KSCE Journal of Civil Engineering (2016) 20(6):2492-2498

DOI 10.1007/s12205-015-0510-1

Structural Engineering

pISSN 1226-7988, eISSN 1976-3808
www.springer.com/12205

Exact Deformation of a Rectangular Plate with a Central Circular Hole under In-plane Loads

Jae-Hoon Kang*

Received July 2, 2015/Accepted October 11, 2015/Published Online November 30, 2015

Abstract

Exact solutions for stresses, strains, and displacements of a perforated rectangular plate by a central circular hole under both linearly varying in-plane normal stresses on two opposite edges and in-plane shear stresses acting on its entire outer boundary are investigated using the Airy stress function. The hoop stresses arising at the edge of the circular hole are also calculated and plotted. Stress concentration factors (the maximum non-dimensional hoop stresses) depending upon the size of the circular hole and the in-plane loading condition are tabularized.

Keywords: *perforated plate, circular hole, airy stress function, exact solution, hoop stress, stress concentration factor*

1. Introduction

Numerous researchers have studied the mechanical behaviors of perforated plates, with main considerations being classified into four categories; stress concentration (Savin, 1961; Muskhelishvili, 1963; Miyata, 1970; Timoshenko and Goodier, 1970; Peterson, 1974; Iwaki and Miyao, 1980; Theocaris and Petrou, 1987; Mal and Singh, 1991; Fu, 1996; Yang and He, 2002; Zhang *et al.*, 2002; She and Guo, 2007; Li *et al.*, 2008; Yang *et al.*, 2008; Yu *et al.*, 2008; Radi, 2011; Kang, 2014; Woo *et al.*, 2014), vibration, buckling and fatigue. The various methods have been used to study them. The Finite Element Method (FEM) is the most widely used for this perforated plate problems. Various methods other than FEM have been used like the complex variable method, three-dimensional stress analysis, the Ritz method, the boundary element method, the differential quadrature element method, semi-analytical solution method, experimental method, conjugate load/displacement method, and Galerkin averaging method. Most of the perforated holes form three kinds of circular, elliptical, or rectangular cutout.

In the present study, exact solutions for stresses, strains, displacements, and stress concentration factors of a rectangular plate with a central circular hole subjected to both linearly varying in-plane normal stresses on two opposite edges and in-plane shear stresses acting on its entire outer boundary are investigated using the Airy stress function. Most of previous analyzers have dealt with the perforated plate under uni-axial or bi-axial tension. Timoshenko and Goodier (1970) presented exact solutions for a rectangular plate with a central circular hole under uni-axial tension. Mal and Singh (1991) obtained exact solutions for the perforated plate under in-plane pure shear loading. Exact solutions for perforated plates with a central circular hole subjected to both linearly varying in-plane normal and shear stresses have not been reported. The hoop stresses arising at the edge of the circular hole are computed and plotted. Comparisons are made for the stress concentration factors depending upon in-plane loading types and size of the circular hole.

2. Airy Stress Function

Figure 1 shows a perforated rectangular plate of lateral dimensions $L \times h$ by a central circular hole of radius of a under both linearly varying in-plane normal stresses on two opposite edges at $x = \pm L/2$ and in-plane shear stresses τ_0 acting on its entire outer boundary. Also the positive sign conventions for stress components in polar coordinates (r, θ) is presented. The plate is assumed to be very large compared with the circular

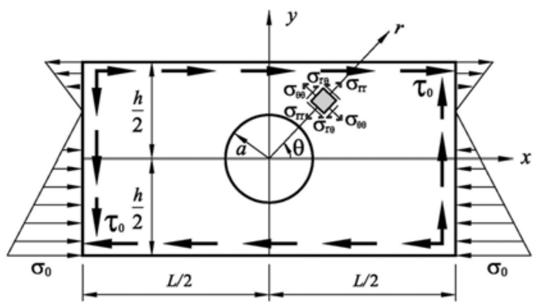

Fig. 1. A Rectangular Plate with a Central Circular Hole under Both Linearly Varying In-plane Normal Stresses on Two Opposite Edges and In-plane Shear Stresses

*Member, Professor, Dept. of Architectural Engineering, Chung-Ang University, Seoul 06974, Korea (Corresponding Author, E-mail: jhkang@cau.ac.kr)

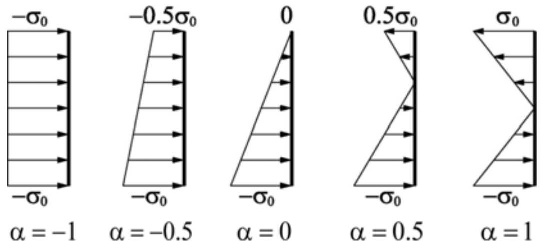

Fig. 2. Examples of In-plane Loading σ_{xx} along the Edge x = –L/2

hole. The origins of rectangular (x, y) and polar (r, θ) coordinate systems are located at the center of the plate.

First of all, considering a rectangular plate with no hole subjected to linearly varying in-plane normal stresses on two opposite edges and in-plane shear stresses τ_0 acting on its entire outer boundary, the stress components become

$$\sigma_{xx}^0 = \frac{\partial^2\phi^0}{\partial y^2} = \frac{\sigma_0(1+\alpha)}{h}y + \frac{\sigma_0}{2}(\alpha-1)$$

$$\sigma_{xy}^0 = -\frac{\partial^2\phi^0}{\partial x\partial y} = \tau_0$$

$$\sigma_{yy}^0 = \frac{\partial^2\phi^0}{\partial x^2} = 0 \tag{1}$$

where ϕ^0 is a fundamental Airy stress function, σ_0 is the intensity of compressive stress at $y = -h/2$, and α is a numerical loading factor. By changing α, we can obtain various particular cases. For example, by taking $\alpha = -1$ we have the case of uniformly distributed compressive force. When $\alpha = 0$, the compressive force varies linearly from $-\sigma_0$ at $y = -h/2$ to zero at $y = +h/2$. For $\alpha = 1$ we obtain the case of pure in-plane bending. With other α in the range $-1 < \alpha < 1$, we have a combination of bending and compression. Examples of these cases are shown in Fig. 2. For $\alpha < -1$ or $\alpha > 1$ the problems arising are identical with ones having $-1 < \alpha < 1$. The fundamental Airy stress function ϕ^0 satisfies the governing equation $\nabla^4\phi^0 = \nabla^2(\nabla^2\phi^0) = 0$ with no body forces, where the Laplacian operator ∇^2 is expressed as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2}$$

and ∇^4 is the bi-harmonic differential operator defined by

$$\nabla^4 = \nabla^2(\nabla^2) = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4} \tag{3}$$

in rectangular coordinates. By means of the relation of the Airy stress function and the stress components in Eq. (1), the fundamental Airy function ϕ^0 can be assumed as

$$\phi^0 = \frac{\sigma_0(1+\alpha)}{6h}y^3 + \frac{\sigma_0}{4}(\alpha-1)y^2 - \tau_0 xy + Ax + By + C \tag{4}$$

where A, B, and C are arbitrary integration constants. A linear function of x or y and a constant in the Airy stress function are trivial terms which do not generate any stresses and strains (Fu, 1996). Eliminating the trivial terms in Eq. (4), the fundamental Airy stress function ϕ^0 results in

$$\phi^0 = \frac{\sigma_0(1+\alpha)}{6h}y^3 + \frac{\sigma_0}{4}(\alpha-1)y^2 - \tau_0 xy \tag{5}$$

Using the relations of

$$x = r\cos\theta,\ y = r\sin\theta \tag{6}$$

and the formulas of multiple angles

$$\sin^2\theta = \frac{1-\cos2\theta}{2},\ \sin^3\theta = \frac{3\sin\theta - \sin3\theta}{4} \tag{7}$$

Equation (5) can be converted into the bi-harmonic functions in polar coordinates as

$$\phi^0 = \frac{\sigma_0}{24}\left[\frac{(1+\alpha)}{h}(3r^3\sin\theta - r^3\sin3\theta) - 3(\alpha-1)(r^2\cos2\theta - r^2)\right] - \frac{\tau_0}{2}r^2\sin2\theta \tag{8}$$

which satisfies the governing equation $\nabla^4\phi^0 = \nabla^2(\nabla^2\phi^0) = 0$, where ∇^2 is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{9}$$

and ∇^4 is defined by

$$\nabla^4 = \nabla^2(\nabla^2) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2} + \frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) \tag{10}$$

in polar coordinates. From the below relations between stresses and the Airy stress function in polar coordinates, the stresses in the rectangular plate with no hole under linearly varying in-plane normal stresses and in-plane shear stresses τ_0 acting on its entire outer boundary can be calculated as

$$\sigma_{rr}^0 = \frac{1}{r}\frac{\partial\phi^0}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi^0}{\partial\theta^2} = \frac{\sigma_0}{4}\left[\frac{1+\alpha}{h}(\sin\theta + \sin3\theta)r + (\alpha-1)(\cos2\theta+1)\right] + \tau_0\sin2\theta$$

$$\sigma_{r\theta}^0 = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi^0}{\partial\theta}\right) = \frac{\sigma_0}{4}\left[\frac{1+\alpha}{h}(\cos3\theta - \cos\theta)r + (1-\alpha)\sin2\theta\right] + \tau_0\cos2\theta$$

$$\sigma_{\theta\theta}^0 = \frac{\partial^2\phi^0}{\partial r^2} = \frac{\sigma_0}{4}\left[\frac{1+\alpha}{h}(3\sin\theta - \sin3\theta)r - (\alpha-1)(\cos2\theta-1)\right] - \tau_0\sin2\theta \tag{11}$$

Returning to the original problem of a rectangular plate with a central circular hole, the total Airy function ϕ becomes

$$\phi = \phi^0 + \phi^* \tag{12}$$

where ϕ^* is an Airy stress function to cancel unwanted traction due to ϕ^0 on the edge of the circular hole ($r = a$). The normal stress σ_{rr} and shear stress $\sigma_{r\theta}$ on the edge of the circular hole ($r = a$) must be free as below

$$\sigma_{rr}\big|_{r=a} = [\sigma_{rr}^0 + \sigma_{rr}^*]_{r=a} = 0$$

$$\sigma_{r\theta}\big|_{r=a} = [\sigma_{r\theta}^0 + \sigma_{r\theta}^*]_{r=a} = 0 \tag{13}$$

Table 1. Stresses of Bi-harmonic Functions ϕ

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$\ln r$	$1/r^2$	0	$-1/r^2$
$r^2 \ln r$	$2\ln r+1$	0	$2\ln r+3$
$r^3\sin\theta$	$2r \sin\theta$	$-2r \cos\theta$	$6r \sin\theta$
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\ln r\sin\theta$	$\sin\theta/r$	$-\cos\theta/r$	$\sin\theta/r$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$2\sin\theta/r^3$
$r^2\cos2\theta$	$-2\cos2\theta$	$2\sin2\theta$	$2\cos2\theta$
$r^4\cos2\theta$	0	$6r^2\sin2\theta$	$12r^2\cos2\theta$
$\cos e\theta/r^2$	$-6\cos2\theta/r^4$	$-6\sin2\theta/r^4$	$6\cos2\theta/r^4$
$\cos2\theta$	$-4\cos2\theta/r^2$	$-2\sin2\theta/r^2$	0
$r^3\sin3\theta$	$-6r\sin3\theta$	$-6r\cos3\theta$	$6r\sin3\theta$
$r^5\sin3\theta$	$-4r^3\sin3\theta$	$12r^3\cos3\theta$	$20r^3\sin3\theta$
$\sin3\theta/r^3$	$-12\sin3\theta/r^3$	$12\cos3\theta/r^5$	$12\sin3\theta/r^5$
$\sin3\theta/r$	$-10\sin3\theta/r^3$	$6\cos3\theta/r^3$	$2\sin3\theta/r^3$
$r^2\sin2\theta$	$-2\sin2\theta$	$-2\cos2\theta$	$2\sin2\theta$
$\sin2\theta$	$-4\sin2\theta/r^2$	$2\cos2\theta/r^2$	0
$r^4\sin2\theta$	0	$-6r^2\cos2\theta$	$12r^2\sin2\theta$
$\sin2\theta/r^2$	$-6\sin2\theta/r^4$	$6\cos2\theta/r^4$	$6\sin2\theta/r^4$

Therefore, σ^*_{rr} and $\sigma^*_{r\theta}$ on $r = a$ must have terms of $\sin\theta$, $\sin 3\theta$, $\cos 2\theta$, $\sin 2\theta$ or a constant and have $\cos\theta$, $\cos 3\theta$, $\sin 2\theta$, or $\cos 2\theta$ respectively, in order to eliminate the stresses on $r = a$ due to ϕ^0 in Eqs. (11). Tables 1 and 2 show the potential candidates of the bi-harmonic functions for the present problem identified as r^2, $\ln r$, $r^2 \ln r$, $r^3 \sin\theta$, $r\theta\cos\theta$, $r \ln r \sin\theta$, $\sin\theta/r$, $r^2 \cos 2\theta$, $r^4 \cos 2\theta$, $r^3 \sin 3\theta$, $r^5 \sin 3\theta$, $\sin^3\theta/r^3$, $\sin 3\theta/r$, $r^3 \sin 2\theta$, $\sin 2\theta$, $r^4 \sin 2\theta$, and $\sin2\theta/r^2$ from the tables by Dundurs (Fu, 1996), which contain stresses and displacements of certain bi-harmonic functions in polar coordinates. However, the five terms of $r^3 \sin\theta$, $r^3 \sin 3\theta$, $r^2 \cos2\theta$, r^2, and $r^2\sin 2\theta$ in the fundamental Airy stress function ϕ^0 of Eq. (8) must be excluded in ϕ^* in order not to disturb the traction boundary conditions in Eq. (1). The terms of $r\ln r\sin\theta$ and $r\theta\cos\theta$ give rise to multi-valued displacements in u_r and u_θ. Singularity at infinity arises in stresses or displacements owing to the term of $r^5\sin 3\theta$, $r^4\cos 2\theta$, and $r^4\sin 2\theta$. The term of $r^2 \ln r$ gives multi-valueness in u_θ and singularity in normal stress at infinity. Eliminating the inappropriate terms, the total Airy stress function ϕ in Eq. (12) can be assumed as

$$\phi = \frac{\sigma_0}{24}\left[\frac{(1+\alpha)}{h}(3r^3\sin\theta - r^3\sin3\theta) - 3(\alpha-1)(r^2\cos2\theta - r^2)\right.$$
$$+ C_1a^2\ln r + C_2\frac{a^3\sin\theta}{r} + C_3\frac{a^4\cos2\theta}{r^2} + C_4a^2\cos2\theta + C_5\frac{a^5\sin3\theta}{r^3}$$
$$\left. C_6 + \frac{a^3\sin3\theta}{r}\right] - \frac{\tau_0}{2}\left[r^2\sin2\theta + C_7a^2\sin2\theta + C_8\frac{a^4\sin2\theta}{r^2}\right] \tag{14}$$

where C_1~C_8 are arbitrary integration constants which are determined by the traction boundary conditions at $r = a$. In order to make the constants C_1~C_8 dimensionless, they are multiplied by $a^2, \ldots, a^5$. Applying the following stress free boundary conditions on the edge of the circular hole at $r = a$,

$$\sigma_{rr}\big|_{r=a} = \left[\frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right]_{r=a} = 0$$

$$\sigma_{r\theta}\big|_{r=a} = \left[-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\right]_{r=a} = 0 \tag{15}$$

the unknown constants are computed as

$$C_1 = 6(1-\alpha),\ C_2 = 3\beta(1+\alpha),\ C3 = 3(1-\alpha),\ C_4 = -6(1-\alpha),$$
$$C_5 = -2\beta(1+\alpha),\ C_6 = 3\beta(1+\alpha),\ C_7 = -2,\ C_8 = 1 \tag{16}$$

where β is a ratio of a/h. Thus the total Airy stress function ϕ

Table 2. Displacements of Bi-harmonic Functions ϕ

ϕ	$2\mu u_r$	$2\mu u_\theta$
r^2	$(\kappa-1)r$	0
$\ln r$	$-1/r$	0
$r^2\ln r$	$(\kappa-1)r\ln r-r$	$(\kappa+1)r\theta$
$r^3\sin\theta$	$(\kappa-2)r^2\sin\theta$	$-(\kappa+2)r^2\cos\theta$
$r\theta\cos\theta$	$[(\kappa-1)\theta\cos\theta-(\kappa+1)\ln r\sin\theta+\sin\theta]/2$	$[-(\kappa-1)\theta\sin\theta-(\kappa+1)\ln r\cos\theta+\cos\theta]/2$
$r\ln r\sin\theta$	$[-(\kappa+1)\theta\cos\theta+(\kappa-1)\ln r\sin\theta-\sin\theta]/2$	$[(\kappa+1)\theta\sin\theta+(\kappa-1)\ln r\cos\theta-\cos\theta]/2$
$\sin\theta/r$	$\sin\theta/r^2$	$-\cos\theta/r^2$
$r^2\cos2\theta$	$-2r\cos2\theta$	$2r\sin2\theta$
$r^4\cos2\theta$	$-(3-\kappa)r^3\cos2\theta$	$(3+\kappa)r^3\sin2\theta$
$\cos2\theta/r^2$	$2\cos2\theta/r^3$	$2\sin2\theta/r^3$
$r^3\sin3\theta$	$-3r^2\sin2\theta$	$-3r^2\cos2\theta$
$r^5\sin3\theta$	$-(4-\kappa)r^4\sin3\theta$	$-(4+\kappa)r^4\cos3\theta$
$\sin3\theta/r^3$	$3\mathrm{sing}3\theta/r^4$	$-3\cos3\theta/r^4$
$\sin3\theta/r$	$(2+\kappa)\sin3\theta/r^2$	$-(2-\kappa)\cos3\theta/r^2$
$r^2\sin2\theta$	$-2r\sin2\theta$	$-2r\cos2\theta$
$\sin2\theta$	$(\kappa+1)\sin2\theta/r$	$(\kappa-1)\cos2\theta/r$
$r^4\sin2\theta$	$-3(3-\kappa)r^3\sin2\theta$	$-(3+\kappa)r^3\cos2\theta$
$\sin2\theta/r^2$	$2\sin2\theta/r^3$	$-2\cos2\theta/r^3$

finally becomes

$$\phi = \frac{\sigma_0}{24}\left[\frac{\beta(1+\alpha)}{a}(3r^3\sin\theta - r^3\sin3\theta) + 3(1-\alpha)(r^2\cos2\theta - r^2)\right.$$
$$6a^2(1-\alpha)\ln r + 3a^3\beta(1+\alpha)\frac{\sin\theta}{r} + 3a^4(1-\alpha)\frac{\cos2\theta}{r^2}$$
$$\left.-6a^2(1-\alpha)\cos2\theta - 2a^5\beta(1+\alpha)\frac{\sin3\theta}{r^3} + 3a^3\beta(1+\alpha)\frac{\sin3\theta}{r}\right] \tag{17}$$
$$-\frac{\tau_0}{2}\left[r^2\sin2\theta - 2a^2\sin2\theta + \frac{a^4\sin2\theta}{r^2}\right]$$

3. Stresses

Substituting the total Airy stress function ϕ in Eq. (17) into the relations between the stresses and the Airy stress function ϕ in polar coordinates in Eq. (11), the stresses can be calculated as

$$\sigma_{rr} = \sigma_0\left[\beta(1+\alpha)\left\{\left(\frac{r}{4a} - \frac{5a^3}{4r^3} + \frac{a^5}{r^5}\right)\sin3\theta + \frac{1}{4}\left(\frac{r}{a} - \frac{a^3}{r^3}\right)\sin\theta\right\}\right.$$
$$\left.-(1-\alpha)\left\{\left(\frac{1}{4} - \frac{a^2}{r^2} + \frac{3a^4}{4r^4}\right)\cos2\theta + \frac{1}{4}\left(1 - \frac{a^2}{r^2}\right)\right\}\right] + \tau_0\left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right)\sin2\theta$$

$$\sigma_{r\theta} = \sigma_0\left[\beta(1+\alpha)\left\{\left(\frac{r}{4a} + \frac{3a^3}{4r^3} - \frac{a^5}{r^5}\right)\cos3\theta - \frac{1}{4}\left(\frac{r}{a} - \frac{a^3}{r^3}\right)\cos\theta\right\}\right.$$
$$\left.+\frac{(1-\alpha)}{4}\left(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right)\sin2\theta\right] + \tau_0\left(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right)\cos2\theta$$

$$\sigma_{\theta\theta} = \sigma_0\left[-\beta(1+\alpha)\left\{\left(\frac{r}{4a} - \frac{a^3}{4r^3} + \frac{a^5}{r^5}\right)\sin3\theta - \frac{1}{4}\left(3\frac{r}{a} + \frac{a^3}{r^3}\right)\sin\theta\right\}\right.$$
$$\left.+\frac{1}{4}(1-\alpha)\left\{\left(1 + \frac{3a^4}{r^4}\right)\cos2\theta - 1 - \frac{a^2}{r^2}\right\}\right] - \tau_0\left(1 + \frac{3a^4}{r^4}\right)\sin2\theta \tag{18}$$

The limiting case of a plate with no hole ($a \to 0$, $\beta \to 0$), the stresses is calculated as

$$\sigma_{rr} = -\frac{\sigma_0}{4}(1-\alpha)(\cos2\theta + 1)\,\tau_0\sin2\theta$$
$$\sigma_{r\theta} = \frac{\sigma_0}{4}(1-\alpha)\sin2\theta + \tau_0\cos2\theta$$
$$\sigma_{\theta\theta} = \frac{\sigma_0}{4}(1-\alpha)(\cos2\theta - 1) - \tau_0\sin2\theta \tag{19}$$

Timoshenko and Goodier (1970) analyzed the exact stresses in a rectangular plate with a central circular hole under uni-axial uniform compression ($\alpha = -1$, $\tau_0 = 0$). Substituting $\alpha = -1$ and $\tau_0 = 0$ into the stress components in Eqs. (18) results in

$$\sigma_{rr} = -\frac{\sigma_0}{2}\left[1 - \frac{a^2}{r^2} + \left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right)\cos2\theta\right]$$
$$\sigma_{r\theta} = \frac{\sigma_0}{2}\left(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right)\sin2\theta$$
$$\sigma_{\theta\theta} = -\frac{\sigma_0}{2}\left[1 + \frac{a^2}{r^2} - \left(1 + \frac{3a^4}{r^4}\right)\cos2\theta\right] \tag{20}$$

which are exactly same with those by Timoshenko and Goodier (1970).

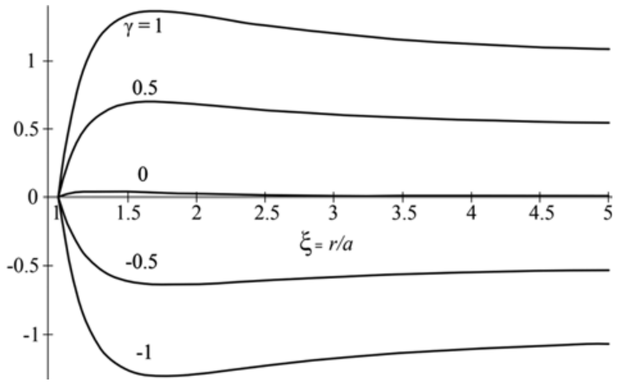

Fig. 3. Non-dimensional Shear Stress $\sigma_{r\theta}/\sigma_0$ for $\alpha = 1$, $\beta = 0.1$ and $\theta = 0°$

Mal and Singh (1991) presented the exact stresses in a rectangular plate with a central hole under in-plane pure shear ($\sigma_0 = 0$). Substituting $\sigma_0 = 0$ into Eqs. (18), the stress components become

$$\sigma_{rr} = \tau_0\left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right)\sin2\theta$$
$$\sigma_{r\theta} = \tau\left(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right)\cos2\theta$$
$$\sigma_{\theta\theta} = -\tau_0\left(1 + \frac{3a^4}{r^4}\right)\sin2\theta \tag{21}$$

which exactly coincide with those by Mal and Singh (1991).

As a numerical example, Fig. 3 shows the non-dimensional shear stress $\sigma_{r\theta}/\sigma_0$ for $\alpha = 1$, $\beta = 0.1$ and $\theta = 0°$ with $\gamma = \tau_0/\sigma_0 = -1, -0.5, 0, 0.5$, and 1 where $\xi = r/a$.

4. Displacements

Using the table by Dundurs (Fu, 1996), displacement components can be easily calculated by selecting and summing the displacement corresponding to each term in the total Airy stress functions ϕ in Eq. (17). The displacement components are as below

$$\mu_r = \frac{\sigma_0}{16\mu}\left[\beta(1+\alpha)\left\{\left(\frac{(\kappa-2)}{a}r^2 + \frac{a^3}{r^2}\right)\sin\theta + \left(\frac{r^2}{a} - \frac{2a^5}{r^4} + \frac{a^3(\kappa+2)}{r^2}\right)\sin3\theta\right\}\right.$$
$$\left.-2(1-\alpha)\left\{\left(r - \frac{a^4}{r^3} + \frac{a^2(\kappa+1)}{r}\right)\cos2\theta + \frac{\kappa-1}{2}r + \frac{a^2}{r}\right\}\right]$$
$$+\frac{\tau_0}{2\mu}\left[r + \frac{a^2(\kappa+1)}{r} - \frac{a^4}{r^3}\right]\sin2\theta$$
$$u_\theta = \frac{\sigma_0}{16\mu}\left[\beta(1+\alpha)\left\{\left(\frac{r^2}{a} + \frac{2a^5}{r^4} + \frac{a^3(\kappa-2)}{r^2}\right)\cos3\theta - \left(\frac{2(\kappa+2)}{a}r^2 + \frac{a^3}{r^2}\right)\cos\theta\right\}\right.$$
$$\left.+2(1-\alpha)\left(r + \frac{a^4}{r^3} + \frac{a^2(\kappa-1)}{r}\right)\sin2\theta\right] + \frac{\tau_0}{2\mu}\left[r + \frac{a^2(\kappa-1)}{r} + \frac{a^4}{r^3}\right]\cos2\theta \tag{22}$$

where μ is shear modulus and κ is a secondary elastic constant

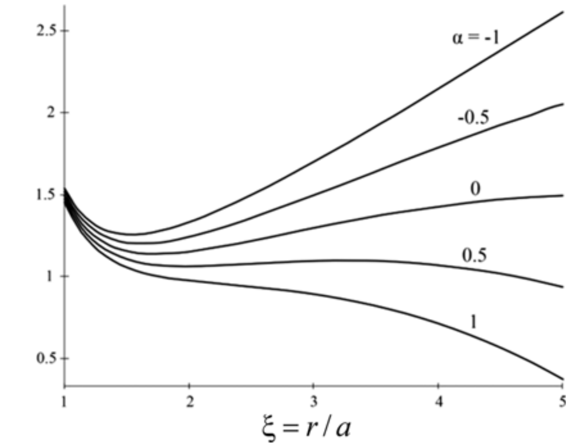

Fig. 4. Non-dimensional Displacement $\mu_{u\theta}/a\sigma_0$ for β = 0.1, γ = 0.1, ν = 0.3, and θ = 0°

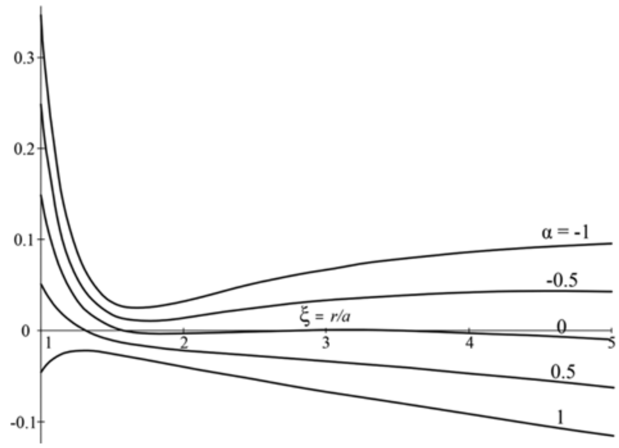

Fig. 5. Non-dimensional Strain $\mu\varepsilon_{rr}/\sigma_0$ for β = 0.1 and θ = 90°

defined by

$$\kappa = \frac{2-\nu}{1+\nu} \tag{23}$$

for the plane stress problems.

The limiting case of a plate with no hole ($a \to 0$, $\beta \to 0$), the displacements are calculated as below:

$$\mu_r = -\frac{\sigma_0 r}{8\mu}(1-\alpha)\left(\cos 2\theta + \frac{\kappa-1}{2}\right) + \frac{\tau_0}{2\mu} r \sin 2\theta$$
$$\mu_\theta = \frac{\sigma_0 r}{8\mu}(1-\alpha)\sin 2\theta + \frac{\tau_0}{2\mu} r\cos 2\theta \tag{24}$$

As a numerical example, Fig. 4 shows the non-dimensional displacement $\mu u_\theta/a\sigma_0$ for β = 0.1, γ = 0.1, ν = 0.3, and θ = 0°.

5. Strains

Substituting the displacement components in Eqs. (22) into the following well-known relations of strain-displacement in polar coordinates

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r},\ \varepsilon_{r\theta} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right],\ \varepsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) \tag{25}$$

the strains can be computed as

$$\varepsilon_{rr} = \frac{\sigma_0}{8\mu}\left[\beta(1+\alpha)\left\{\left(\frac{\kappa-2}{a}r - \frac{a^3}{r^3}\right)\sin\theta + \left(\frac{r}{a} + 4\frac{a^5}{r^5} - \frac{a^3(\kappa+2)}{r^3}\right)\sin 3\theta\right\}\right.$$
$$\left. -(1-\alpha)\left\{\left(1 + \frac{3a^4}{r^4} - \frac{a^2(\kappa+1)}{r^2}\right)\cos 2\theta + \frac{\kappa-1}{2} - \frac{a^2}{r^2}\right\}\right] + \frac{\tau_0}{2\mu}\left[1 - (1+\kappa)\frac{a^2}{r^2} + \frac{3a^4}{r^4}\right]\sin 2\theta$$

$$\varepsilon_{r\theta} = \frac{\sigma_0}{8\mu}\left[\beta(1+\alpha)\left\{\frac{r}{a} + \frac{3a^3}{r^3} - \frac{4a^5}{r^5}\right\}\cos 3\theta - \beta(1+\alpha)\left\{\left(\frac{\kappa}{4} + \frac{3}{2}\right)\frac{r}{a} - \frac{a^3}{r^3}\right\}\cos\theta\right.$$
$$\left. +(1-\alpha)\left\{1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right\}\sin 2\theta\right] + \frac{\tau_0}{2\mu}\left[1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right]\cos 2\theta$$

$$\varepsilon_{\theta\theta} = \frac{\sigma_0}{8\mu}\left[\beta(1+\alpha)\left\{\frac{(3\kappa+2)r}{2a} + \frac{a^3}{r^3}\right\}\sin\theta - \beta(1+\alpha)\left\{\frac{r}{a} + \frac{(\kappa-4)a^3}{r^3} + \frac{4a^5}{r^5}\right\}\sin 3\theta\right.$$
$$\left. +(1-\alpha)\left\{1 + \frac{(\kappa-3)a^2}{r^2} + \frac{3a^4}{r^4}\right\}\cos 2\theta - (1-\alpha)\left(\frac{\kappa-1}{2} + \frac{a^2}{r^2}\right)\right] - \frac{\tau_0}{2\mu}\left[1 - (3-\kappa)\frac{a^2}{r^2} + \frac{3a^4}{r^4}\right]\sin 2\theta \tag{26}$$

The limiting case of a plate with no hole ($a \to 0$, $\beta \to 0$), the strains become as below:

$$\varepsilon_{rr} = -\frac{\sigma_0}{8\mu}(1-\alpha)\left(\cos 2\theta + \frac{\kappa-1}{2}\right) + \frac{\tau_0}{2\mu}\sin 2\theta$$
$$\varepsilon_{r\theta} = \frac{\sigma_0}{8\mu}(1-\alpha)\sin 2\theta + \frac{\tau_0}{2\mu}\cos 2\theta$$
$$\varepsilon_{\theta\theta} = \frac{\sigma_0}{8\mu}(1-\alpha)\left(\cos 2\theta - \frac{\kappa-1}{2}\right) - \frac{\tau_0}{2\mu}\sin 2\theta \tag{27}$$

As a numerical example, Fig. 5 shows the non-dimensional strain $\mu\varepsilon_{rr}/\sigma_0$ for β = 0.1 and θ = 90°, which is independent of γ.

6. Stress Concentration Factor

The circumferential stress $\sigma_{\theta\theta}$ arising at the edge of the circular hole ($r = a$) is called the hoop stress $\sigma_{\text{Hoop}} = \sigma_{\theta\theta}|_{r=a}$. By means of Eq. (18), the non-dimensional hoop stress results in

$$\left.\frac{\sigma_{\theta\theta}}{\sigma_0}\right|_{r=a} = \frac{1}{2}(1-\alpha)(2\cos 2\theta - 1) - \beta(1+\alpha)(\sin 3\theta - \sin\theta) - 4\gamma\sin 2\theta \tag{28}$$

where γ is the ratio of stresses defined by $\gamma = \tau_0/\sigma_0$, β is the ratio of a/h, and α is the loading factor (cf. Fig. 2). It is interesting to

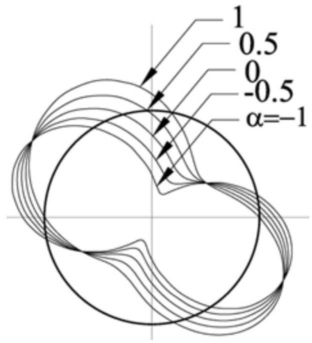

Fig. 6. Non-dimensional Hoop Stresses $\sigma_{\text{Hoop}}/\sigma_0$ for β = 0.2 and γ = 0.5

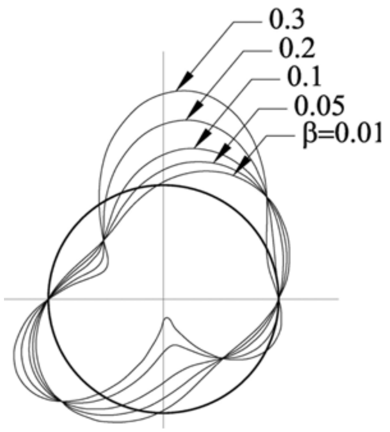

Fig. 7. Non-dimensional Hoop Stresses σ_{Hoop}/σ_0 for α = 1 and γ = −0.1

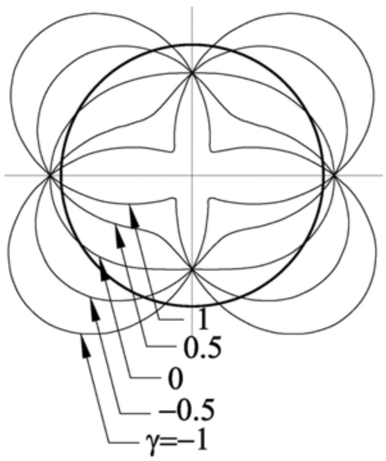

Fig. 8. Non-dimensional Hoop Stresses σ_{Hoop}/σ_0 α = 0 and β = 0.1

Table 3. The Stress Concentration Factors

β	γ	-1	-0.5	0	0.5	1
0.05	0	-3	-2.3	-1.6	-0.9	±0.2
	1	-5.472	-5.037	-4.643	-4.297	±4.003
	2	-9.246	-8.897	-8.572	-8.273	±8.001
0.2	0	-3	-2.45	-1.9	-1.35	±0.8
	1	-5.472	-5.083	-4.714	-4.367	±4.046
	2	-9.246	-8.919	-8.606	-8.306	±8.021

notice that the hoop stress is irrespective of shear modulus μ and Poisson's ratio ν. Figs. 6-8 show the non-dimensional hoop stresses σ_{Hoop}/σ_0. In Fig. 6, β = 0.2 and γ = 0.5 are held constant while α is varied from −1 to 1. In Fig. 7, α = 1 and γ = −0.1 are held constant while β is varied from 0.01 to 0.3. In Fig. 8, α = 0 and β = 0.1 are held constant while γ is varied from −1 to 1.

The stress concentration factor is the maximum non-dimensional hoop stress defined by the ratio of the maximum hoop stress to a nominal stress $(\sigma_{Hoop})_{max}/\sigma_0$. The stress concentration factors varying with the values of α, β, and γ are in given in Table 3.

7. Conclusions

Exact solutions for stresses, strains, and displacement of a rectangular plate with a central circular hole under both linearly varying in-plane normal stresses on two opposite edges and in-plane pure shear stresses acting on its entire outer boundary are investigated using the Airy stress function. The hoop (circumferential) stresses arising at the edge of the hole and the stress concentration factor depending upon the in-plane loading conditions and the size of the hole are plotted and computed.

Taking the multi-valueness and the singularity in stresses and displacements into account, once proper bi-harmonic functions $\phi(r, \theta)$ for a certain problem, which satisfy both the governing equation ($\nabla^4\phi = 0$) with no body forces in polar coordinates and traction boundary conditions, are selected from the table presented by Dundurs (Fu, 1996), which comprises stresses and displacements of certain bi-harmonic functions, the stress and displacement components according to the bi-harmonic functions can be easily selected from the table, and then strain components can be calculated from the relations of strain-displacement in Eqs. (25).

References

Fu, L.-S. (1996). *A first course in elasticity*, The Ohio State University.

Iwaki, T. and Miyao, K. (1980). "Stress concentrations in a plate with two unequal circular holes." *Int. J. Eng. Sci.*, Vol. 18, No. 8, pp. 1077-1090.

Kang, J.-H. (2014). "Exact solutions of stresses, strains, and displacements of a perforated rectangular plate by a central circular hole subjected to linearly varying in-plane normal stresses on two opposite edges." *Int. J. Mech. Sci*., Vol. 84, pp. 18-24.

Li, F., He, Y. T., Fan, C. H., Li, H. P., and Zhang, H. X. (2008). "Investigation on three-dimensional stress concentration of LY12-CZ plate with two equal circular holes under tension." *Materials Sci. Eng.*, A. Vol. 483-484, No. 15, pp. 474-476.

Mal, A. K. and Singh, S. J. (1991). *Deformation of elastic solids*, Prentice-Hall, Englewood Cliffs.

Miyata, H. (1970). "Finite elastic deformations of an infinite plate perforated by two circular holes under biaxial tension." *Ingenieur-Archive*, Vol. 39, No. 4, pp. 209-218.

Muskhelishvili, N. I. (1963). *Some basic problems of the mathematical theory of elasticity*, Noordhoff, Groningen, The Netherlands.

Peterson, R. E. (1974). *Stress concentration factor*, John Wiley and Sons, New York.

Radi, E. (2011). "Path-independent integrals around two circular holes in an infinite plate under biaxial loading conditions." *Int. J. Eng. Sci*., Vol. 49, No. 9, pp. 893-914.

Savin, G. N. (1961). Stress *concentration around holes*, Pergamon Press, New York.

She, C. M. and Guo, W. L. (2007). "Three-dimensional stress concentrations at elliptic holes in elastic isotropic plates subjected to tensile stress." *Int. J. Fatig.*, Vol. 29, No. 2, pp. 330-335.

Theocaris, P. S. and Petrou, L. (1987). "The order of singularities and the stress intensity factors near corners of regular polygonal holes." *Int. J. Eng. Sci*., Vol. 25, No. 7, pp. 821-832.

Timoshenko, S. P. and Goodier, J. N. (1970). *Theory of elasticity*, 3rd Ed., McGraw-Hill, New York.

Woo, H.-Y., Leissa, A. W., and Kang, J.-H. (2014). "Exact solutions for stresses, strains, displacements, and the stress concentration factors of a perforated rectangular plate by a circular hole subjected to in-plane bending moment on two opposite edges." *J. Eng. Mech*., Vol. 140, No. 6, pp. 1-8.

Yang, L. H. and He, Y. Z. (2002). "Stress field analysis for infinite plate with rectangular opening." *J. Harbin Eng. Univ*., Vol. 23, No. 2, pp. 106-110 in Chinese.

Yang, Z., Kim, C. B., Cho, C., and Beom, H. G. (2008). "The concentration of stress and strain in finite thickness elastic plate containing a circular hole." *Int. J. Solids Struct*., Vol. 45, Nos. 3-4, pp. 713-731.

Yu, P. S., Guo, W. L., She, C. M., and Zhao, J. H. (2008). "The influence of Poisson's ratio on thickness-dependent stress concentration at elliptic holes in elastic plates." *Int. J. Fatig*., Vol. 30, No. 1, pp. 165-171.

Zhang, T., Liu, T. G., Zhao, Y., and Liu, J. X. (2002). "Analysis of stress field of finite plates weakened by holes." *J. Huazhong Univ. Sci. Tech*., Vol. 30, pp. 87-89 in Chinese.