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# The Influence of Mass of Two-Parameter Elastic Foundation on Dynamic Responses of Beams Subjected to a Moving Mass

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## Abstract

The influence of mass of two-parameter elastic foundation on dynamic responses of beams subjected to a moving mass is presented in this paper. The analytical model of the foundation is characterized by shear layer connecting with elastic foundation modelled by linear elastic springs based on Winkler model and the mass of foundation is directly proportional with deformation of the springs. By using finite element method and principle of the dynamic balance, the governing equation of motion is derived and solved by the Newmark's time integration procedure. The numerical results are compared with those obtained in the literature showing reliability of a computer program. The influence of parameters such as moving mass, stiffness and mass of foundation on dynamic responses of the beam is discussed.

Keywords: *dynamic analysis of beam, two-parameter foundation, moving mass, foundation mass*

## 1. Introduction

The Winkler modeling, one of the most fundamental elastic foundation models was suggested quite early in 1867 and has been applied so much in behavior analysis models of structures resting on foundation. In this model, the elastic foundation stiffness is considered as a continuous distribution of linear elastic springs, whose constraint reaction per unit length at each point of the foundation is directly proportional to the deflection of the foundation itself. It can be seen that the Winkler foundation model is very simple and has quite many studies related to response of the structure on Winkler foundation model (Abohadima, 2009, Eisenberger, 1987; Gupta, 2006; Lee, 1998; Malekzadeh, 2003; Mohanty, 2012; Ruge, 2007). Beside the Winkler foundation model, a few different foundation models were established to describe more real response of structure resting on foundation such as two-parameter foundation (Çalım, 2012; Eisenberger, 1994; Matsunaga, 1999; Chen, 2004; Kargarnovin, 2004), three-parameter foundation (Avramidis, 2006; Morfidis, 2010), viscous-elastic foundation (Çalım, 2009), variable elastic foundation (Eisenberger, 1994; Kacar, 2011) or tensionless elastic foundation (Konstantinos, 2013). All most the foundation models introduced above did not consider the effects of foundation mass on dynamic responses of structures resting on foundation. In reality, the foundation has mass density, so that vertical inertia force due to this mass has existed in vibration of the beam. Hence, the dynamic responses of structures on foundations should be considered with attending of this force. But, all most the researches in the literature were not attention to the effects of the foundation mass.

From these literatures and continuously attention to the influence of mass of foundation on dynamic responses of structures, the paper studies the influence of mass of two-parameter elastic foundation on the dynamic response of beam subjected to a moving mass using finite elemnet method. The analytical model of the foundation is characterized by shear layer connecting with elastic foundation modelled by linear elastic springs based on Winkler model and the mass of foundation is directly proportional with deformation of the springs. The governing equation of motion is derived by principle of dynamic balance based on finite element method of Euler-Bernoulli element and solved by the Newmark's time integration procedure. The effects of parameters such as the moving mass, stiffness and mass of foundation on the dynamic responses of the beam are investigated.

## 2. Formulation

### 2.1 Beam Model

A simple support Euler-Bernoulli beam resting on the two-parameter elastic foundation is shown in Fig. 1. In this Figure, $L$, $A$, $I$, $E$, $\rho$ are the beam length, cross-sectional area, moment of inertia, Young's modulus and mass density, respectively. The model of foundation is characterized by the Winkler elastic

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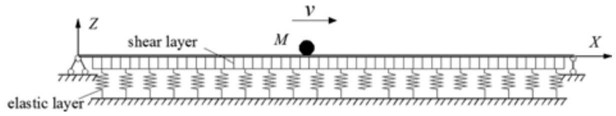


Fig. 1. The Beam Resting on the Foundation Subjected to a Moving Mass

foundation $k_w$ (first-parameter foundation) and shear layer $k_s$ (second-parameter foundation). The foundation has mass density $\rho_f$ and the mass density ratio is defined as the ratio of the mass density of the foundation to the mass density of the beam $\mu = \rho_f/\rho$. The moving mass $M$ moves in the axial direction of the beam with constant velocity $v$ and the mass ratio is defined as the ratio of the mass of the moving mass to the mass of the beam $R = M/\rho AL$ (Stanisic *et al*., 1969).

### 2.2 Finite Element Procedure

A two-node beam element resting on the foundation, having length $l$, each node having two global degrees of freedom including displacements and rotation about an axis normal to the plane ($x$, $z$) is shown in Fig. 2. At any time $t$, the position of the moving mass is $x_m = vt$ and the left end of the beam element in global coordinate (node $i^{th}$) is to be

$$x_i = \mathrm{Int}[x_m/l]l \tag{1}$$

One can find the element number $i^{th} = \mathrm{Int}[x_m/l]+1$, nodes $i^{th}$ and $i+1^{th}$, which the moving mass is applied to at any time $t$, therefore, $\xi$ can be rewritten in terms of the global instead of the local

$$\xi(t) = x_m - i^{th}l \tag{2}$$

By means of finite element method, the consistent element mass matrix $[\mathbf{M}]_e$ and stiffness matrix $[\mathbf{K}]_e$ as a summation of the stiffness matrices due to the beam bending $[\mathbf{K}]_b$, the elastic foundation stiffness $[\mathbf{K}]_w$ and shear layer stiffness $[\mathbf{K}]_s$ can be developed from strain energy and kinetic energy expressions (Chopra, 2001) as follows

$$[\mathbf{M}]_e = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{3}$$

$$[\mathbf{K}]_e = [\mathbf{K}]_b + [\mathbf{K}]_w + [\mathbf{K}]_s \tag{4}$$

with

$$[\mathbf{K}]_b = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}; \tag{5}$$

$$[\mathbf{K}]_w = k_w\int_0^l [\mathbf{N}_w]^T[\mathbf{N}_w]d\xi$$

$$[\mathbf{K}]_s = k_s\int_0^l [\mathbf{N}_s]^T[\mathbf{N}_s]d\xi$$

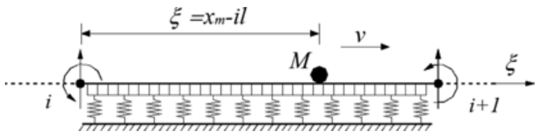


Fig. 2. The Beam Element Resting on the Foundation Subjected to a Moving Mass

where $[\mathbf{N}_w]$, $[\mathbf{N}_s]$ are the matrices of interpolation functions for displacements and rotation in the local coordinate $\xi$, respectively, studied in many researches related to finite element method.

### 2.3 Mass of Foundation

Based on finite element method, the functions of dynamic displacement $u_i(\xi,t)$ and acceleration $u_i(\xi,t)$ of element $i^{th}$ expressed in terms of the nodal displacement $\{\mathbf{u}_e(t)\}$ and acceleration vector $\{\ddot{\mathbf{u}}_e(t)\}$ in each time step are given by

$$\begin{aligned} u_i(\xi,t) &= [\mathbf{N}_w]\{\mathbf{u}_e(t)\} \\ \ddot{u}_i(\xi,t) &= [\mathbf{N}_w]\{\ddot{\mathbf{u}}_e(t)\} \end{aligned} \tag{6}$$

Considering continuous contact between the beam and foundation during vibration of the beam, and the mass of foundation is directly proportional with vertical displacement of the beam shown in Fig. 3, the mass of foundation per unit length of the beam element which influent dynamic response of the beam can be expressed as follows

$$m_{i,f}(\xi) = \kappa\rho_f H(\xi)u_i(\xi) \tag{7}$$

with $\kappa > 0$, dimensionless parameter used to describe the influence of mass of foundation abilily; $H(\xi) = 1$ when $u_i(\xi) \geq 0$ and $H(\xi) = -1$ when $u_i(\xi) < 0$. The unit contact reaction between the beam and foundation caused influence of unit foundation mass is given by

$$f_{i,m}(\xi) = m_{i,f}(\xi)\ddot{u}_i(\xi) \tag{8}$$

Under moving mass, the beam and foundation have vertical motion so the mass of foundation develops an inertia forces acting on the beam; this force acts as an external force on the beam during vibration. Therefore, the dynamic response of the beam has logically changed.

By means of finite element method, the element external force vector in each time step can be expressed as

$$\{\mathbf{F}\}_{e,f} = \int_0^l [\mathbf{N}_w]^T f_{i,m}(\xi)d\xi \tag{9}$$

### 2.4 Governing Equation of Motion

By assuming the no-jump condition for the moving mass, at

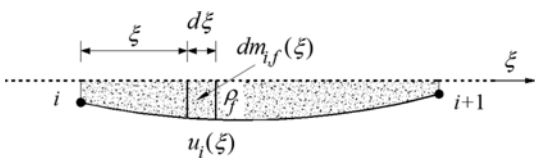


Fig. 3. The Mass of Foundation on the Beam Element

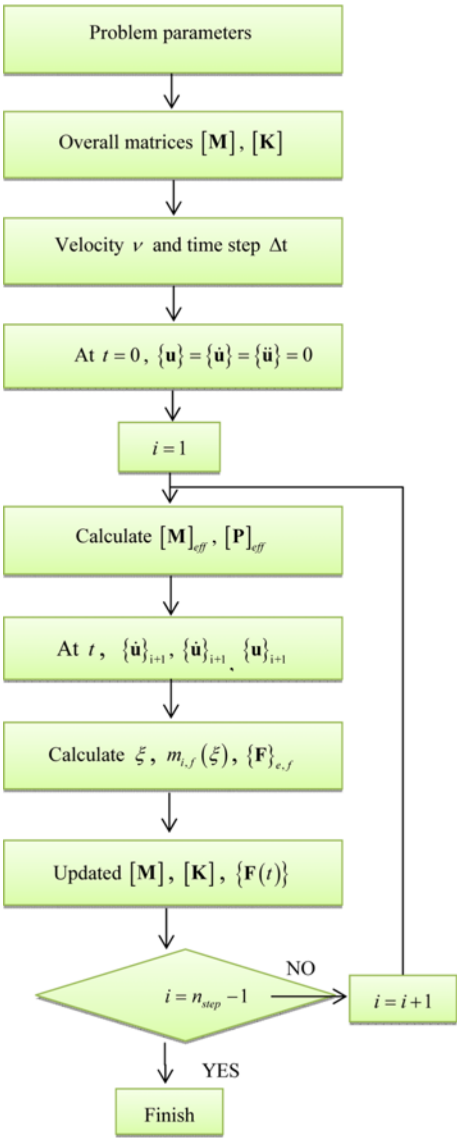


Fig. 4. The Flowchart for Numerical Procedures

any time $t$, the governing differential equation of the beam element resting on two-parameter foundation subjected to a moving mass $M$ without material damping can be written as

$$[\mathbf{M}]_e\{\ddot{\mathbf{u}}_e\}+[\mathbf{K}]_e\{\mathbf{u}_e\} = \{\mathbf{F}\}_{e,f}-[\mathbf{N}_{w,\xi}]^T f_c \tag{10}$$

where $[\mathbf{N}_{w,\xi}]$ is the values of the matrix of interpolation function, and $f_c$ is contact force between the beam resting on the foundation and the moving mass depended on the coordinate $\xi(t)$ of the position of the moving mass on the beam element at the time $t$, given by

$$f_c = (M\ddot{u}(\xi,t)+Mg)\delta(\xi-vt+i^{th}l) \tag{11}$$

with $\delta(\xi-vt+i^{th}l)$ is the Dirac delta function. Substituting Eq. (6) and Eq. (11) into Eq. (10) and rearrangement of this equation gives as

$$([\mathbf{M}]_e+[\mathbf{N}_{w,\xi}]^T M[\mathbf{N}_{w,\xi}])\{\ddot{\mathbf{u}}_e\}+[\mathbf{K}]_e\{\mathbf{u}_e\} = \{\mathbf{F}\}_{e,f}-[\mathbf{N}_{w,\xi}]^T Mg \tag{12}$$

Using the finite element method, the governing equation of motion of the entire system is written as

$$[\mathbf{M}]\{\ddot{\mathbf{u}}\}+[\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{F}(t)\} \tag{13}$$

where $[\mathbf{M}]$, $[\mathbf{K}]$ are the mass and stiffness matrices of the system, respectively; the vectors $\{\ddot{\mathbf{u}}\}$, $\{\dot{\mathbf{u}}\}$, $\{\mathbf{u}\}$ are the acceleration, velocity and displacement vectors, respectively; and $\{\mathbf{F}(t)\}$ is the external load vector. The Newmark method (Chopra, 2001) is used for integrating the Eq. (13) to analyze the dynamic response of the beam.

## 3. Numerical Results

### 3.1 Verified Examples

Before studying numerical results, in order to check the accuracy of the above formulation and the computer program using MATLAB software developed, the results of the present study are compared with those obtained in the literature.

The first example considers a simple support Euler-Bernoulli beam resting on two-parameter elastic foundation with dimensionless parameters of Winkler elastic foundation stiffness $K_1 = k_w L^4/EI$ and shear layer stiffness $K_2 = k_s L^2/\pi^2 EI$. The first dimensionless natural frequency of the beam is compared with results in the literature shown in Table 1. As seen from this Table, the present results are in good agreement with those of Matsunaga (1999).

In order to verify the present dynamic responses due to the moving mass, the dynamic deflections of a simply-supported beam without foundation under a moving mass from the computer program, formulation of this study and Stanisic (1969) are plotted in Fig. 5 with geometric property of the beam $L/h$ = 20 and the constant velocity $v$ = 25 m/s. For the various the mass ratio $R$ = 0.1 and $R$ = 0.25, the displacements of the beam are shown in Figs. 5(a) and 5(b). The comparisons show that the present dynamic deflections are in good agreement; the difference with very small relative error of solution of the present study from finite element method and Stanisic from series form with truncated error may be due to the omission of the terms truncated error in Fourier finite sine transformation. From this results, the comments of the response of the beam due to moving

Table 1. The First Dimensionless Natural Frequencies of Beam Comparison with Previously Published Results

| L/h=10 | $K_2$ | $K_1$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0 | 10 | $10^2$ | $10^3$ | $10^4$ | $10^5$ |
| Matsunaga, 1999 | 0 | 9.8696 | 10.3638 | 14.0502 | 33.1272 | 100.4859 | 316.3817 |
| Present | | 9.8696 | 10.3638 | 14.0502 | 33.1272 | 100.4859 | 316.3817 |
| Matsunaga, 1999 | 1 | 13.9577 | 14.3115 | 17.1703 | 34.5661 | 100.9694 | 316.5356 |
| Present | | 13.9577 | 14.3115 | 17.1703 | 34.5661 | 100.9694 | 316.5356 |

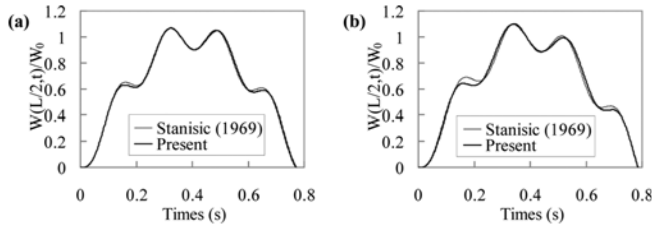


Fig. 5. The Dimensionless Transverse Dynamic Deflections of Beam under the Moving Mass: (a) $R$ = 0.1, (b) R = 0.25 (——) Present, (—) (Stanisic, 1969)

mass are similar in the previous example.

Through above examples, the numerical results from the computer program based on the suggested formulation show good agreement with those presented in literature. Therefore, the program can be used to analyze the influence of mass of foundation on the dynamic responses of the beam subjected to a moving mass in the next parts.

### 3.2 The Influence of Mass of Foundation

The influence of mass of foundation on the dynamic responses of the beam subjected to a moving mass is analysed by the numerical investigation in this part. The moving mass $M$ moves in the axial direction of the beam with constant velocity $v$. The following material and geometric properties of the beam are adopted as: $E$ = 206.10$^9$ N/m$^2$, $\rho$ = 7860 kg/m$^3$ (from steel material), $h$ = 0.01 m and $L$ = 2 m. These properties of the beam are selected to advantage in setting up the experiment in next

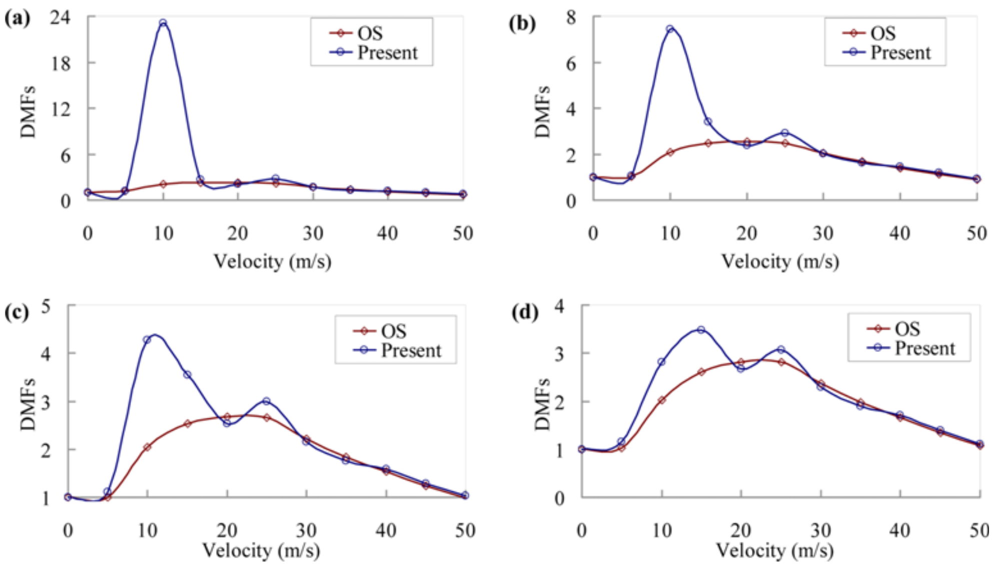


Fig. 6. The Influence of Winkler Elastic Stiffness Parameter on DMFs of the Beam with the Velocity of the Moving Mass for $\mu$ = 1, $\kappa$ = 1, $R$ = 1.5, $K_2$ = 1: (a) $K_1$ = 10, (b) $K_1$ = 50, (c) $K_1$ = 75, (d) $K_1$ = 100

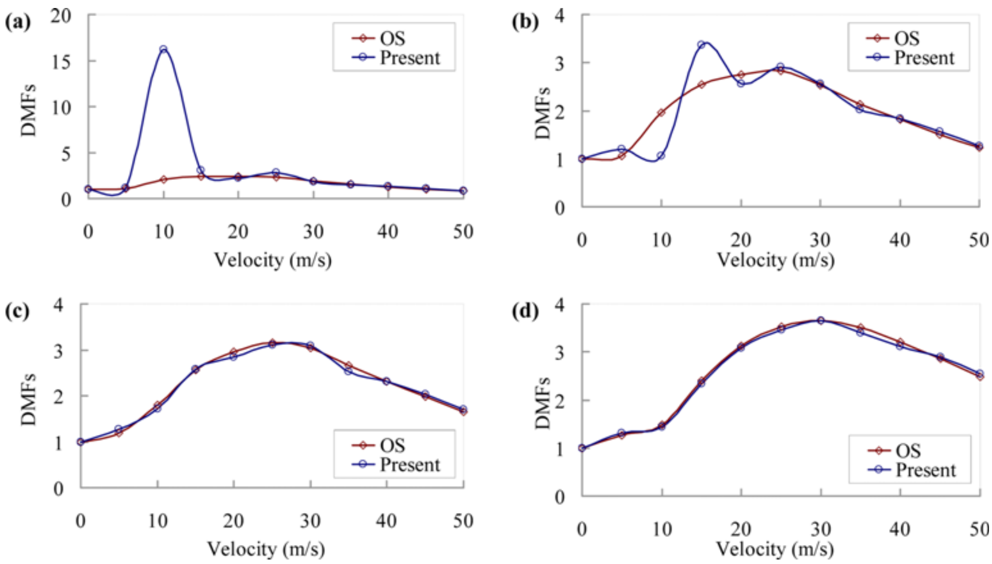


Fig. 7. The Influence of Shear Layer Stiffness Parameter on DMFs of the Beam with the Velocity of the Moving Mass for $\mu$ = 1, $\kappa$ =1, $R$ = 1.5, $K_1$ = 25: (a) $K_2$ = 1, (b) $K_2$ = 2, (c) $K_2$ = 3, (d) $K_2$ = 5

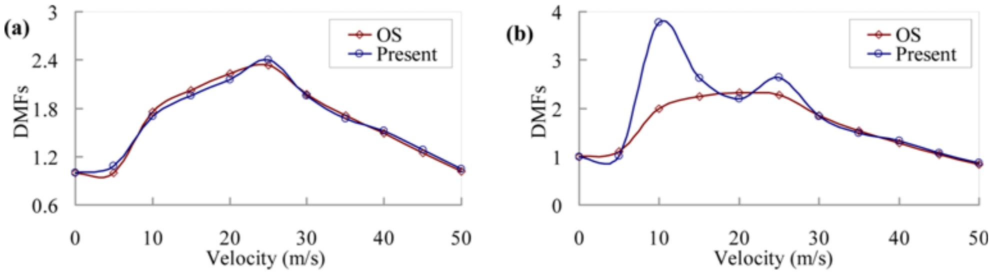


Fig. 8. The Influence of Ratio Mass on DMFs of the Beam with the Velocity of the Moving Mass for $\mu$ = 1, $\kappa$ = 1, $K_1$ = 10, $K_2$ = 1: (a) $R$ = 0.75, (b) $R$ = 1.25

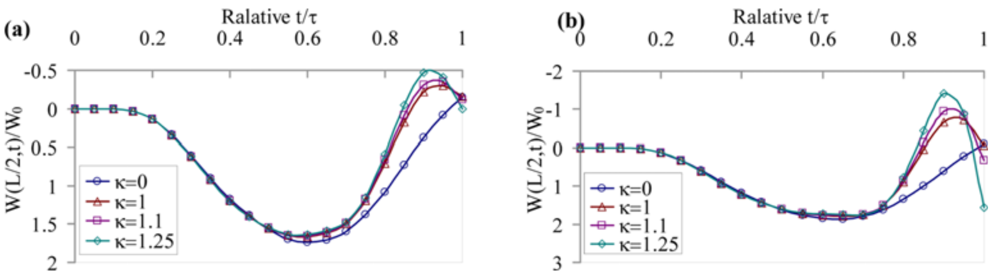


Fig. 9. The Influence of Dimensionless Parameter $\kappa$ on Dimensionless Vertical Dynamic Displacements of the center of the beam for $\upsilon$ = 10 m/s, $\mu$ = 1, $K_1$ = 25, $K_2$ = 1: (a) $R$ = 0.75, (b) $R$ = 1

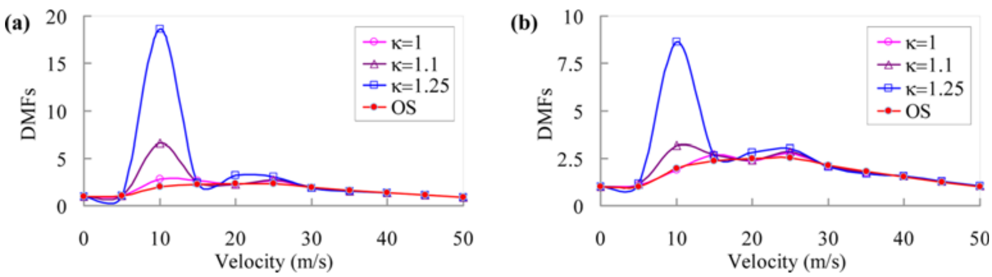


Fig. 10. The Influence of Dimensionless Parameters $\kappa$ on DMFs of the Beam with the Velocity of the Moving Mass for $\mu$ = 1, R = 1.25, $K_2$ = 1: (a) $K_1$ = 20, (b) $K_1$ = 50

steps and are not affecting to the relative results compared from the solutions. The parameters to measure the dynamic responses of the beam based on Dynamic Magnification Factor (DMF) which is defined as the ratio of maximum dynamic deflection to maximum static deflection at the center of the beam are carried out. The numerical results obtained according to the present study are compared with Ordinary Solution (OS) without the influence of mass of foundation.

The DMFs (with and without mass of foundation) for different values of Winkler and shear layer elastic foundation stiffness parameters with various velocities of the moving mass are plotted in Figs. 6, 7. The comparisons show that the mass of foundation is significant effects and increases the DMFs of the beam for a range of low velocity. In range of higher velocity of the moving mass, the results of the present solution and ordinary solution are similar. From the Figs. 6(d) and 7(c) and 7(d), while the values of the stiffness of the foundation (according to stiffness of global system) increase significantly, the dynamic responses of the beam also decrease. It can be seen that the influence of the mass of foundation on the DMFs of the beam is not really significant and the results of the two solutions are quite similar.

In the next results, Fig. 8 plots the influence of the mass ratio $R$ (depending on the moving mass) on dynamic magnification factors of the beam with the velocity of the moving mass. The observation in this case is same with previous ones. Moreover, the values $R$ to be significantly extended, the dynamic responses of the beam are also increasing so the influence of the mass of foundation on the results is really significant and the results of the two solutions are difference shown clearly in Figs. 8(c) and 8(d).

In the last results, the influence of the properties of the mass of foundation including the dimensionless parameter $\kappa$ and ratio density $\mu$ is studied. The times history of dimensionless vertical displacement of the center of the beam and dynamic magnification factors are shown in Figs. 9, 10 for the dimensionless parameter $\kappa$ and Figs. 11, 12 for various ratio density $\mu$. The dynamic responses of the beam have significant difference and sensitivity between the present study and ordinary solution in many cases. Furthermore, the comparisons show that the responses of the beam have the significant increase due to the effect of mass of

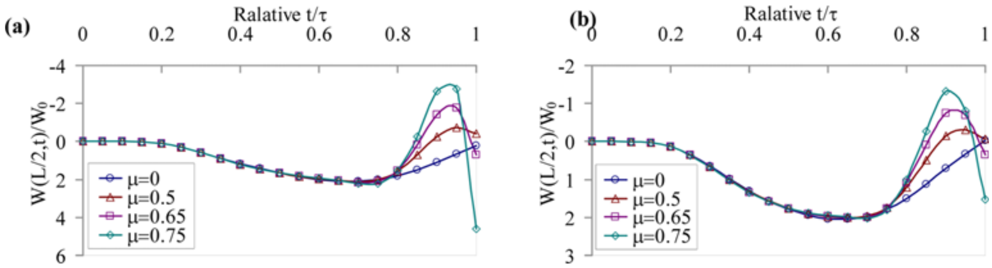


Fig. 11. The Influence of Ratio Density on Dimensionless Vertical Dynamic Displacements of the Center of the Beam for $\upsilon$ = 10 m/s, $\kappa$ = 1.2, $R$ = 1.5, $K_2$ = 1: (a) $K_1$ = 25, (b) $K_1$ = 75

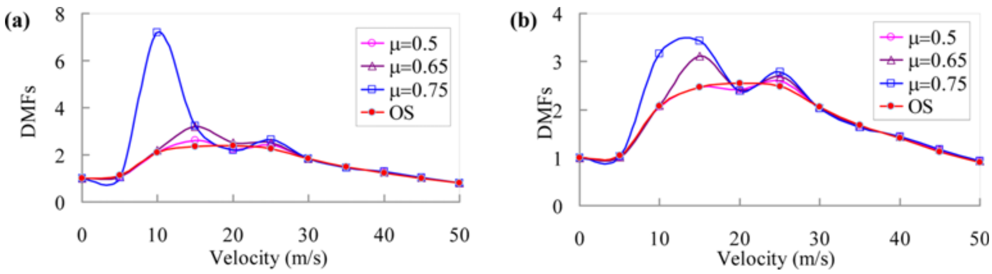


Fig. 12. The Influence of Ratio Density μ on DMFs of the *b*eam with the Velocity of the Moving Mass for $\kappa$ = 1.2, $R$ = 1.5, $K_2$ = 1: (a) $K_1$ = 20, (b) $K_1$ = 50

foundation.

## 4. Conclusions

The influence of mass of two-parameter elastic foundation on dynamic responses of the beam subjected to a moving mass has been studied in this paper. The mass of foundation is directly proportional with vertical displacement of the springs. The comparisons between present solution and ordinary solution without the influence of mass of foundation show that the dynamic responses of the beam are quite different and the influence of mass of foundation is increasing the dynamic responses than the ordinary solution for a range of low velocity of the moving mass.

## References

Abohadima, S. and Taha, M. H. (2009). "Dynamic analysis of nonuniform beams on elastic foundations." *The Open Applied Mathematics Journal*, Vol. 3, No. 1, pp. 40-44, DOI: 10.2174/1874114200903010040.

Avramidis, I. E. and Morfidis, K. (2006). "Bending of beams on three-parameter elastic foundation." *International Journal of Solids and Structures*, Vol. 43, No. 2, pp. 357-375, DOI: 10.1016/j.ijsolstr.2005.03.033.

Çalım, F. F. (2009). "Dynamic analysis of beams on viscoelastic foundation." *European Journal of Mechanics - A/Solids*, Vol. 28, No. 3, pp. 469-476, DOI: 10.1016/j.euromechsol.2008.08.001.

Çalım, F. F. (2012). "Forced vibration of curved beams on two-parameter elastic foundation." *Applied Mathematical Modelling*, Vol. 36, No. 3, pp. 964-973, DOI: 10.1016/j.apm.2011.07.066.

Chen, W. Q., Lü, C. F., and Bian, Z. G. (2004). "A mixed method for bending and free vibration of beams resting on a Pasternak elastic foundation." *Applied Mathematical Modelling*, Vol. 28, No. 10, pp. 877-890, DOI: 10.1016/j.apm.2004.04.001.

Chopra, A. K. (2001). Dynamics of Structures, 2nd edition, Prentice-Hall.

Eisenberger, M. (1994). "Vibration frequencies for beams on variable one- and two-paramter elastic foundations." *Journal of Sound and Vibration*, Vol. 176, No. 5, pp. 577-584, DOI: 10.1006/jsvi.1994.1399.

Eisenberger, M. and Clastornik, J. (1987). "Vibrations and buckling of a beam on a variable Winkler elastic foundation." *Journal of Sound and Vibration*, Vol. 115, No. 2, pp. 233-241, DOI: 10.1016/0022-460X(87)90469-X.

Gupta, U. S., Ansari, A. H., and Sharma, S. (2006). "Buckling and vibration of polar orthotropic circular plate resting on Winkler foundation." *Journal of Sound and Vibration*, Vol. 297, Nos. 3-5, pp. 457-476, DOI: 10.1016/j.jsv.2006.01.073.

Kacar, A., Tan, H. T., and Kaya, M. O. (2011). "A note free vibration analysis of beams on variable winkler elastic foundation by using the differential transform method." *Mathematical and Computational Applications*, Vol. 16, No. 3, pp. 773-783.

Kargarnovin, M. H. and Younesian, D. (2004). "Dynamics of Timoshenko beams on Pasternak foundation under moving load." *Mechanics Research Communications*, Vol. 31, No. 6, pp. 713-723, DOI: 10.1016/j.mechrescom.2004.05.002.

Konstantinos, S. P. and Dimitrios, S. S. (2013). "Buckling of beams on elastic foundation considering discontinuous (unbonded) contact." *International Journal of Mechanics and Applications,* Vol. 3, No. 1, pp. 4-12, DOI: 10.5923/j.mechanics.20130301.02.

Lee, H. P. (1998). "Dynamic response of a Timoshenko beam on a Winkler foundation subjected to a moving mass." *Applied Acoustics*, Vol. 55, No. 3, pp. 203-215, DOI: 10.1016/S0003-682X(97)00097-2.

Malekzadeh, P., Karami, G., and Farid, M. (2003). "DQEM for free vibration analysis of Timoshenko beams on elastic foundations." *Comput. Mech.*, Vol. 31, Nos. 3-4, pp. 219-228, DOI: 10.1007/s00466-002-0387-y.

Matsunaga, H. (1999). "Vibration and buckling of deep beam-coulmns on two parameter elastic foundations." *Journal of Sound and*

*Vibration*, Vol. 228, No. 2, pp. 359-376, DOI: 10.1006/jsvi.1999.2415.

Mohanty, S. C., Dash, R. R., and Rout, T. (2012). "Parametric instability of a functionally graded Timoshenko beam on Winkler's elastic foundation." *Nuclear Engineering and Design*, Vol. 241, No. 8, pp. 2698-2715, DOI: 10.1016/j.nucengdes.2011.05.040.

Morfidis, K. (2010). "Vibration of Timoshenko beams on three-parameter elastic foundation." Computers and Structures. Vol. 88, Nos. 5-6, pp. 294-308, DOI: 10.1016/j.compstruc.2009.11.001.

Ruge, P. and Birk, C. (2007). "A comparison of infinite Timoshenko and Euler-Bernoulli beam models on Winkler foundation in the frequency- and time-domain." *Journal of Sound and Vibration*, Vol. 304, Nos. 3-5, pp. 932-947, DOI: 10.1016/j.jsv.2007.04.001.

Stanisic, M. M. and Hardin, J. C. (1969). "On the response of beams to an arbitrary number of concentrated moving masses." *Journal Franklin Inst.*, Vol. 287, No. 2, pp. 115-23, DOI: 10.1016/0016-0032(69)90120-3.