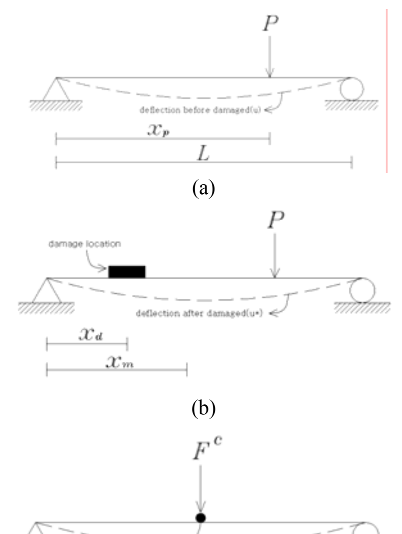


Fig. 1. Damage detection approach: (a) deflected curve before being damaged, (b) deflected curve after being damaged, (c) additional displacement caused by the additional force on undamaged beam.

forces at the measurement points. Thus, at first this study estimates the unknown force to describe the damaged deflection based on the measured displacement data.

Let us consider a structural system as a finite element model described by a displacement vector $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}^T$. The static equilibrium equation of the structure at the undamaged state is expressed by

$$\mathbf{F} = \mathbf{K}\mathbf{u} \tag{1}$$

where $\mathbf{K}$ is an $n \times n$ positive-definite stiffness matrix and $\mathbf{F}$ denotes an $n \times 1$ applied static force vector. Then the displacement vector $\mathbf{u}$ can be calculated as

$$\mathbf{u} = \mathbf{K}^{-1}\mathbf{F} \tag{2}$$

If the displacements calculated by Eq. (2) correspond to the measured displacements, they exhibit the undamaged displacements. However, if the structure has been damaged by environmental or accidental loads, the structural stiffness matrices are changed and the calculated displacements at the intact state do not correspond with the measured displacements. The difference between the intact and damaged displacements can be defined as the output error expressed by

$$\mathbf{e} = \mathbf{B}\mathbf{K}^{-1}\mathbf{F} - \hat{\mathbf{u}} \tag{3}$$

where $\hat{\mathbf{u}}$ denotes the m measured degrees of freedom and $\mathbf{B}$ is an $m \times n$ Boolean matrix that extracts the measured response $\hat{\mathbf{u}}$ from the complete displacement vector $\mathbf{u}$. The algorithm for the damage detection is to minimize the output error of Eq. (3).

Assuming that the mass of the structure is unchanged under the damaged state, the damage should be only evaluated by the flexural rigidity of the structure. Expressing the variation in the stiffness matrix by an amount $\delta\mathbf{K}$, the equilibrium equation of the damaged structure can be expressed as

$$\mathbf{K}^*\mathbf{u}^* = \left(\mathbf{K} + \delta\mathbf{K}\right)\mathbf{u}^* = \mathbf{F} \tag{4}$$

where $\mathbf{K}^*$ and $\mathbf{u}^*$ denote the stiffness matrix and displacement vector at the damaged state, respectively. Taking the first-order approximation of the displacement vector $\mathbf{u}^*$, it can be written by

$$\mathbf{u}^* = \left(\mathbf{K} + \delta\mathbf{K}\right)^{-1}\mathbf{F} \approx \left(\mathbf{K}^{-1} - \mathbf{K}^{-1}\delta\mathbf{K}\mathbf{K}^{-1}\right)\mathbf{F} \tag{5}$$

The displacement data at the measurement points are not complete to detect the damage. This study expands the measured deflection data to estimate the data at unmeasured locations. With the m $(m < n)$ measuring displacements of the structure, it can be deduced that the damaged displacements must be larger than the displacements of the undamaged structure because the damage and the static behavior of the damaged structure is partially restricted by the measured displacements. It is investigated that the deflection difference between damaged and undamaged beams comes from the action of the additional force to be determined. Assuming the measured deflection data at the measurement points as static constraints to describe the damaged beam, the static deflection of the damaged beam is obtained from the static equilibrium equation of a constrained beam.

The m constraints by the measured displacements are expressed as

$$\phi_i\left(\mathbf{u}^*\right) = c_i, \quad i = 1, \quad 2, \quad \cdots, \quad m \tag{6}$$

where $c_i's$ are the measured displacements and Eq. (6) can be expressed in matrix form of

$$\mathbf{A}\mathbf{u}^* = \mathbf{c} \tag{7}$$

where $\mathbf{A}$ is an $m \times n$ coefficient matrix and $\mathbf{c}$ is an $n \times 1$ vector.

The static behavior of the damaged beam can be described by the constrained behavior to satisfy both the static equilibrium equation of the undamaged beam and the constraint equations. This study minimized the change of the displacement vector at the undamaged and damaged states

$$\text{Minimize} \quad J = \mathbf{R}\left\|\mathbf{u}^* - \mathbf{u}\right\|^2 \tag{8}$$

where $\mathbf{R}$ is a weighting matrix. Although there are many approaches to minimize Eq. (8), most of them depend on numerical approaches with unknown multipliers. This study utilized the analytical method presented by Eun et al. [13] as shown in the Appendix. Using the structural stiffness $\mathbf{K}$ as the weighting matrix $\mathbf{R}$ in Eq. (8)and minimizing the same form as Eq. (8), the analytical method is explicitly derived as follows:

$$\mathbf{u}^* = \mathbf{u} + \boldsymbol{\delta}\mathbf{u} = \mathbf{u} + \mathbf{K}^{-1/2}\left(\mathbf{A}\mathbf{K}^{-1/2}\right)^+\left(\mathbf{c} - \mathbf{A}\mathbf{u}\right) \tag{9}$$

where '+' is the Moore-Penrose generalized inverse and $\boldsymbol{\delta}\mathbf{u}$ denotes the displacement variation due to the damage.

Premultiplying both sides of Eq. (9) by the stiffness matrix $\mathbf{K}$, it yields

$$\mathbf{Ku}^* = \mathbf{Ku} + \mathbf{K}^{1/2}\left(\mathbf{AK}^{-1/2}\right)^+\left(\mathbf{c} - \mathbf{Au}\right) \tag{10}$$

where the second term on the right-hand side denotes the additional force vector, $\mathbf{F}^c$, to be necessary for obtaining the same displacements at the measuring locations besides the initially applied force **F**. It indicates that the damage is caused by the action of such unexpected force. Comparing Eqs. (5) and (9), the displacement variation $\delta\mathbf{u}$ is defined as

$$\delta\mathbf{u} = -\mathbf{K}^{-1}\left(\delta\mathbf{K}\right)\mathbf{u} = \mathbf{K}^{-1/2}\left(\mathbf{AK}^{-1/2}\right)^+\left(\mathbf{c} - \mathbf{Au}\right) \tag{11}$$

And the additional force vector to act at the measuring points is calculated as

$$\mathbf{F}^c = \mathbf{K}^{1/2}\left(\mathbf{AK}^{-1/2}\right)^+\left(\mathbf{c} - \mathbf{Au}\right) \tag{12}$$

If the structure is not damaged, the displacement variation $\delta\mathbf{u} = \mathbf{u}^* - \mathbf{u}$ must be zero and the additional forces are not required. It can be observed that the additional force increases with the displacement difference $\mathbf{c} - \mathbf{Au}$ at the measuring points or with the increase in damaged displacements.

From Eq. (9), the displacements and additional forces of the damaged structure should be derived as

$$\mathbf{u}^* = \mathbf{u} - \mathbf{K}^{-1}(\delta\mathbf{K})\mathbf{K}^{-1}\mathbf{F} \tag{13a}$$

$$\mathbf{F}^c = -(\delta\mathbf{K})\mathbf{K}^{-1}\mathbf{F} \tag{13b}$$

As indicated by Eqs. (13), the damaged displacement vector $\mathbf{u}^*$ and the additional force $\mathbf{F}^c$ increase with the deterioration of structural stiffness. That is, the damaged displacements are apparently related to the measuring locations, the damage location and degree. It can be expected that the damaged deflection can be properly described when the additional forces act at the measuring locations to yield the displacement difference $\boldsymbol{\delta}\mathbf{u}$.

Structures can be accurately modeled as discrete systems, and finite element analysis can be used to approximate the static behavior of the damaged and undamaged systems. The damage introduced into the flexural beam to be modeled as finite elements leads to local changes in the shape of displacement curvature. Pandey et al. (1991) stated that once the displacement shapes of a damaged and of the corresponding undamaged structure are identified, the curvature at each location j on the structure is numerically obtained by a central difference approximation:

$$\phi_j'' = \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{h^2} \tag{14}$$

where h is the distance between the measurement points j-1 and j+1.

The damage is located at the location to exhibit the largest difference between the displacement curvatures of the damaged and undamaged structures as follows:

$$\Delta\phi'' = \phi_d'' - \phi'' \tag{15}$$

where ϕ_d and ϕ denote the displacements of the damaged and of the undamaged structures, respectively. The displacement curvature at damaged locations will abruptly increase due to the deterioration of flexural rigidity and the damages will be detected. The following section evaluates the validity of the proposed method through a comparison of the experimental and analytical results.

3. Beam test and damage detection

An experiment of simple cantilever beams under the loading arrangement shown in Fig. 2 was carried out to illustrate the proposed damage detection method. It was assumed that each beam for the analytical approach was modeled by using the finite element method with 50 beam elements that its length is 20mm. The beams were assumed as steel with an elastic modulus of 1.95×10^5MPa.

Fifteen beam specimens were tested, and the test variables included the damage location and size. The specimens are summarized in Table 1 and Fig. 3. The experiment was made on 1m cantilever beams with a gross cross-section of $75\text{mm}\times9\text{mm}$ and the damage section of $25\text{mm}\times9\text{mm}$. The damage degree of the section was established as the 67% strength loss of the initial second moment of inertia. The damage of the beams was located at 200, 400, 600 and 800mm, respectively, from the fixed end, and they had a single damage or multiple damages. Specimens were also tested for evaluating the damage effects of the beams with multiple damages such that one damage is positioned at 200mm from the fixed end and the other is

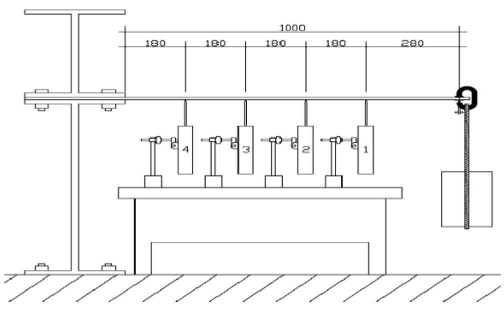


Fig. 2. Set-up of beam test (unit; mm).

Table 1. Specimens and measured displacements.

Specimens	Deflection (mm)			
	x=720mm	x=540mm	x=360mm	x=180mm
W/O D	21.98	13.55	6.72	2.00
S200-3	24.17	14.97	7.35	2.05
S400-3	23.04	14.03	6.73	1.97
S600-3	21.82	13.19	6.41	1.83
S800-3	22.54	13.91	6.88	2.03
S200-20	25.44	15.82	7.72	1.99
S400-20	23.66	14.40	6.77	2.01
S600-20	22.03	13.31	6.47	1.84
S800-20	22.44	13.89	6.83	2.03
M2,400-3	24.78	15.19	7.27	2.04
M2,600-3	24.98	15.38	7.52	2.04
M2.800-3	24.10	14.84	7.22	1.93
M2,400-20	26.60	16.27	7.60	1.95
M2,600-20	26.39	16.26	7.97	2.01
M2,800-20	25.49	15.81	7.67	1.94

S 200–3
- damage size(3, 20mm)
- damage location from fixed end (200,400,600,800mm) 2,400 (200 and 400mm)
- single (S) and multiple (M) damages

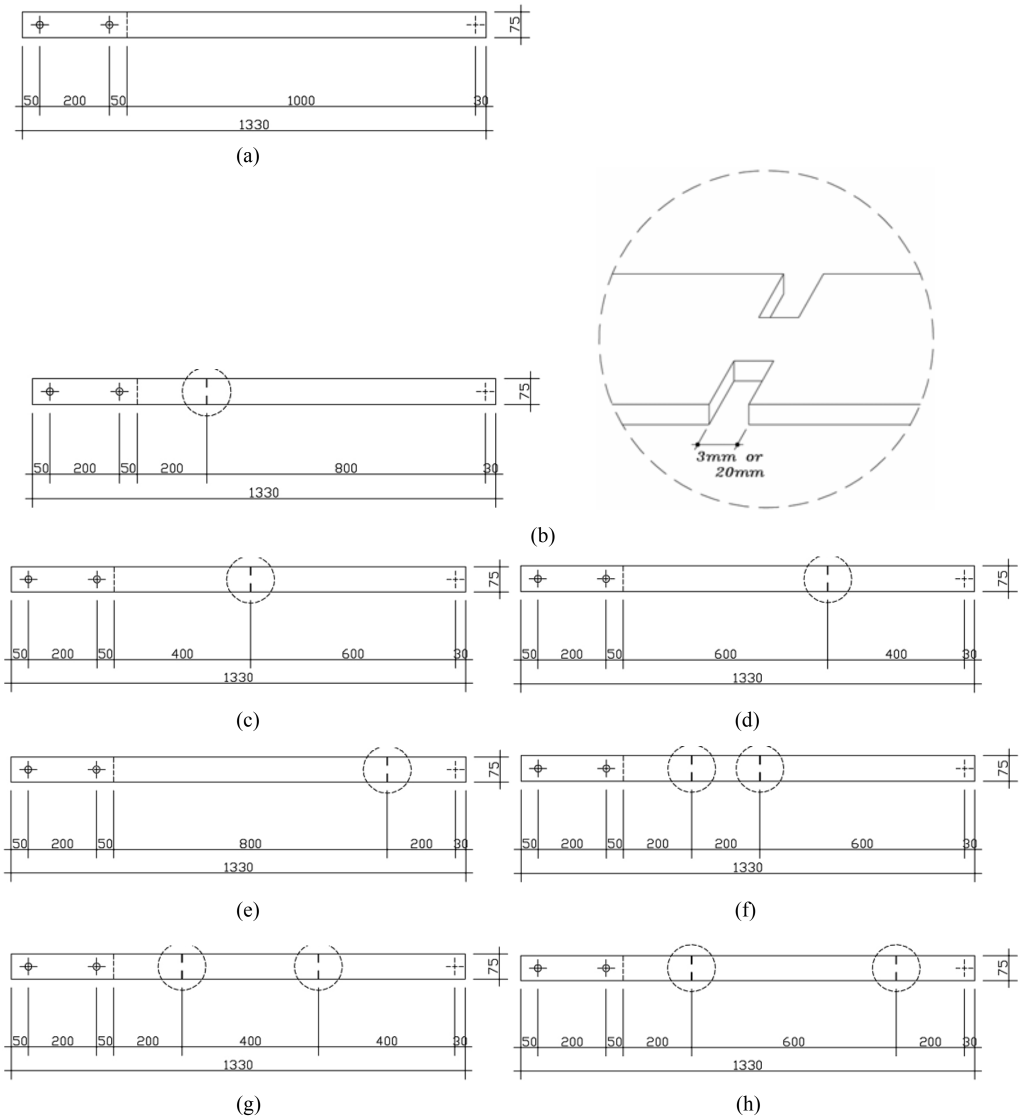


Fig. 3. Damage location and specimens (unit; mm); (a) W/O D, (b) S200-3, S200-20, (c) S400-3, S400-20, (d) S600-3, S600-20, (e) S800-3, S800-20, (f) M2,400-3, S2,400-20, (g) M2,600-3, M2,600-20, (h) M2,800-3, M2,800-20.

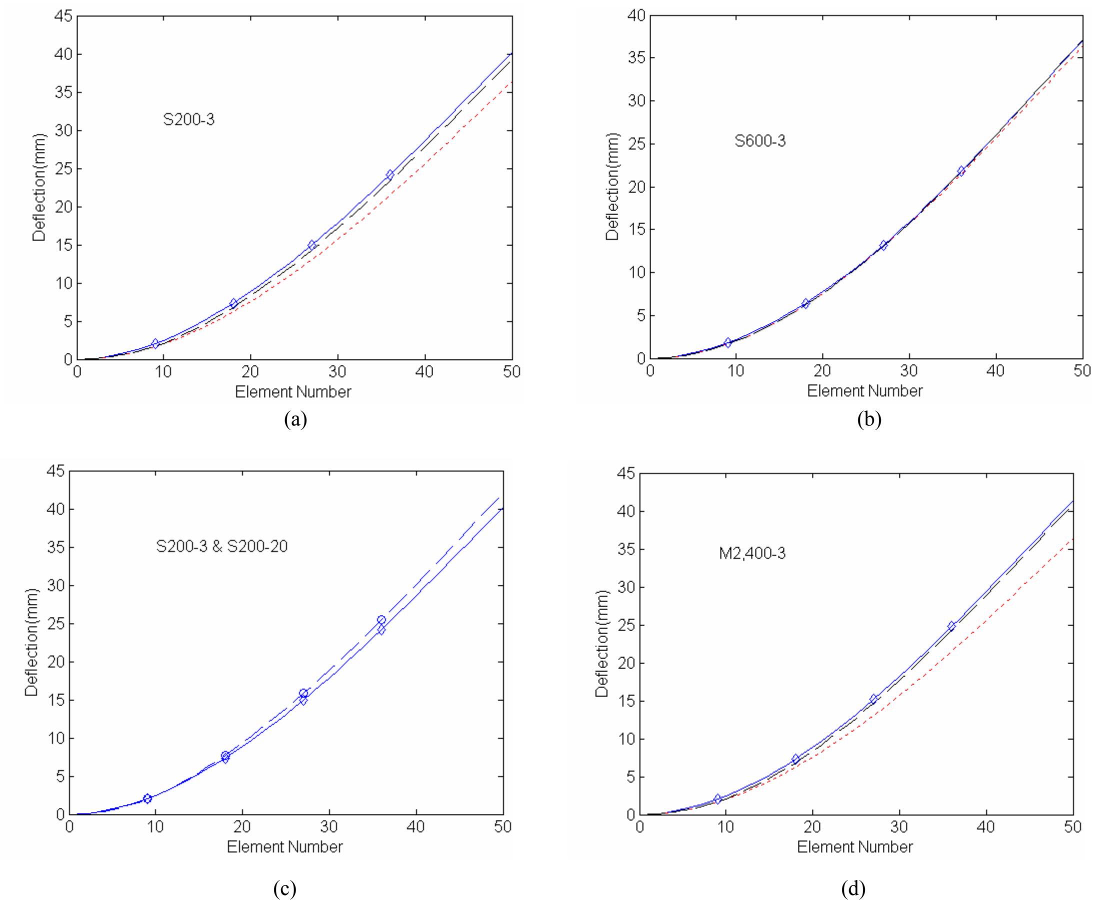


Fig. 4. Beam deflection; (a) S200-3, (b) S600-3, (c) S200-3 and S200-20, (d) M2,400-3. The diamond and solid line indicates the deflection data obtained from the experiment, the dashed line represents the analytical results obtained from finite element model, and the dotted line the deflection of the undamaged beam.

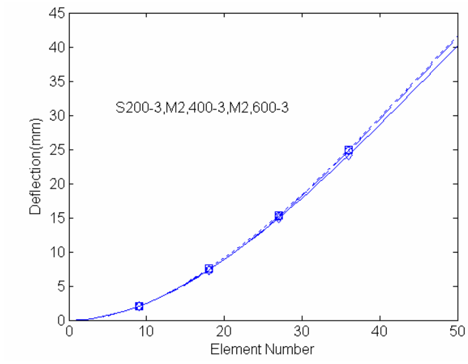
positioned at 400, 600 or 800mm, respectively. The beams were subjected to a concentrated force 97N at the free end and the corresponding deflections were measured by four LVDTs installed at locations of 180, 360, 540 and 720mm, respectively, from the fixed end.

The deflection data at four points measured by the LVDTs are listed in Table 1. Fig. 4 compares the deflections of the undamaged and damaged beams. One of analytical results indicates the deflection of the undamaged beam and the other is the one obtained by the finite element model of the damaged beam. The plots represent that the test data do not absolutely correspond with the analytical results because of the inaccuracy in the finite element modeling. The measured and analytical data are unlikely to be equal due to measurement noise and model inadequacies. Major problems arise because of the large number of degrees of freedom in the analytical model, the limited number of transducers used to measure the response of the structure, and modeling inaccuracies.

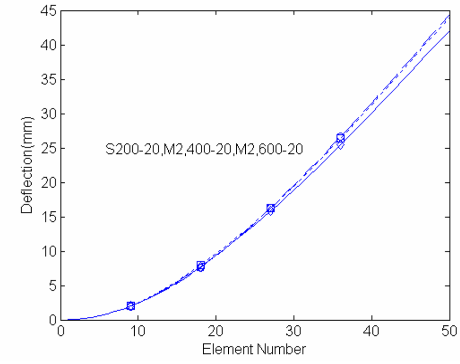
Comparing Figs. 4(a) and (b), it is observed that the deflection of the damaged beam slightly increases due to the damage, and the deflection of the damaged beam decreases as the distance between the damage location and the fixed end increases. It is understood that the observation comes from the difference in the load-carrying capacity of the damaged beam. The load-carrying capacity of the beam is gradually deteriorated as the damage location approaches the fixed end. Fig. 4(c) exhibits that the deflection of damaged beam increases with the damage size due to the difference in the load-carrying capacity.

Comparing Figs. 4(a) and (d) to represent the static behavior of the beams with a single damage and multiple damages, a little difference between two plots was observed. It means that the load-carrying capacity of the damaged beam with the multiple damages largely depends on the damage that initially occurs from the fixed end.

Figs. 5(a) and (b) compare the static behavior of the damaged beams according to the number of damage locations along the beam. The plots show that the beams with multiple damages lead to a slight increase in the deflection. It is also explained that the difference is due to the load-carrying capacity that the beams have. Comparing the maximum deflection of the beams, it is found that the deflection difference is

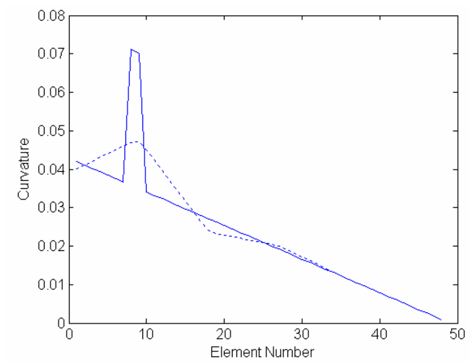


(a)

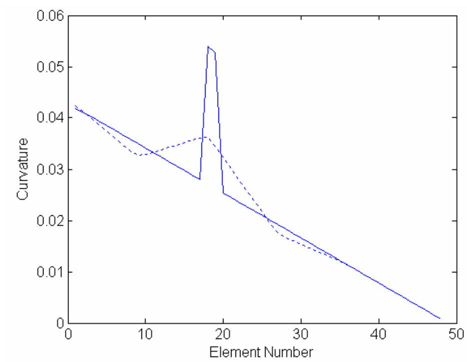


(b)

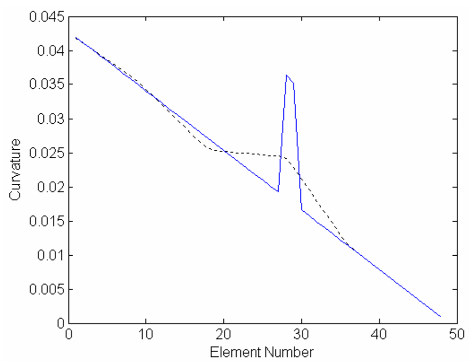
Fig. 5. Comparison of beam deflection; (a) S200-3, M2,400-3, M2,600-3, (b) S200-20, M2,400-20, M2,600-20.



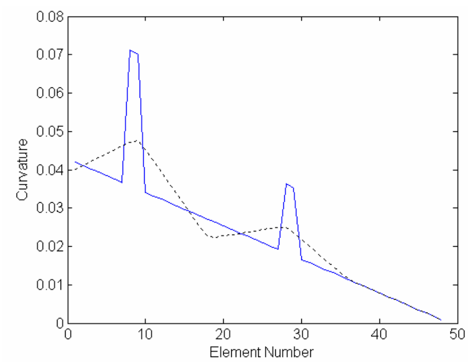
(a)



(b)



(c)



(d)

Fig. 6. Displacement curvature of damaged beam; (a) S200-3, (b) S400-3, (c) S600-3, (d) M2,600-3, The dotted line indicates the results calculated from the proposed method with the measured displacements, and the solid line represents the displacement curvature of the damaged beam.

reduced as the second damage moves to the free end.

From Figs. 4 and 5, it is recognized that it is impossible to detect the damage location by the static deflection only. Despite local damage, it is recognized from Figs. 4 and 5 that the damage detection cannot be established by the deflection shape itself. For the damage detection, this study utilized the displacement curvature to be calculated by Eq. (14) from the deflection data determined by the proposed method. Neglecting the variation between the test data and the analytical results by the finite element model, Fig. 6 compares the displacement curvature of the beams. The analytical curvature in the plots was obtained from the finite element model to utilize the real measurement displacement data as the displacement constraints. By the plots, it is observed that the damage is detected by the displacement curvature calculated by the proposed method and identified by the locally dull increase in the shape of the displacement curvature in the neighbor of the damage location, although it exhibits a different shape with the actual curvature. It can be concluded that the proposed method can effectively detect the damage of the beams with a single damage and multiple damages based on the measured displacement data with the assumption of neglecting the variation between the test data and the analytical results.

4. Conclusion

Establishing the displacements measured at measurement points of a damaged beam as displacement constraints and minimizing the displacement difference between undamaged and damaged beams, this study estimated the deflection at full set of finite element degrees of freedom of the damaged beams. It was recognized that the damage detection cannot be established by the deflection shape itself. And utilizing the displacement curvature based on the calculated displacements, this study developed a static approach for the damage detection. Comparing the analytical results of damaged beam modeled by finite element and the experimental results on a simple cantilever beam test, the validity of the proposed method for damage detection was illustrated. It could be concluded that the proposed method can effectively detect the damage of the beams with a single damage and multiple damages based on the exact displacement measurement with the assumption of neglecting the variation between the test data and the analytical results.

Acknowledgement

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Appendix

The equilibrium equation for unconstrained structure described by a displacement vector $\hat{\mathbf{u}} = \begin{bmatrix} \hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_n \end{bmatrix}^T$ can be written as

$$\mathbf{F} = \mathbf{K}\hat{\mathbf{u}} \tag{A1}$$

where $\mathbf{F}$ is the $n \times 1$ nodal force vectors, and $\mathbf{K}$ is the $n \times n$ positive definite stiffness matrix. Assume that the structure is subjected to m displacement constraints

$$\phi_i(\mathbf{u}) = c_i\,,\quad i = 1,\quad 2,\quad \cdots \quad m\,. \tag{A2}$$

or $\mathbf{A}\mathbf{u} = \mathbf{b}$, (A3)

where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{u}$ is an $n \times 1$ actual displacement vector. The two displacement vectors exhibit different displacements due to the constraints.

Defining the displacements at actual state as $\mathbf{u} = \hat{\mathbf{u}} + \Delta\mathbf{u}$, the change of the potential energies can be written as

$$\begin{aligned} \Delta\Pi = &\frac{1}{2}\left(\hat{\mathbf{u}} + \Delta\mathbf{u}\right)^T \mathbf{K}\left(\hat{\mathbf{u}} + \Delta\mathbf{u}\right) \\ &- \left(\hat{\mathbf{u}} + \Delta\mathbf{u}\right)^T \mathbf{F} - \frac{1}{2}\hat{\mathbf{u}}^T\mathbf{K}\hat{\mathbf{u}} + \hat{\mathbf{u}}^T\mathbf{F} \end{aligned}\,. \tag{A4}$$

The equilibrium equation of the constrained system and the constraint equation can be modified as

$$\mathbf{F} = \mathbf{K}\hat{\mathbf{u}} + \mathbf{K}\Delta\mathbf{u} \tag{A5}$$

$$\mathbf{A}\left(\hat{\mathbf{u}} + \Delta\mathbf{u}\right) = \mathbf{b} \tag{A6}$$

Expressing the constraint force vector $\mathbf{F}^{\mathbf{c}}$, Eq. (A5) is written as

$$\mathbf{F} = \mathbf{K}\hat{\mathbf{u}} + \mathbf{F}^{\mathbf{c}} \tag{A7}$$

For this derivation, Eq. (A6) is modified as

$$\mathbf{A}\Delta\mathbf{u} = \mathbf{b} - \mathbf{A}\hat{\mathbf{u}}\,. \tag{A8}$$

In order to utilize Eq. (A8) into Eq. (A4), Eq. (A8) is modified as

$$\mathbf{A}\mathbf{K}^{-1/2}\mathbf{K}^{1/2}\Delta\mathbf{u} = \mathbf{b} - \mathbf{A}\hat{\mathbf{u}}\,. \tag{A9}$$

From the fundamental property of generalized inverse matrix and its solution*, the solution with respect to $\mathbf{K}^{1/2}\Delta\mathbf{u}$ can be obtained as

$$\begin{aligned} \mathbf{K}^{1/2}\Delta\mathbf{u} = &\left(\mathbf{A}\mathbf{K}^{-1/2}\right)^+ \left(\mathbf{b} - \mathbf{A}\hat{\mathbf{u}}\right) \\ &+ \left[\mathbf{I} - \left(\mathbf{A}\mathbf{K}^{-1/2}\right)^+ \left(\mathbf{A}\mathbf{K}^{-1/2}\right)\right]\mathbf{y} \end{aligned}\,, \tag{A10}$$

where '+' denotes the generalized inverse matrix and $\mathbf{y}$ is an arbitrary vector.

Minimizing the variation of total energy with respect to $\Delta\mathbf{u}$, it follows that

$$\frac{\Delta\Pi}{\Delta\mathbf{u}} = \mathbf{K}^{1/2}\left(\hat{\mathbf{u}} + \Delta\mathbf{u}\right) - \mathbf{F} = 0\,. \tag{A11}$$

Eq. (A11) can be rewritten as

$$\mathbf{K}^{1/2}\Delta\mathbf{u} = \mathbf{F} - \mathbf{K}^{1/2}\hat{\mathbf{u}}\,. \tag{A12}$$

From Eqs. (A10) and (A12), the following equation is obtained.

$$\begin{aligned} \mathbf{K}^{1/2}\Delta\mathbf{u} = &\left(\mathbf{A}\mathbf{K}^{-1/2}\right)^+ \left(\mathbf{b} - \mathbf{A}\hat{\mathbf{u}}\right) \\ &+ \left[\mathbf{I} - \left(\mathbf{A}\mathbf{K}^{-1/2}\right)^+ \left(\mathbf{A}\mathbf{K}^{-1/2}\right)\right]\mathbf{y} \\ &= \mathbf{F} - \mathbf{K}^{1/2}\hat{\mathbf{u}} \end{aligned} \tag{A13}$$

Letting $\mathbf{Q} = \left[\mathbf{I} - \left(\mathbf{A}\mathbf{K}^{-1/2}\right)^+ \left(\mathbf{A}\mathbf{K}^{-1/2}\right)\right]$, solving Eq. (A13) with respect to $\mathbf{y}$, and the fundamental properties of generalized inverse matrix**, it follows that

$$\mathbf{y} = \left(\mathbf{A}\mathbf{K}^{-1/2}\right)^+ \left(\mathbf{A}\mathbf{K}^{-1/2}\right)\mathbf{z}\,, \tag{A14}$$

where $\mathbf{z}$ is another arbitrary vector.

Substitution of Eq. (A14) into Eq. (A13) and arranging the result, it follows that

$$\mathbf{K}^{1/2}\Delta\mathbf{u} = \left(\mathbf{A}\mathbf{K}^{-1/2}\right)^+ \left(\mathbf{b} - \mathbf{A}\hat{\mathbf{u}}\right) \tag{A15}$$

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* The general solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A}$ is $m \times n$ matrix, $\mathbf{x}$ and $\mathbf{b}$ are $n \times 1$ and $m \times 1$ vectors, respectively, can be written as

$$\mathbf{x} = \mathbf{A}^+\mathbf{b} + \left[\mathbf{I} - \mathbf{A}^+\mathbf{A}\right]\mathbf{d}\,,$$

where $\mathbf{I}$ is $n \times n$ identity matrix and $\mathbf{d}$ is $n \times 1$ arbitrary vector.

**

$$\left(\mathbf{A}\mathbf{K}^{-1/2}\right)^+ \left(\mathbf{A}\mathbf{K}^{-1/2}\right)\left(\mathbf{A}\mathbf{K}^{-1/2}\right)^+ = \left(\mathbf{A}\mathbf{K}^{-1/2}\right)^+,\quad \mathbf{Q}^+ = \left[\mathbf{I} - \left(\mathbf{A}\mathbf{K}^{-1/2}\right)^+ \left(\mathbf{A}\mathbf{K}^{-1/2}\right)\right]^+ = \mathbf{Q}\,,\quad \mathbf{Q}^+\mathbf{Q} = \mathbf{Q}$$

and premultiplying both sides of Eq. (A15) by $\mathbf{K}^{1/2}$, the constraint force is obtained as

$$\mathbf{F}^c = \mathbf{K}\Delta\mathbf{u} = \mathbf{K}^{1/2}\left(\mathbf{A}\mathbf{K}^{-1/2}\right)^+\left(\mathbf{b} - \mathbf{A}\hat{\mathbf{u}}\right). \tag{A16}$$

Substituting Eq. (A16) into Eq. (A7), the equilibrium equation of constrained structure is derived as

$$\mathbf{F} = \mathbf{K}\hat{\mathbf{u}} + \mathbf{K}^{1/2}\left(\mathbf{A}\mathbf{K}^{-1/2}\right)^+\left(\mathbf{b} - \mathbf{A}\hat{\mathbf{u}}\right) \tag{A17}$$

or $$\mathbf{u} = \hat{\mathbf{u}} + \mathbf{K}^{-1/2}\left(\mathbf{A}\mathbf{K}^{-1/2}\right)^+\left(\mathbf{b} - \mathbf{A}\hat{\mathbf{u}}\right). \tag{A18}$$