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An Improved Solution for Beam on Elastic Foundation using Quintic Displacement Functions

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Abstract

The theory of beam on elastic foundation is a simple and popular analytic approach for computing the response of laterally loaded piles. For a laterally loaded pile with constant subgrade reaction coefficient, an analytical solution can be easily deduced based on the theory of beam on elastic foundation when the load distribution and boundary condition are simple. However, when the subgrade reaction coefficient increases linearly with the depth or when the constraint condition is complex, an approximate solution can only be obtained by numerical method. At present, the node-spring simulation method and the modifying stiffness matrix method are two main solution methods for beam on elastic foundation with a nonuniform distribution of subgrade reaction coefficient, but a large number of elements are necessary for obtaining a sufficient calculation accuracy. Based on the Winkler elastic foundation model, an improved Finite Element (FE) method for the laterally loaded pile on an elastic foundation with a linearly distributed modulus of the subgrade reaction is proposed. A quintic displacement function is proposed as an approximate solution, and the weighted residual method is used for solving differential equations. The corresponding element stiffness matrix and nodal force vector are derived, and a more accurate nodal displacement, element internal force and displacement distribution can then be obtained by employing fewer elements. Three beams on elastic foundations under different boundaries and loading conditions are taken as typical examples to compare the difference of the calculation accuracy between the improved method and the node-spring simulation method. A laterally loaded pile is analyzed by the improved method, and the numerical results show that two elements for one soil layer can provide a sufficient calculation accuracy.

Keywords: *beam on elastic foundation, subgrade reaction coefficient, FE method, weighted residual method, quintic displacement function*

1. Introduction

The analytical model of beam on an elastic foundation can be used to resolve many engineering problems. One typical engineering application is to calculate the response of laterally loaded piles. Pile foundations are widely used to resist lateral loads in building structures, offshore structures, bridges, locks and dams, and retaining walls, etc. The lateral loads on piles are derived from earth pressures, inclined loads, wind, waves, earthquakes, etc. A number of different methods have been proposed to predict the lateral load-displacement response of piles. The subgrade reaction method is most extensively used because the nonlinearity of soil and variation of subgrade reaction with depth can be taken into account relatively easily. In this method, the laterally loaded pile is idealized as a beam on an elastic foundation loaded transversely and restrained by independent elastic springs acting along the length of the beam. Hetenyi (1974) presented closed-form solutions of laterally loaded piles under a variety of loading conditions and end restraints when the modulus of the subgrade reaction along the depth was assumed to be constant. Matlock and Reese (1960) proposed generalized iterative solutions for rigid and flexible piles under lateral loads based on a uniformly and linearly varying with depth subgrade modulus. Davisson and Salley (1970) performed tests to develop design criteria for pile foundations in sand and found that the subgrade modulus linearly increases with depth. Randolph (1981) obtained algebraic expressions of the response of flexible piles under lateral loading by FE method. Dinev (2012) proposed a closed-form analytical solution of a beam on an elastic foundation by a variational formulation of the total potential energy functional. Suchart *et al.* (2013) presented a nonlinear Winkler-based beam element with improved displacement shape functions that was capable of representing the nonlinear interaction mechanics between the beam and the foundation. Sánchez and Roesset (2013) proposed a more accurate model of a beam on elastic foundation for laterally loaded piles and used a consistent boundary matrix to evaluate the computing accuracy.

When the modulus of the subgrade reaction varies as a function

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of the deflection, the p-y curve method is widely used in the design. The p-y curve method can conveniently and easily incorporate nonlinear soil reaction-deflection relations between the lateral load (p) and the lateral displacement (y) at a point in the pile. In this method, the mechanical behavior of soils is represented by a series of nonlinear springs attached to laterally loaded piles, and a finite difference method is used to find a solution. Many approaches for estimating p-y curves have been developed by Matlock (1970) for soft clay, Reese *et al.* (1974) for sand, Reese and Welch (1975) for stiff clays above the water table, Reese *et al.* (1975) for stiff clays below the water table, Sullivan *et al.* (1980) for any clay soil, Reese (1997) for weak rock, and Kodikara *et al.* (2010) for undrained clay. However, the p–y curves were developed based on the limited results of full-scale lateral load tests or centrifuge model tests. These empirically derived p-y curves could not incorporate the effects of various pile properties and soil continuity (Heidari *et al.*, 2014).

Nowadays, numerical methods for the solution of structural elements have been developed rapidly based on meshfree approximations and isogeometric analysis. Nguyen-Thanh *et al.* (2011) presented a novel approach for isogeometric analysis of thin shells using polynomial splines over hierarchical T-meshes (PHT-splines). Areias *et al.* (2013) adopted an alternative approach for the analysis of non-linear shells based on mixed forms of the spatial metric, spherical linear interpolation for quadrilaterals and covariant fixed frames to ensure the satisfaction of all patch tests. Areias and Rabczuk (2013) proposed a simple and efficient algorithm for FEM-based computational fracture of plates and shells with both brittle and ductile materials on the basis of edge rotation and load control. Amiri *et al.* (2014) presented a phase-field model for fracture in Kirchoff-Love thin shells using the Local Maximum Entropy (LME) meshfree method. Nguyen-Thanh *et al*. (2015) developed an extended isogeometric element formulation (XIGA) for analysis of through-the-thickness cracks in thin shell structures. Thus it can be seen that the finite element method has been widely used as a numerical tool to solve the structural problems.

However, the subgrade modulus method is the most popular method because it is stated in a simple mathematical form and very convenient to use in engineering calculations. Therefore, the elastic subgrade reaction method, in which the modulus of the subgrade reaction is assumed to increase linearly with depth, was suggested for evaluating the deformation of piles under lateral loads in the Chinese code JGJ94 (2008) because of its simplicity and reasonable accuracy. As stated above, for laterally loaded piles with constant subgrade reaction coefficients, an analytical solution can be easily deduced based on the theory of beam on elastic foundation when the load distribution and boundary condition are simple. Otherwise, the finite difference method or FE method is required to obtain an approximate solution when the subgrade reaction coefficient linearly increases with depth or when the constraint conditions are complex. The finite difference method was used by some researchers to find numerical solutions of the problems (Matlock and Reese, 1960). However, this finite difference method were out of date some decades ago due to the emergence of the finite element method. In this study, an improved FE method for laterally loaded piles on elastic foundations with a linearly distributed modulus of the subgrade reaction is proposed based on the Winkler elastic foundation model.

2. Existing Methods for Beam on Winkler Elastic Foundation

FE method is a powerful tool for analyzing the response of laterally loaded piles and is widely used in computer-aided calculations. Some large universal packages, such as ABAQUS, PLAXIS, ANSYS and SAP2000, are typical applications of FE method in engineering practice. At present, there are mainly two solution methods for beams on elastic foundations with nonuniform distributions of subgrade reaction coefficients using FE method: the node-spring simulation method and the modifying stiffness matrix method.

2.1 Node-spring Simulation Method

In the node-spring simulation method (Bowles, 1974; Harden and Hutchinson, 2009), the elastic foundation is idealized by discrete springs and the displacement at any point is assumed to be independent to displacements at any other points. This method can simplify the calculation process and is suitable for manual modeling using universal FE softwares such as ABAQUS, PLAXIS, and ANSYS. Two professional software packages, SAP2000 and FRWS (calculating deep foundation pit support in China), also address the continuum of soil for laterally loading piles by this method because of its high efficiency. This method can determine an approximate solution, but the satisfactory accuracy can only be obtained from the refined elements. Consider a straight beam supported by an elastic foundation along its entire length and subjected to a vertical concentrated load of 400 kN acting at the right end (shown in Fig. 1), in which *EI* denotes the flexural rigidity of the beam. The subgrade reaction coefficient along the depth is a constant 40,000 kN/m^3, and the width of the beam is 1 m.

SAP2000 and FRWS are applied to obtain the numerical solutions of the above beam. The beam is divided into different numbers of elements to show its influence on the calculation accuracy. The spring supports are applied on all nodes and used to simulate

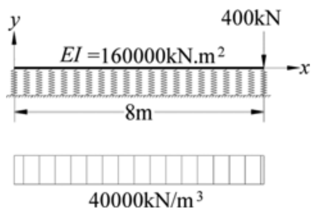

Fig. 1. Beam on an Elastic Foundation with Uniformly Distributed Subgrade Reaction Coefficients

Table 1. Comparison between the Theoretical Solution and the Node-spring Simulation Method

Items	Numbers of elements	Deflection at x = 8 m (mm)	Maximum moment at x = 6.43 m (kN.m)	Maximum shear force at x = 4.88 m (kN)
Theoretical solution	/	-10.0	-257.7	-82.5
FRWS	32	-10.0	-256.1	-81.4
	16	-9.8	-252.2	-78.6
	8	-9.2	-228.8	-68.6
	4	-7.6	-191.0	-19.0
SAP2000	32	-10.0	-255.4	-81.6
	16	-9.8	-251.4	-79.7
	8	-9.3	-221.1	-72.5
	4	-7.6	-142.5	-57.0

the support of elastic foundation. The stiffness of every spring is equal to the product of the subgrade reaction coefficient, the width of the beam and the element length. The numerical solutions with different numbers of elements by SAP2000 and FRWS are presented in Table 1. The corresponding theoretical solutions (Hetenyi, 1974) are also given for comparison. From Table 1, it should be noted that the calculation accuracy of both FRWS and SAP2000 depends on the number of elements. When the number of elements is relatively small, the numerical results and the theoretical results have a great difference. For this beam on an elastic foundation, an acceptable accuracy in engineering practice can be only obtained when the number of elements is greater than or equal to 16. In fact, the acceptable number of elements should be determined according to the length of the beam, cross section of the beam, property of the soil, etc. This is mainly because every node has one concentrated spring element, which results in a discontinuous distribution of shear forces. The numerical results are approximate values obtained by smoothing processing.

2.2 Exact Stiffness Matrix Method

The exact stiffness matrix method is another common method used to obtain the numerical solution of a beam on an elastic foundation (Eisenberger and Yankelevsky, 1985; Kim and Kim, 2012). In this method, the reaction matrix of the elastic foundation is derived using the principle of virtual work or variational work and added to the beam element stiffness matrix (Chen, 2000, 2001). Thus, the exact beam element stiffness matrix is formed. Although this method has higher accuracy than the node-spring simulation method, the manual computation workload is very heavy. For an elastic foundation beam under complex loads and restraints, it is difficult to obtain numerical results without computational software. Furthermore, the universal softwares almost all adopt the node-spring simulation method for its simplicity. At the same time, although the node displacement and force acquired by this method are accurate, the element internal displacement and force are not accurate because the displacement function is only relative to the beam and fails to take the elastic foundation into account. Moreover, for beams on an elastic foundation with linearly distributed subgrade reaction coefficients and lateral loading, the displacements and internal forces on the nodes are solved through converting element loads into nodal loads because the analytical solutions of displacements and internal forces on elements cannot be obtained according to existing studies. Therefore, the relatively accurate displacement and internal force distribution along the whole beam can only be plotted by meshing more elements and smoothing the nodal results.

As mentioned above, for beams on elastic foundations with linear subgrades, a large number of elements are needed to obtain a sufficient calculation accuracy with both the node-spring simulation method and the exact stiffness matrix method. Their distribution curves of displacements and internal forces can only be plotted approximately according to the nodal results. In this study, a method with fewer elements is proposed to improve the calculation accuracy of the laterally loaded pile using the theory of beam on elastic foundation. The general solution of a pile with linearly distributed subgrade reaction coefficients and lateral element loads has been developed. Based on this, smooth and accurate distribution curves of displacements and internal forces can be obtained.

3. Improved Method

The two-node beam element is widely used in structural analyses because of its simple pre- and post-processing. However, when lateral loads act on beam elements, the distribution of the element's internal forces cannot be correctly described because the displacement function is expressed as a cubic polynomial curve. In this study, a quintic displacement function and corresponding element stiffness matrix is proposed to obtain the approximate element internal forces and displacements of beam on elastic foundation with linearly distributed subgrade reaction coefficients and pressures. A more accurate nodal displacement, element internal force and displacement distribution can then be obtained by employing fewer elements. The computational efficiency is greatly improved and the post-processing can be immediately simplified.

When the subgrade reaction coefficients are distributed nonlinearly, the beam on an elastic foundation can be simulated segmentally by dividing elements. If there are concentrated forces acting on the beam, beam elements should be divided at points of force application. If there are nonlinear distributed pressures acting on the beam, the proposed quintic displacement

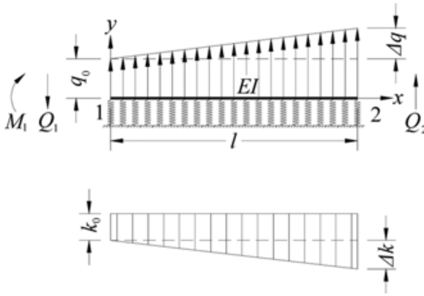

Fig. 2. Beam on an Elastic Foundation with Linearly Distributed Subgrade Reaction Coefficients

function and corresponding element stiffness matrix can be applied by dividing the beam into several beam elements. Therefore, only a beam element on an elastic foundation with linearly distributed subgrade reaction coefficients and pressures is considered in this paper. Consider a prismatic beam with an element scheme as shown in Fig. 2, where EI is the flexural rigid; l is the length of the beam; q_0 represents the load intensity at Point 1; Δq is the difference of the load intensity between Point 1 and Point 2; k_0 represents the subgrade reaction coefficient at Point 1; Δk is the difference of the subgrade reaction coefficients between Point 1 and Point 2; Q_1 and Q_2 represent the shear forces at Point 1 and Point 2, respectively; and M_1 and M_2 represent the bending moments of Point 1 and Point 2, respectively (forces acting in the direction of the arrow are positive forces).

3.1 Displacement Function

The response of a beam on an elastic foundation with linearly distributed subgrade reaction coefficients can be determined by solving the following equilibrium equations:

$$-\frac{dQ}{dx} = qb - kbw \tag{1}$$

$$-\frac{dM}{dx} = Q \tag{2}$$

where q is the linearly distributed pressure acting on the beam element ($q = q_0 + \xi\Delta q$); b is the width of the beam; k is the linearly distributed subgrade reaction coefficient for an elastic foundation ($k = k_0 + \xi\Delta k$); ξ is the relative coordinate ($\xi = x/l$); Q and M represent the shear force and the bending moment of the beam element, respectively (the positive direction is shown in Fig. 2); and w represents the deflection of the beam element (considering the upward direction as positive).

When the slope and the deflection are very small, the bending moment and the shear force in the beam can be expressed as follows:

$$M(x) = EI\frac{d^2w}{dx^2} \tag{3}$$

$$Q(x) = -EI\frac{d^3w}{dx^3} \tag{4}$$

By substituting the relative coordinate ξ into Eq. (3) and Eq. (4), we have

$$M(\xi) = \frac{EI}{l^2}\frac{d^2w}{d\xi^2} \tag{5}$$

$$Q(\xi) = -\frac{EI}{l^3}\frac{d^3w}{d\xi^3} \tag{6}$$

Thus, the equilibrium equation Eq. (1) in terms of displacements can be rewritten as

$$\frac{EI}{l^4}\frac{d^4w}{d\xi^4} = q_0b + \xi\Delta qb - (k_0 + \xi\Delta k)bw \tag{7}$$

3.1.1 Displacement Field Caused by Joint Displacements

For a beam on an elastic foundation with linearly distributed subgrade reaction coefficients (shown in Fig. 3), the displacement field caused by joint displacements can be obtained by solving the following equilibrium equation:

$$\frac{EI}{l^4}\frac{d^4w_a}{d\xi^4} + (k_0 + \xi\Delta k)bw_a = 0 \tag{8}$$

The corresponding boundary conditions are determined as the following:

$$w_a(0) = w_1,\ w_a(1) = w_2,\ w_a'(0) = l\theta_1,\ w_a'(1) = l\theta_2 \tag{9}$$

where w_a represents the deflection of an element caused by the joint displacements; w_1 and w_2 represent the deflection of an element at Point 1 and Point 2, respectively; θ_1 and θ_2 represent the rotation of an element at Point 1 and Point 2, respectively.

The analytical solution of the fourth-order differential equation shown in Eq. (8) is difficult to be obtained under the boundary constraints shown in Eq. (9). It is generally assumed that the approximate solution can be expressed as a polynomial. For an elastic foundation with linearly distributed subgrade reaction coefficients, the subgrade reaction acting on the above beam is at least linearly distributed and the shear force of the element is at least quadratically distributed. Therefore, the approximate solution of the deflection of the beam can be most simply expressed with a quintic polynomial and written as

$$w_a(\xi) = \sum_{i=0}^{5} a_i\xi^5 \tag{10}$$

where a_i (i=0; 1; …, 5) are six unknown constants. Substituting the boundary conditions Eq. (9) into Eq. (8), we obtain four equilibrium equations. To obtain the six undetermined coefficients in Eq. (10), two other equations need to be established.

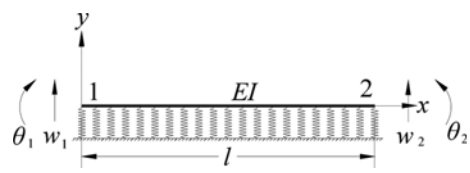

Fig. 3. Beam on an Elastic Foundation under Joint Displacement

The weighted residual method is a general mathematical tool applicable for solving all types of differential equations. The other two equations are then established in two sub-domains, [0, 1/2] and [1/2, 1], by this method as follows:

$$\begin{cases} \int_0^{\frac{1}{2}} \left[\frac{EI}{l^4}\frac{d^4 w_a}{d\xi^4} + (k_0 + \Delta k\xi) b w_a \right] d\xi = 0 \\ \int_{\frac{1}{2}}^{1} \left[\frac{EI}{l^4}\frac{d^4 \dot{w}}{d\xi^4} w_a + (k_0 + \Delta k\xi) b w_a \right] d\xi = 0 \end{cases} \quad (11)$$

Note that the six undetermined coefficients a_i (i = 0; 1; … ;5) are functions of the parameters w_1, w_2, θ_1 and θ_2. Thus, Eq. (10) can be rewritten as follows:

$$w_a(\xi) = \sum_{i=0}^{5} a_i \xi^5 = [N_1 \;\; N_2 \;\; N_3 \;\; N_4] \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \mathbf{N}\mathbf{a}^e \quad (12)$$

where N_1, N_2, N_3 and N_4 are shape functions as given in Appendix 1; **N** is the matrix of shape functions; and $\mathbf{a}^e$ is the nodal displacement vector. It can be easily verified that these shape functions will be reduced into the shape functions of the general two-node beam element when both k_0 and Δk are equal to zero. Separating the variable ξ, the shape function matrix N can be further written as

$$\mathbf{N}^T = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \begin{bmatrix} c_{10} & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{20} & c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{30} & c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\ c_{40} & c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \end{bmatrix} \begin{Bmatrix} 1 \\ \xi \\ \xi^2 \\ \xi^3 \\ \xi^4 \\ \xi^5 \end{Bmatrix} = \mathbf{Ct} \quad (13)$$

where **C** is the coefficient matrix of the shape function matrix **N** in which matrix element c_{ij} (i = 1, 2, 3, 4; j = 0, 1, …, 5) can be extracted from the shape functions presented in Appendix 1 and **t** is a vector formed by the relative coordinate variable.

3.1.2 Displacement Field Caused by Linearly Distributed Loads

For a beam on an elastic foundation with linearly distributed subgrade reaction coefficients and fixed-end constraints (shown in Fig. 4), the displacement field produced by external loads can be obtained by solving the following fourth-order differential equation:

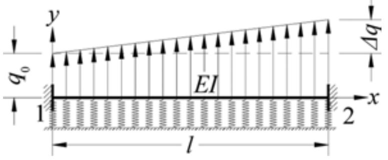

Fig. 4. Beam on an Elastic Foundation under External Loads

$$\frac{EI}{l^4}\frac{d^4 w_b}{d\xi^4} = q_0 b + \xi \Delta q b - (k_0 + \xi \Delta k) b w_b \quad (14)$$

The solution of this equation must satisfy the following boundary conditions:

$$w_0(0) = w_b(1) = w_b{}'(0) = w_b{}'(1) = 0 \quad (15)$$

where w_b is the deflection of the element produced by the linearly distributed load.

From Eq. (15), it can be easily observed that all boundary conditions can be satisfied when the displacement function includes the factor $\xi^2(1-\xi)^2$. For simplicity, the approximate solution is expressed as a quintic polynomial, and the other factor is then determined to be a linear polynomial. The deflection function of beam on elastic foundation with fixed-end constraints under the linearly distributed load is constructed as

$$w_b(\xi) = \xi^2 (1-\xi)^2 (b_0 + b_1 \xi) \quad (16)$$

where b_0 and b_1 are unknown constants that can be determined by the weighted residual method. The weight function is expressed by the least-square method, and the residual is then evaluated as

$$I = \int_0^1 \left[\frac{EI}{l^4}\frac{d^4 w_b}{d\xi^4} + (k_0 + \Delta k \xi) b w_b - (q_0 + \Delta q \xi) b \right]^2 d\xi \quad (17)$$

The unknown constants b_0 and b_1 can be obtained from $\partial I/\partial b_0 = 0$ and $\partial I/\partial b_1 = 0$. The calculation expressions are clearly shown in Appendix 2. It can be easily verified that the displacement function will be reduced into the deflection function of the beam clamped at two ends under a trapezoidal distributed load when both k_0 and Δk are equal to zero.

Separating the variable ξ, Eq. (16) can be rewritten in the following form:

$$\begin{aligned} w_b(\xi) &= [0 \;\; 0 \;\; b_0 \;\; b_1{-}2b_0 \;\; b_0{-}2b_1 \;\; b_1] \\ &\quad \{1 \;\; \xi \;\; \xi^2 \;\; \xi^3 \;\; \xi^4 \;\; \xi^5\}^T = \mathbf{Dt} \end{aligned} \quad (18)$$

where **D** is the coefficient matrix of the deflection function under the linearly distributed load and **t** is a vector formed by the relative coordinate variable.

3.2 Element Stiffness Matrix and Nodal Force

Based on the principle of virtual work, we have

$$\frac{EI}{l^3}\int_0^1 w_a'' \delta w_a'' d\xi + l\int_0^1 k b w_a \delta w_a d\xi - l\int_0^1 q b \delta w_a d\xi = 0$$

Substituting Eq. (10) into the above equation results gives

$$\frac{EI}{l^3}\int_0^1 \mathbf{N}''\mathbf{a}^e \cdot \mathbf{N}''\delta\mathbf{a}^e d\xi + l\int_0^1 k b \mathbf{N}\mathbf{a}^e \cdot \mathbf{N}\delta\mathbf{a}^e d\xi - l\int_0^1 q b \mathbf{N}\delta\mathbf{a}^e d\xi = 0$$

Further simplifying and substituting $q = q_0 + \xi\Delta q$ and $k = k_0 + \xi\Delta k$ into the above equation, we can obtain

$$\left[\frac{EI}{l^3}\int_0^1 \mathbf{N}''^T\mathbf{N}'' d\xi + l\int_0^1 (k_0 + \Delta k\xi) b \mathbf{N}^T\mathbf{N} d\xi\right]\mathbf{a}^e = l\int_0^1 (q_0 + \Delta q\xi) b \mathbf{N}^T d\xi$$

Rewriting it into a matrix equation, we have

$$\mathbf{K}^e\mathbf{a}^e = \mathbf{P}^e$$

where $\mathbf{K}^e$ is the element stiffness matrix of a beam on an elastic foundation with linearly distributed subgrade reaction coefficients and $\mathbf{P}^e$ is the nodal force vector acting on the nodes, which is equivalent to the linearly distributed load applied on the element in terms of the work done by a virtual displacement.

The stiffness matrix $\mathbf{K}^e$ can be further expressed in the following form:

$$\mathbf{K}^e = \frac{EI}{l^3}\int_0^1 \mathbf{N}''^T\mathbf{N}''d\xi + k_0bl\int_0^1 \mathbf{N}^T\mathbf{N}d\xi + \Delta kbl\int_0^1 \mathbf{N}^T\mathbf{N}\xi d\xi = \mathbf{K}_b^e + \mathbf{K}_{f1}^e + \mathbf{K}_{f2}^e \quad (19)$$

where $\mathbf{K}_b^e$ is the stiffness matrix of beam element, $\mathbf{K}_{f1}^e$ is the stiffness matrix of an elastic foundation with a rectangular distribution of subgrade reaction coefficients, and $\mathbf{K}_{f2}^e$ is the stiffness matrix for the elastic foundation with a triangular distribution of subgrade reaction coefficients. The calculation expressions of $\mathbf{K}_b^e$, $\mathbf{K}_{f1}^e$, and $\mathbf{K}_{f2}^e$ can be found in Appendix 3.

The nodal force vector $\mathbf{P}^e$ can be further expressed as follows:

$$P^e = q_0bl\int_0^1 N^T d\xi + \Delta qbl\int_0^1 N^T \xi d\xi = P_1^e + P_2^2 \quad (20)$$

where $\mathbf{P}_1^e$ is the nodal force vector acting on the nodes, which is equivalent to the rectangularly distributed load applied on the beam element in terms of the work done by a virtual displacement and $\mathbf{P}_2^e$ is the nodal force vector acting on the nodes, which is equivalent to the triangular distributed load applied on the beam element in terms of the work done by a virtual displacement. The detailed calculation processes of these two vectors are also presented in Appendix 3.

3.3 Displacement and Internal Force Distribution of the beam Element

For a beam on an elastic foundation with linearly distributed subgrade reaction coefficients (shown in Fig. 2), the total deflection of an element is the superposition of two displacement fields:

$$w(\xi) = w_a(\xi) + w_b(\xi) \quad (21)$$

Substituting Eq. (12) and Eq. (18) into Eq. (21), the deflection of the beam element can be obtained by

$$w = w_a + w_b = \mathbf{Na} + \mathbf{Dt} = \mathbf{t}^T(\mathbf{C}^T\mathbf{a} + \mathbf{D}^T) = (\mathbf{a}^T\mathbf{C} + \mathbf{D})\mathbf{t} \quad (22)$$

According to Eq. (3) and Eq. (4), the moment and shear force of an element in the beam element can be calculated by

$$M = \frac{EI}{l^2}w'' \quad (23)$$

$$Q = -\frac{EI}{l^3}w''' \quad (24)$$

4. Verification of the Improved Method

To verify the calculation accuracy of the improved method, three beams with different boundary conditions, external loadings and subgrade reaction coefficients are taken as typical examples. Their stiffness matrixes and nodal force vectors are obtained from Eq. (20) and Eq. (21), respectively. Based on Eq. (22), Eq.

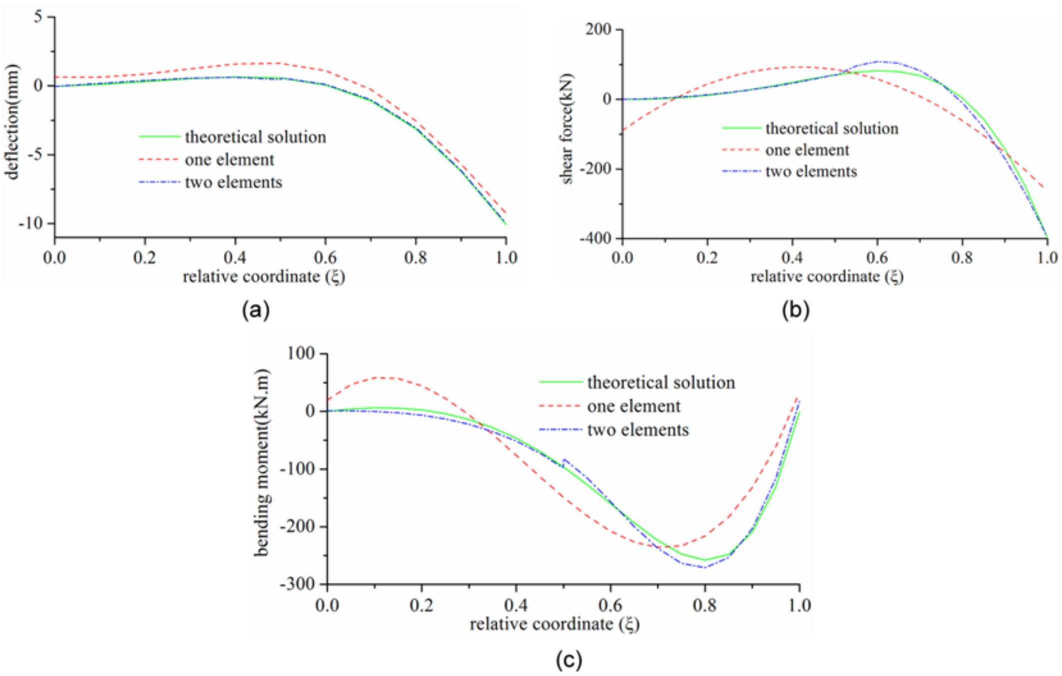

Fig. 5. Comparison of the Theoretical Solution and the Numerical Solutions for Example I: (a) Deflection, (b) Shear Force, (c) Bending Moment

(23) and Eq. (24), their corresponding deflections and internal forces are calculated using both the improved method and the theoretical method (Hetenyi, 1974).

4.1 Numerical Example I

The beam on an elastic foundation shown in Fig. 1 is analyzed using the improved method. One and two elements are used to verify the accuracy of the numerical results. The deflection, shear force and bending moment diagrams are shown with dashed lines and dash-dotted lines in Fig. 5, respectively. Their corresponding theoretical solutions are presented with solid lines in the same figure. From Fig. 5, it can be observed that the results of one element have a large relative error when compared with the theoretical solutions, and the results of two elements can agree well with the theoretical solutions. In other words, the approximate solutions by the improved method can obtain a sufficient calculation accuracy when two elements are used to simulate one beam.

4.2 Numerical Example II

One beam on an elastic foundation with linearly distributed subgrade reaction coefficients is taken as the second typical example to verify the improved method (shown in Fig. 6). The flexural rigidity *EI* is assumed to be infinite, and the maximum subgrade reaction coefficient is 10,000 kN/m^3. A vertical concentrated load of 100 kN is applied on its right end. The theoretical solutions are shown in Fig. 7 with solid lines, and the numerical results using one element are presented with dashed lines in the same figure. From Fig. 7, we can see that the results of one element agree very well with the theoretical solutions. It indicates that the approximate solutions by the improved method have a sufficient calculation accuracy even when only one element is used.

4.3 Numerical Example III

One beam on an elastic foundation with uniformly distributed subgrade reaction coefficients is taken as the third typical example to verify the improved method (shown in Fig. 8). The flexural rigidity *EI* is 4×10^4 kN·m^2, and the subgrade reaction coefficient is a constant 10,000 kN/m^3. A uniformly distributed load of 200 kN/m is applied. One and two elements are used to compare the accuracy of the numerical results. The deflection, shear force and bending moment diagrams are shown with dashed lines or dash-dotted lines in Fig. 9, respectively. Their corresponding theoretical solutions are presented with solid lines

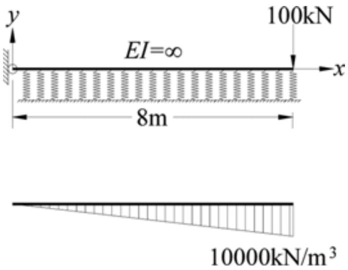

Fig. 6. Numerical Example II - Concentrated Load

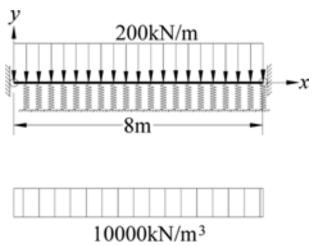

Fig. 8. Numerical Example III - Uniformly Distributed Load

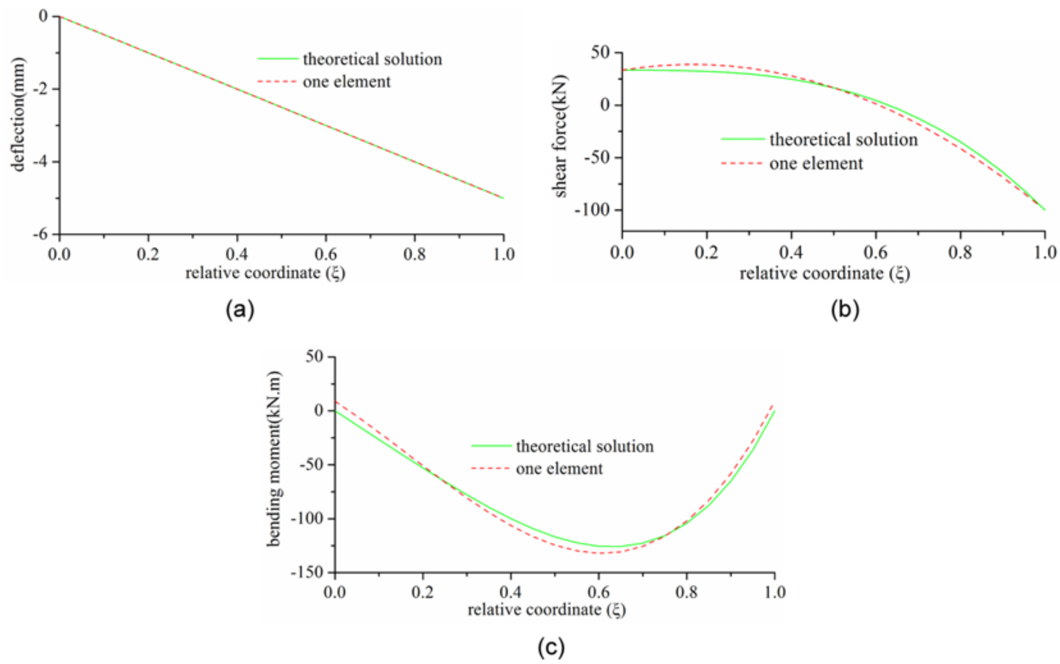

Fig. 7. Comparison of the Theoretical Solution and the Numerical Solutions for Example II: (a) Deflection, (b) Shear Force, (c) Bending Moment

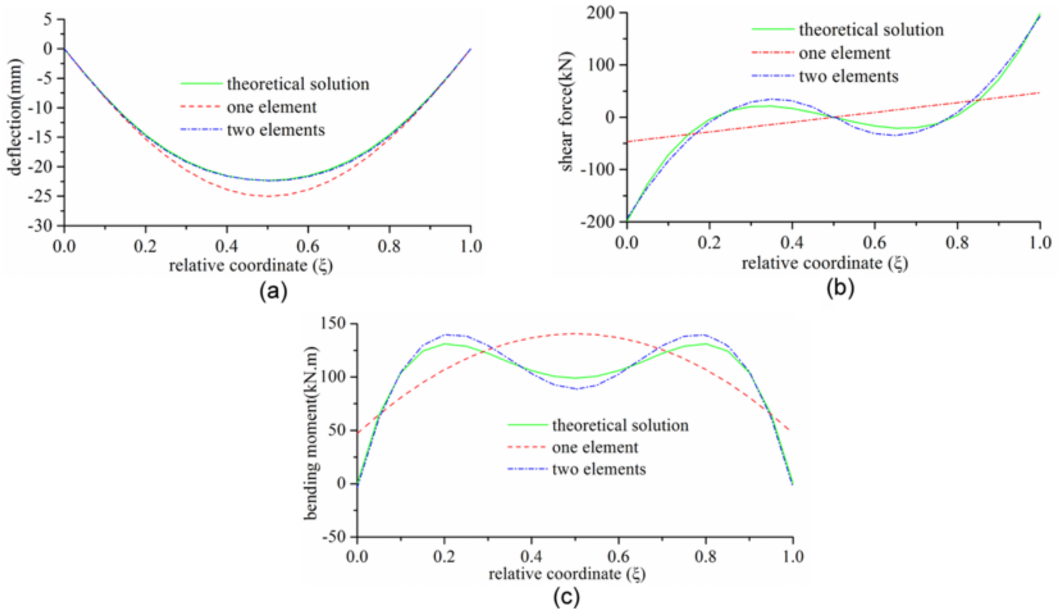

Fig. 9. Comparison of Theoretical Solution and Numerical Solutions For Example III: (a) Displacement, (b) Shear Force, (c) Bending Moment

in the same figure. From Fig. 9, we can see that the results of one element have a large relative error when compared with the theoretical solutions and the calculation accuracy of two elements improves significantly. It indicates that the approximate solutions by the improved method can obtain an acceptable calculation accuracy when two elements are used.

5. Application of the Improved Method

From three typical verification examples, we can see that the sufficient calculation accuracy can be achieved by the improved method for one beam using two elements. The improved method is then applied to analyze the response of a laterally loaded pile. The length of the pile is 20 m, and the diameter is 0.6 m. The flexural rigidity EI is 2×10^5 kN·m^2. The subgrade reaction coefficients and the distribution of soils are presented in Fig. 10. Considering that the subgrade reaction coefficient for a laterally loaded pile depends on the characteristic of the soil, the pile segment in one soil layer is meshed into two beam elements. The total number of elements is only six. The horizontal spring supports are applied on all nodes to simulate the support of soils. The axial deformation of the pile are not neglected.

The head displacement and the bending moment at z = 2 m by the improved method are shown in Fig. 11 with dashed lines. By contrast, the corresponding results by the node-spring simulation method with different numbers of elements are shown in the same figure with solid lines. From Fig. 11, it should be noted that the numerical results acquired by the improved method are almost critical values acquired by the node-spring simulation method as the number of elements increases. This means that a sufficient accuracy of the numerical results for laterally loaded piles can be obtained by the improved method when the pile segment in one soil layer is simulated by two elements.

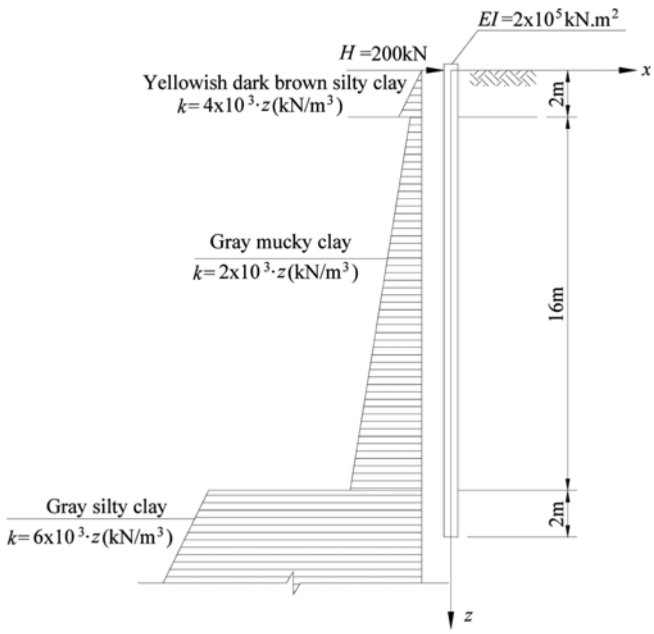

Fig. 10. Laterally Loaded Piles with Linearly Distributed Subgrade Reaction Coefficients

6. Conclusions

Based on the theory of beams on elastic foundations, an improved FE method is proposed to determine the response of laterally loaded piles with linearly distributed subgrade reaction coefficients. A quintic displacement function is proposed as an approximate solution of a beam on an elastic foundation with

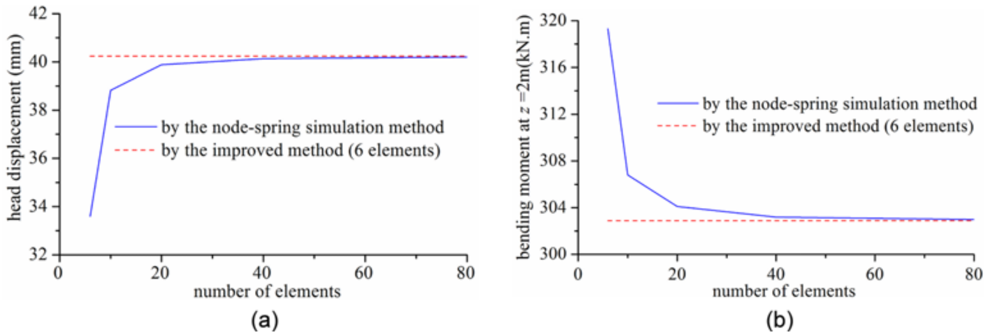

Fig. 11. Comparison of the Node-spring Simulation Method and the Improved Method for a Laterally Loaded Pile: (a) Head Displacement, (b) Bending Moment at z = 2 m

linearly distributed subgrade reaction coefficients. The weighted residual method is used to solve the differential equations. The corresponding element stiffness matrix and nodal force vector are derived, and the approximate solution is obtained by employing fewer elements. Three typical beams on elastic foundations and a laterally loaded pile with linearly distributed subgrade reaction coefficients are analyzed by the improved method and the node-spring simulation method. Based on the numerical results, the following conclusions can be drawn:

1. For a two-node beam element, the displacement function expressed in a quintic polynomial is feasible for a beam on an elastic foundation even with linearly distributed subgrade reaction coefficients.
2. For a two-node beam element, the proposed FE method can obtain a sufficient calculation accuracy when only two beam elements are used to simulate the pile segment in one soil layer.
3. Compared to the node-spring simulation method, the approximate solution of laterally loaded piles with linearly distributed subgrade reaction coefficients by the improved method can meet the engineering accuracy requirements using fewer beam elements.

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Appendix 1. The Shape Functions

$$N_1 = 1+\frac{1}{h}\Big[-\Big(\frac{174182400E^2I^2}{l^8}+\frac{529200EI\Delta kb}{l^4}+\frac{1784160EIk_0b}{l^4}$$
$$+96\Delta k^2b^2+597k_0\Delta kb^2+672k_0^2b^2\Big)\xi^2+\Big(\frac{116121600E^2I^2}{l^8}$$
$$+\frac{937440EI\Delta kb}{t^4}+2128k_0^2b^2+\frac{3810240EIk_0b}{t^4}+285\Delta k^2b^2$$
$$+1852k_0\Delta kb^2)\xi^3-\Big(\frac{514080EI\Delta kb}{l^4}+\frac{2721600EIk_0b}{l^4}+297\Delta k^2b^2$$
$$+1983k_0\Delta kb^2+2310k_0^2b^2\Big)\xi^4+\Big(\frac{60480EI\Delta kb}{l^4}+\frac{604800EIk_0b}{l^4}$$
$$+105\Delta k^2b^2+714k_0\Delta kb^2+840k_0^2b^2\Big)\xi^5\Big]$$

$$N_2 = l\xi+\frac{1}{h}\Big[-\Big(\frac{116121600E^2I^2}{l^8}+\frac{171360EI\Delta kb}{l^4}+\frac{413280EIk_0b}{l^4}$$
$$+20\Delta k^2b^2+101k_0\Delta kb^2+105k_0^2b^2\Big)\xi^2+\Big(\frac{58060800E^2I^2}{l^8}$$
$$+\frac{206640EI\Delta kb}{l^4}+\frac{614880EIk_0b}{l^4}+45\Delta k^2b^2+237k_0\Delta kb^2$$
$$+252k_0^2b^2\Big)\xi^3-\Big(\frac{80640EI\Delta kb}{l^4}+\frac{352800EIk_0b}{l^4}+42\Delta k^2b^2$$
$$+227k_0\Delta kb^2+245k_0^2b^2\Big)\xi^4+\Big(\frac{60480EIk_0kb^2}{l^4}+14\Delta k^2b^2$$
$$+77k\Delta k_0b^2+84k_0^2b^2\Big)\xi^5\Big]$$

$$N_3 = \frac{1}{h}\Big[\Big(\frac{174182400E^2I^2}{l^8}-\frac{438480EI\Delta kb}{l^4}-\frac{635040EIk_0b}{l^4}$$
$$+78\Delta k^2b^2+303k_0\Delta kb^2+252k_0^2b^2\Big)\xi^2-\Big(\frac{116121600E^2I^2}{l^8}$$
$$-\frac{514080EI\Delta kb}{l^4}+1288k_0^2b^2-\frac{1028160EIk_0b}{l^4}+375\Delta k^2b^2$$
$$+1516k_0\Delta kb^2\Big)\xi^3+\Big(\frac{514080EI\Delta kb}{l^4}+\frac{302400EIk_0b}{l^4}$$
$$+531\Delta k^2b^2+2193k_0\Delta kb^2+189k_0^2b^2\Big)\xi^4-\Big(\frac{544320EI\Delta kb}{l^4}$$
$$+\frac{604800EIk_0b}{l^4}+231\Delta k^2b^2+966k_0\Delta kb^2+840k_0^2b^2\Big)\xi^5\Big]$$

$$N_4 = \frac{1}{h}\Big[-\Big(\frac{5806088E^2I^2}{l^8}-\frac{45360EI\Delta kb}{l^4}-\frac{80640EIk_0b}{l^4}$$
$$+6\Delta k^2b^2+24k_0\Delta kb^2+21k_0^2b^2\Big)\xi^2+\Big(\frac{58060800E^2I^2}{l^8}$$
$$-\frac{75600EIk_bb}{l^4}-\frac{191520EIk_0b}{l^4}+30\Delta k^2b^2+125k_0\Delta kb^2+112k_0^2b^2\Big)\xi^3$$
$$-\Big(\frac{30240EI\Delta kb}{l^4}-\frac{50400EIk_0b}{l^4}+45\Delta k^2b^2+192k_0\Delta kb^2$$
$$+175k_0^2b^2\Big)\xi^4+\Big(\frac{60480EI\Delta kb}{l^4}+\frac{60480EIk_0b}{l^4}+21\Delta k^2b^2$$
$$+91k_0\Delta kb^2+84k_0^2b^2\Big)\xi^5\Big]$$

where

$$h = \frac{58060800E^2I^2}{l^8}+\frac{45360EI\Delta kb}{l^4}+\frac{90720EIk_0b}{l^4}+3\Delta k^2b^2$$
$$+14k_0\Delta kb^2+14k_0^2b^2$$

Appendix 2. The Displacement Function Coefficients under Linearly Distributed Loads

$$b_0 = \frac{1}{g}\Bigg[\frac{115075344384000E^3I^3\Delta q + 287688360960000E^3I^3q_0}{l^{12}}\cdot b$$

$$+\Bigg(\frac{81055054080E^2I^2\Delta k\Delta q + 150693903360E^2I^2k_0\Delta q}{l^8}$$

$$+\frac{251156505600E^2I^2\Delta kq_0 + 468064396800E^2I^2k_0q_0}{l^8}\Bigg)\cdot b^2$$

$$+\Bigg(\frac{103783680EIk_0^2q_0 - 28911168EIk\Delta k\Delta q + 131459328EIk_9\Delta kq_0}{l^4}$$

$$-\frac{6782688EI\Delta k^2\Delta q + 36900864EIk_0^2\Delta q - 38244096EI\Delta k^2q_0}{l^4}\Bigg)\cdot b^3$$

$$-(231\Delta k^3\Delta q + 3432k_0^3\Delta q - 1848\Delta k^3q_0 - 12012k_0^3q_0)$$

$$+4290k_0^2\Delta k\Delta q + 1584k_0\Delta k^2\Delta q - 20592k_0^2\Delta kq - 11418k_0\Delta kq_0)\cdot b^4\Bigg]$$

$$b_1 = \frac{1}{g}\Bigg[\frac{57537672192000E^3I^3\Delta qb + 66213987840E^2I^2\Delta k\Delta qb^2}{l^{12}}$$

$$+\frac{166676590080E^2I^2k\Delta q - 3248614400E^2I^2\Delta kq_0}{l^8}\cdot b^2$$

$$+\frac{32123520EI\Delta k^2\Delta q + 177585408EIk_0^2\Delta q}{l^4}\cdot b^3$$

$$-\frac{27675648EI\Delta k^2q_0 - 149909760EIk_0\Delta k\Delta q}{l^4}\cdot b^3$$

$$+-\frac{55351296EIk_0\Delta kq_0}{l^4}\cdot b^3 + (1716\Delta k^3\Delta q + 18876k_0^3\Delta q$$

$$+25740k_0^2\Delta k\Delta q + 11154k_0\Delta k^2\Delta q - 5148k_0^2\Delta kq_0)\cdot b^4\Bigg]$$

where

$$g = \frac{6904520663040000E^4I^4}{l^{16}} + \frac{10411578777600E^3I^3\Delta kb}{l^{12}}$$

$$+\frac{20823157555200E^3I^3k_0b}{l^{12}} + \frac{6101111808E^2I^2\Delta k^2b^2}{l^8}$$

$$+\frac{23801057280E^2I^2k_0\Delta kb^2}{l^8} + \frac{23801057280E^2I^2k_0^2b^2}{l^8}$$

$$+\frac{582912EI\Delta k^3b^3}{l^4} + \frac{3719232EIk_0\Delta k^2b^3}{l^4} + \frac{7660224EIk_0^2\Delta kb^3}{l^4}$$

$$+\frac{5106816EIk_0^3b^3}{l^4} + 35\Delta k^4b^4 + 280k_0\Delta k^3b^4 + 852k_0^2\Delta k^2b^4$$

$$+1144k_0^3\Delta kb^4 + 572k_0^4b^4$$

Appendix 3. The Element Stiffness Matrix and the Nodal Force Vector

$$K_b^e = \frac{EI}{l^3}\int_0^1 N''^T N'' d\xi = \frac{EI}{l^3}\int_0^1 C\mathbf{t}''\mathbf{t}''^T C^T d\xi = \frac{EI}{t^3}C\left(\int_0^1 \mathbf{t}''\mathbf{t}''^T d\xi\right)C^T$$

$$= C\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 6 & 8 & 10 \\ 0 & 0 & 6 & 12 & 18 & 24 \\ 0 & 0 & 8 & 18 & \frac{144}{5} & 40 \\ 0 & 0 & 10 & 24 & 40 & \frac{400}{7} \end{bmatrix}C^T$$

$$K_{f1}^e = k_0 l\int_0^1 N^T N d\xi = k_0 l\int_0^1 C\mathbf{t}\mathbf{t}^T C^T d\xi = k_0 kC\left(\int_0^1 \mathbf{t}\mathbf{t}^T d\xi\right)C^T$$

$$= k_0 lC\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} \end{bmatrix}C^T$$

$$K_{f2}^e = \Delta kl\int_0^1 N^T N\xi d\xi = \Delta kl\int_0^1 C\mathbf{t}\mathbf{t}^T C^T \xi d\xi = \Delta klC\left(\int_0^1 \mathbf{t}\mathbf{t}^T \xi d\xi\right)C^T$$

$$= \Delta klC\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} \\ \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} \end{bmatrix}C^T$$

$$P_1^e = q_0 bl\int_0^1 N^T d\xi = q_0 bl\int_0^1 C\mathbf{t}^T d\xi = q_0 blC\int_0^1 \mathbf{t}^T d\xi = q_0 blC$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix}^T$$

$$P_2^e = \Delta qbl\int_0^1 N^T \xi d\xi = \Delta qbl\int_0^1 C\mathbf{t}^T \xi d\xi = \Delta qblC\int_0^1 \mathbf{t}^T \xi d\xi$$

$$= \Delta qblC\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}$$