KSCE Journal of Civil Engineering (2015) 19(7):2281-2290
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DOI 10.1007/s12205-015-0154-1

Water Engineering

pISSN 1226-7988, eISSN 1976-3808
www.springer.com/12205

Evaluation of Nonlinear Muskingum Model with Continuous and Discontinuous Exponent Parameters

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Received March 25, 2014/Revised August 10, 2014/Accepted September 15, 2014/Published Online May 1, 2015

Abstract

The nonlinear Muskingum model is traditionally calibrated using a constant exponent parameter. A recent study has proposed a discontinuous function of the exponent parameter that has substantially improved model performance. This paper evaluates model performance using continuous and discontinuous parameters, expressed as a function of dimensionless inflow variable. The parameters were represented by discontinuous (two-step) function and continuous (three-coefficient) function, resulting in a five-parameter nonlinear Muskingum model (5P-NLMM) for each scheme. Two continuous functions (logarithmic and exponential) were evaluated using two flood routing procedures: the Modified Euler's (ME) routing procedure and the Fourth-Order Runge-Kutta (FORK) routing procedure. The continuous functions and routing procedures were integrated into the Muskingum model. The five parameters of the model were determined using optimization based on minimizing the deviations from observed outflows. The model was applied to three examples with different hydrograph types. The continuous parameter (with ME or FORK) substantially outperformed the discontinuous parameter for smooth and non-smooth hydrographs, and vice versa for multi-peak hydrograph. Guidelines for model selection for different types of hydrographs are presented.

Keywords: *flood routing, continuous and discontinuous, exponent parameter, optimization, muskingum model, calibration, dimensionless inflow*

1. Introduction

The Muskingum model, originally developed by McCarthy (1938), is one of the commonly used hydrologic routing models. The model is normally represented by the continuity and storage equations.

The storage equation (model) has been traditionally represented by the following forms: (1) a two-parameter linear model, (2) a three-parameter nonlinear model with flow parameter α (NL1), and (3) a three-parameter nonlinear model with exponent parameter β (NL2).

The nonlinear model NL1 has originally been derived by Chow (1959). The linear model is obtained by assuming the parameter α of NL1 to be unity for the sake of simplicity and practical purposes (Chow, 1959; Singh, 1988). The nonlinear model NL2 was suggested by Gill (1978). The model has not been derived, but the exponent parameter β was added to the linear model to account for the nonlinearity of the flood wave in an ad-hoc manner.

The linear model is easy to calibrate both graphically and analytically (Subramanya, 2008; James 2006; Viessman and Lewis, 2003; Singh, 1992). However, model performance in fitting observed outflows or storages is not as good as that of nonlinear models. On the other hand, the calibration of the nonlinear model is more complex and requires specialized nonlinear optimization methods. For this reason, numerous algorithms have been developed over the years to improve model performance. A few studies have examined the nonlinear model NL1 and some studies have compared its performance with that of NL2 using different optimization methods. The algorithms include the standard search by Gavilan and Houck (1985), nonlinear least squares by Yoon and Padmanabhan (1993), genetic algorithms by Mohan (1997), and Lagrange multipliers by Das (2004; 2007).

Most research work during the past several decades, however, has focused on estimating the parameters of NL2 because it has provided better performance than that of NL1. The algorithms include segmented least squares by Gill (1978), pattern search by Tung (1985), harmony search by Kim *et al*. (2001), gradient search by Geem (2006), particle swarm optimization by Chu and Chang (2009), immune clonal selection by Luo and Xie (2010), Nelder-Mead simplex by Barati (2011), parameter-setting-free harmony search by Geem (2011), differential evolution by Xu *et al*. (2012), meta-heuristic algorithm by Orouji *et al*. (2012), and hybrid method by Karahan *et al*. (2013).

The author has noticed that in recent years the improvement to model performance using different algorithms was very small.

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Specifically, the sum of the squared deviations between observed and estimated outflows for Wilson's data was reduced from 36.783 (Kim *et al.* 2001) to 36.768 (Karahan *et al.* 2013). This fact provided motivation to improve model performance by modifying the structure of the exponent parameter and the model itself.

In an earlier paper (Easa 2013), the constant exponent parameter β of the NL2 model was replaced with a variable exponent parameter that improved model fit by up to 35%. The main purpose of that model, which is used here, was to introduce a new concept related to the structure of the lumped hydrologic Muskingum model. The variable exponent parameter was represented by a discontinuous (step) function of a dimensionless inflow variable that corresponded to different number of inflow levels (M). It was found that as the number of inflow levels increases, model performance improves, but the improvement was small beyond $M = 5$. Even for small M, improvement in model performance was substantial, compared with the traditional three-parameter nonlinear Muskingum model (3P NLMM). For example, for $M = 2$ (5P-NLMM), *SSQ* was 27.5 compared with that of 3P-NLMM, representing a reduction of 25.5%.

In addition, the structure of the Muskingum model itself was modified in a recent paper and a four-parameter model was developed (Easa 2014). This new model was derived assuming that channel storage is a power function of the weighted storages of the upstream and downstream sections. It has provided for the first time a physical interpretation of the exponent parameter β of the NL2 model. The purpose of the present paper is to evaluate the performance of the Muskingum model with continuous and discontinuous parameters (the word exponent is implied) using two different routing procedures and provide guidance for practical use. The evaluation is based on 5P-NLMM with discontinuous parameter based on Easa (2013).

The following section presents the optimization model, including formulation of continuous and discontinuous parameters, flood routing procedures, and objective function. Application of the model using three examples and guidelines for model selection are then presented, followed by concluding remarks.

2. Optimization Model Formulation

The continuity equation of the Muskingum model is given by:

$$\frac{dS_t}{dt} = I_t - Q_t \tag{1}$$

where, S_t = Channel storage at time t, I_t and Q_t = Rates of inflow and outflow at time t, respectively

Let the number of time intervals used for flood routing be denoted by N and the inflow, outflow, and weighted storage for time interval j by I_j, Q_j, and S_j, respectively, where $j = 0, 1, ..., N$. First define a dimensionless inflow variable as:

$$u_j = \frac{\mathrm{I_j}}{\mathrm{I_{max}}} \tag{2}$$

where, u_j = Dimensionless inflow variable (0 to 1) for time interval j

I_j = Inflow for time interval j

$\mathrm{I_{max}}$ = Maximum inflow during the routing period

Then, the Muskingum model with variable exponent parameter, is given by:

$$\mathrm{S_j} = \mathrm{K}[\mathrm{wI_j} + (1-\mathrm{w})\mathrm{I_j}]^{\beta(u_j)} \tag{3}$$

where, $\beta(u_j)$ = Variable exponent parameter for time interval j.

2.1 Discontinuous and Continuous Exponent Parameters

In Easa (2013), the discontinuous parameter was represented by the following discontinuous (step) function:

$$\beta(u_j) = \sum_{i=1}^{M} \beta_i \chi_{A_i}(u_j) \quad \text{(Discontinuous, Step Function)} \tag{4}$$

where, A_i = Inflow interval, $i = 1,2 \ldots, M$ and $\chi_{A_i}(u_j)$ = indicator function of A_i given by:

$$\chi_{A_i}(u_j) = \begin{cases} 1 \ \ if\ u_j \in A_i \\ 1 \ \ if\ u_j \notin A_i \end{cases} \tag{5}$$

In the preceding function, the dimensionless inflow range (0 to 1) was divided into M ranges with exponent parameters β_i, $i = 1, 2, \ldots, M$, as shown in Fig. 1 (for $M = 3$). These ranges were defined by $(M-1)$ dimensionless inflow variables v_i, $i = 1, 2, \ldots (M-1)$. Thus, the total number of parameters of the Muskingum model with discontinuous parameter is $2M+1$. For example, for $M = 2$ the total number of parameters is five (K, w, β_1, β_2, v_1) and for $M = 3$ the total number of parameters is seven (K, w, β_1, β_2, β_3, v_1, v_2).

For the continuous parameter, the following functions were selected for evaluation as follows:

$$\beta(u_j) = a + b\ \ln(1 + cu_j) \quad \text{(Continuous, Logarithmic)} \tag{6}$$

$$\beta(u_j) = a + be^{-e^{cu_j}} \quad \text{(Continuous, Exponential)} \tag{7}$$

where, a, b, and c = Coefficients (parameters) to be determined using optimization. Thus, the total number of parameters of the Muskingum model with continuous parameter is five (K,

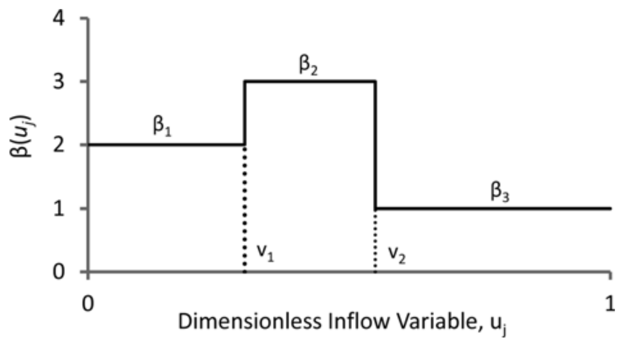

Fig. 1. Illustration of Discontinuous Function of the Variable Exponent Parameter ($M = 3$)

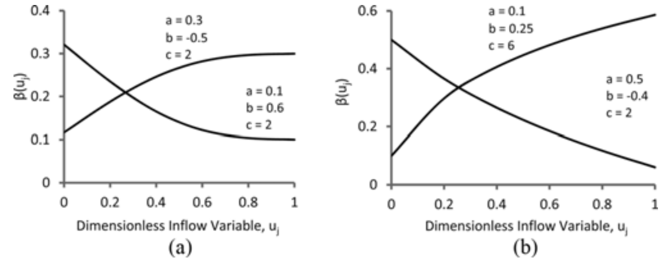

Fig. 2. Variation of Continuous Function of the Variable Exponent Parameter: (a) Exponential Function, (b) Logarithmic Function

w, a, b, c). The functional forms of Eqs. (6) and (7) were selected in this study since they provided the best model performance among several functions evaluated. These functions are illustrated in Fig. 2. As noted, Both functions can provide increasing or decreasing trend for the variation of $\beta(u_j)$ with u_j, depending on flood characteristics. In addition, exponential and logarithmic functions provide mathematically different shapes, as noted in the figure.

2.2 Flood Routing Procedures

Two flood routing procedures were evaluated in this study. The first procedure involved the modified Euler's (ME) method suggested by Tung 1985. The second procedure involved the Fourth-Order Runge-Kutta (FORK) method, suggested by Vatankhah (2010) based on Gerald and Wheatley (1994). Both procedures are based on the hydrologic continuity and storage equations of Eqs. (1) and (3), respectively.

2.2.1 ME Routing Procedure

The ME routing procedure involves the following steps (Tung 1985):

Step 1: Assume values of the three parameters *K*, *w*, and β.

Step 2: Calculate the accumulated storage for $j = 1$ as:

$$S_1 = K[wI_o + (1-w)I_o]^{\beta(u_j)} \tag{8}$$

where, I_o is the initial inflow ($Q_o = I_o$).

Step 3: Calculate the next accumulated storage for $j = 2, 3, \ldots, N$ as:

$$S_j = S_{j-1} + t\, k_{j-1} \tag{9}$$

where k_{j-1} is the storage rate for time interval $j-1$, which is given by:

$$k_{j-1} = -\left(\frac{1}{1-w}\right)\left(\frac{S_{j-1}}{K}\right)^{\frac{1}{\beta(u_{j-1})}} + \left(\frac{1}{1-w}\right)I_{j-1} \tag{10}$$

Step 4: Calculate the outflow For $j = 1, 2, \ldots, N$ as:

$$\overline{Q}_j = \left(\frac{1}{1-w}\right)\left(\frac{S_j}{K}\right)^{\frac{1}{\beta(u_j)}} - \left(\frac{w}{1-w}\right)I_{j-1} \tag{11}$$

Note that most previous authors have used I_{j-1} rather than I_j (Geem 2011) and for $j = 0$, $\overline{Q}_j = Q_o$.

2.2.2 FORK Routing Procedure

The FORK routing procedure involves the following steps:

Step 1: Assume values of the three parameters *K*, *w*, and β.

Step 2: Calculate the accumulated storage for $j = 0$ as:

$$S_j = K[wI_j + (1-w)Q_j]^{\beta(u_j)} \tag{12}$$

Step 3: Calculate the accumulated storage for $j = 1, 2, \ldots, N$ as:

$$S_j = S_{j-1} + \frac{\Delta t}{6}(\mathrm{k1}_{j-1} + 2\mathrm{k2}_{j-1} + 2\mathrm{k3}_{j-1} + \mathrm{k4}_{j-1}) \tag{13}$$

where, the second term of the right side of Eq. (13) is the product of the size of the time interval Δt and an estimated average storage rate based on four storage rates within time interval $j-1$ ($k1_{j-1}$, $k2_{j-1}$, $k3_{j-1}$, and $k4_{j-1}$) which are given by:

$$k1_{j-1} = -\left(\frac{1}{1-w}\right)\left(\frac{S_{j-1}}{K}\right)^{\frac{1}{\beta(u_{j-1})}} + \left(\frac{1}{1-w}\right)I_{j-1} \tag{14}$$

$$k2_{j-1} = -\left(\frac{1}{1-w}\right)\left(\frac{S_{j-1} + 0.5k1_{j-1}\Delta t}{K}\right)^{\frac{1}{\beta(u_{j-0.5})}} + \left(\frac{1}{1-w}\right)\left(\frac{I_{j-1} + I_j}{2}\right) \tag{15}$$

$$k3_{j-1} = -\left(\frac{1}{1-w}\right)\left(\frac{S_{j-1} + 0.5k2_{j-1}\Delta t}{K}\right)^{\frac{1}{\beta(u_{j-0.5})}} + \left(\frac{1}{1-w}\right)\left(\frac{I_{j-1} + I_j}{2}\right) \tag{16}$$

$$k4_{j-1} = -\left(\frac{1}{1-w}\right)\left(\frac{S_{j-1} + k3_{j-1}\Delta t}{K}\right)^{\frac{1}{\beta(u_j)}} + \left(\frac{1}{1-w}\right)I_j \tag{17}$$

where, for $m(u_{j-0.5})$, the inflow $I_{j-0.5} = (I_{j-1} + I_j)/2$.

Step 4: Calculate the outflow for $j = 0, 1, \ldots, N$ as:

$$\overline{Q}_j = \left(\frac{1}{1-w}\right)\left(\frac{S_j}{K}\right)^{\frac{1}{\beta(u_j)}} - \left(\frac{w}{1-w}\right)I_j \tag{18}$$

Note that the calculated outflow for $j = 0$ always equals the observed outflow Q_0.

2.3 Objective Function

The objective function of the optimization model minimizes the sum of the squared deviations between the estimated and observed outflows (outflow criterion). That is:

$$\text{Minimize } SSQ = \sum_{j=1}^{N}(\overline{Q}_j - Q_j)^2 \tag{19}$$

where,

SSQ = Sum of the squared deviations between estimated and observed outflows

Q_j = Observed outflow for time interval j

$\overline{Q}_j$ = Estimated outflow for time interval j as obtained by Eq. (11) or (18)

The proposed optimization model consists of the objective of (19) and the constraints of Eqs. (8)-(11) for the ME procedure or

Table 1. Comparison of the Results for Constant, Continuous, and Discontinuous Exponent Parameters for the Three Application Examples

Exponent Parameter	No. Model Para.	Example 1 (Smooth) SSQ		Example 2 (Non-smooth) SSQ		Example 3 (Multi-Peak) SSQ	
		ME	FORK	ME	FORK	ME	FORK
Constant (C)	3	36.8	62.6	35,240	92,465	71,114	68,189
Continuous (CONT) [a]	5	18.1 (E)	26.5 (L)	34,286 (L)	35,980 (E)	59,402 (L)	63,300 (L)
Discontinuous (DC) (M = 2)	5	27.5	44.9	35,065	45,466	57,391	51,935
% Diff. (CONT - DC)	-	-51.9	-69.4	-2.3	-26.4	3.4	18.0
% Diff. (C - CONT)	-	50.8	57.7	2.7	61.1	16.5	7.2

[a] The letter refers to the best continuous function found: Logarithmic (L) or Exponential (E).

Eqs. (12)-(18) for the FORK procedure. The model, which is nonlinear and non-convex was solved using the *Solver* software in Excel2010 which includes two algorithms for nonlinear optimization: Advanced generalized reduced gradient algorithm and evolutionary algorithm (PRWeb 2014).

3. Application and Guidelines

3.1 Application Examples

The proposed model was applied to three examples with different hydrograph types. Example 1 involved a smooth hydrograph based on (Wilson 1974), Example 2 involved a non-smooth single-peak hydrograph for the River Wye stretch in the United Kingdom (NERC, 1975), and Example 3 involved a multi-peak hydrograph based on Viessman and Lewis. For each example, the Muskingum model was run eight times, four for the ME procedure and four the FORK procedure, each involving constant, continuous (logarithmic and exponential), and discontinuous parameters. The discontinuous parameter was calculated using Eqs. (4) and the continuous parameter was calculated using Eqs. (6) or (7).

3.1.1 Optimal Results

The evaluation results are shown in Table 1. The performance of both continuous and discontinuous exponent parameters is better than that of the constant exponent parameter. This is expected as both involve two more parameters than 3P-NLMM. The continuous parameter improved model performance by 2.7-50.8% for ME procedure and 7.2-61.1% for the FORK procedure, compared with the constant parameter. For smooth and non-smooth hydrographs (Examples 1 and 2), the continuous parameter with ME procedure improved model fit by 2.3% and 51.9%, respectively, compared with the discontinuous parameter. In addition, for these hydrographs, ME procedure consistently outperformed FORK procedure. For multi-peak hydrograph, the performance of the discontinuous parameter was better than that of the continuous parameter (3.4% for ME and 18.0% for FORK).

The optimal continuous and discontinuous parameters for the three examples are shown in Fig. 3 for the FORK procedure. What is interesting is that the trend of the continuous parameter in each example follows a similar trend to that of the dimensionless inflow (or inflow). It seems that the continuous parameter carries the signature of the inflow characteristics. This would provide support for using continuous parameters for other similar flood events in the prediction stage.

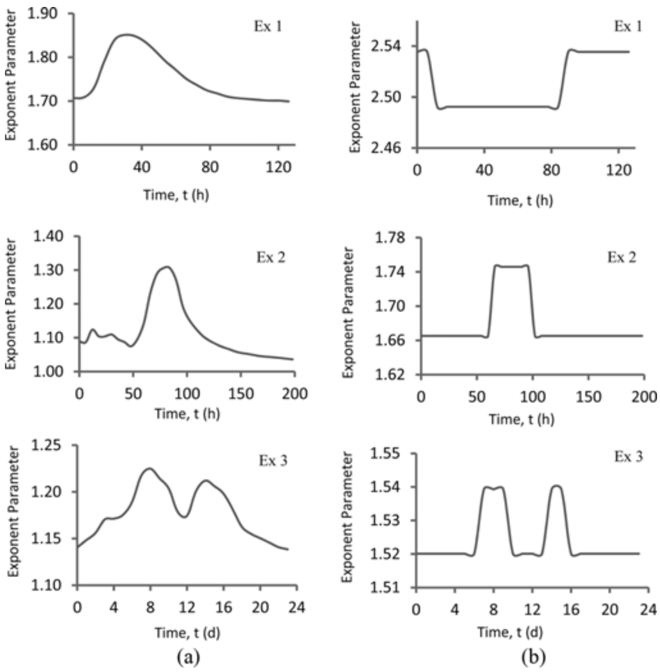

Fig. 3. Variation of the Continuous and Discontinuous Exponent Parameters for the FORK Procedure for the Three Application Examples: (a) Continuous Exponent Parameter, (b) Discontinuous Exponent Parameter

The variation of the discontinuous parameter follows a similar trend. However, the trend of the continuous parameter is smooth while that of the discontinuous parameter is not. The reason for this is that the continuous parameter strictly mimics the shape of the inflow hydrograph since this parameter is a continuous function of the dimensionless inflow variable which is directly related to the inflow ($u_j = I_j/I_{max}$). Thus, the continuous parameter scheme is smooth and similar to the corresponding inflow hydrograph. On the other hand, the discontinuous parameter has dimensionless inflow dividing parameters v_1 to v_{M-1} besides the exponent parameters β_1 to β_M. These dividing parameters can

Table 2. Optimal Results for Continuous Exponent Parameter for the ME Procedure in Example 2 (non-smooth hydrograph)

t	I	Q	I/Imax	β^{b}	S	ΔSt /Δt	Estimated Q
0	154	102	0.134	-	-	-	102
6	150	140	0.131	1.539	1578.1	-7.3	154.0
12	219	169	0.191	1.538	1534.5	122.1	153.3
18	182	190	0.159	1.539	2267.2	-24.1	175.8
24	182	209	0.159	1.539	2122.9	-9.2	191.2
30	192	218	0.168	1.538	2067.6	14.5	185.6
36	165	210	0.144	1.539	2154.9	-43.1	186.0
42	150	194	0.131	1.539	1896.4	-42.7	180.5
48	128	172	0.112	1.539	1639.9	-54.0	164.0
54	168	149	0.147	1.539	1316.1	56.3	144.4
60	260	136	0.227	1.537	1654.1	182.2	153.0
66	471	228	0.411	1.534	2747.2	447.3	196.1
72	717	303	0.626	1.530	5431.2	656.9	261.1
78	1092	366	0.954	1.525	9372.9	1042.6	355.8
84	1145	456	1.000	1.525	15628.7	760.2	428.1
90	600	615	0.524	1.532	20189.9	-422.0	576.6
96	365	830	0.319	1.536	17658.1	-701.2	874.2
102	277	969	0.242	1.537	13451.0	-632.7	837.8
108	227	665	0.198	1.538	9654.5	-500.1	686.2
114	187	519	0.163	1.538	6654.1	-374.7	529.0
120	161	444	0.141	1.539	4405.7	-253.1	392.8
126	143	321	0.125	1.539	2887.1	-154.3	282.6
132	126	208	0.110	1.539	1961.3	-92.9	205.0
138	115	176	0.100	1.540	1404.0	-49.9	155.9
144	102	148	0.089	1.540	1104.6	-36.0	127.4
150	93	125	0.081	1.540	888.5	-23.1	108.7
156	88	114	0.077	1.540	749.9	-12.1	96.0
162	82	106	0.072	1.540	677.2	-11.9	89.0
168	76	97	0.066	1.540	605.5	-11.5	82.6
174	73	89	0.064	1.540	536.4	-5.6	76.2
180	70	81	0.061	1.540	502.6	-5.3	72.9
186	67	76	0.059	1.540	470.5	-5.2	69.8
192	63	71	0.055	1.541	439.2	-6.9	66.6
198	59	66	0.052	1.541	397.7	-6.6	62.3

[a]*SSQ* = 34,286 and the optimal parameters are $K = 0.678$, $w = 0.450$, $a = 1.542$, $b = -0.046$, and $c = 0.454$.

take any values between 0 and 1 to optimize the objective function. Any dimensionless inflow values within these dividing parameters have constant exponent parameters (as indicated by the step function of Fig. 1). Thus, this scheme makes the variation of the discontinuous parameter non-smooth.

Selected optimal results of the ME and FORK procedures with the continuous parameter are presented for different examples. Tables 2 and 3 present the optimal results for the ME procedure in Examples 2 and 3, respectively. Tables 4 and 5 present the optimal results for the FORK procedure in Examples 1 and 3, respectively.

3.1.2 Graphical Illustration

To graphically illustrate the improvement produced by the continuous parameter, the estimated outflows for continuous and discontinuous parameters and the corresponding deviations from the observed outflows are presented in Fig. 4(a) and (b), respectively (Example 2, FORK procedure). As noted, the continuous parameter substantially improved model performance, compared with the discontinuous parameter (26.4% reduction in *SSQ,* see Table 1). In addition, the continuous function also has smoothed out the outflow hydrograph near the peak.

Finally, Fig. 5 shows the estimated outflows for continuous and discontinuous parameters for Example 3 with the ME procedure. Recall from Table 1 that the discontinuous parameter has outperformed the continuous parameter for this multi-peak hydrograph. The better fit of the discontinuous parameter to observed outflows is clearly evident in the figure. As previously noted on Fig. 3, the variations of the continuous and discontinuous parameters were smooth and non-smooth, respectively. The continuous parameter is somewhat restrictive when the hydrograph

Table 3. Optimal Results for Continuous Exponent Parameter for the ME Procedure in Example 1 (smooth hydrograph)

							Estimated
t	I	Q	I/Imax	β^b	S	ΔSt /Δt	Q
0	22	22	0.198	-	-	-	22
6	23	21	0.207	1.826	204.0	1.4	22.0
12	35	21	0.315	1.803	212.3	16.0	23.6
18	71	26	0.640	1.803	308.1	58.1	26.4
24	103	34	0.928	1.803	656.8	81.5	33.5
30	111	44	1.000	1.803	1145.9	70.7	43.3
36	109	55	0.982	1.803	1570.3	52.3	55.9
42	100	66	0.901	1.803	1884.1	29.5	67.1
48	86	75	0.775	1.803	2061.2	4.7	76.0
54	71	82	0.640	1.803	2089.5	-16.8	82.1
60	59	85	0.532	1.803	1988.8	-30.2	84.7
66	47	84	0.423	1.803	1807.5	-41.0	83.5
72	39	80	0.351	1.803	1561.6	-43.8	79.7
78	32	73	0.288	1.803	1299.0	-43.9	73.3
84	28	64	0.252	1.804	1035.4	-38.9	65.4
90	24	54	0.216	1.817	802.2	-32.4	54.8
96	22	44	0.198	1.841	608.1	-23.2	44.5
102	21	36	0.189	1.862	468.8	-15.7	36.3
108	20	30	0.180	1.892	374.5	-10.0	29.6
114	19	25	0.171	1.933	314.6	-5.8	24.4
120	19	22	0.171	1.933	279.8	-3.9	22.9
126	18	19	0.162	1.987	256.3	-1.7	19.3

[a]SSQ = 16.4 and the optimal parameters are K = 0.721, w = 0.274, a = 1.803, b = 12.421, and c = 8.856.

Table 4. Optimal Results for Continuous Exponent Parameter for the FORK Procedure in Example 1 (smooth hydrograph)

											Estimated
t	I	Q	I/Imax	S	k1	k2	k3	k4	kave	β	Q
0	22	22	0.198	134.7	0.0	3.7	2.5	5.7	3.0	1.707	22.0
6	23	21	0.207	137.7	5.4	47.7	34.9	72.9	40.6	1.709	22.1
12	35	21	0.315	178.2	69.7	185.0	159.2	268.1	171.0	1.730	23.4
18	71	26	0.640	349.3	263.9	342.9	331.7	413.9	337.8	1.791	27.0
24	103	34	0.928	687.1	412.4	399.1	400.5	393.6	400.9	1.839	34.3
30	111	44	1.000	1087.9	393.6	351.1	354.7	316.9	353.7	1.851	45.4
36	109	55	0.982	1441.6	317.1	254.7	259.6	201.0	257.8	1.848	56.2
42	100	66	0.901	1699.4	201.3	125.7	131.6	59.9	129.3	1.835	66.4
48	86	75	0.775	1828.7	60.2	-10.9	-5.1	-72.3	-7.4	1.814	76.0
54	71	82	0.640	1821.3	-71.9	-120.1	-115.8	-160.3	-117.3	1.791	83.0
60	59	85	0.532	1704.0	-160.0	-199.7	-195.8	-231.5	-197.1	1.771	85.7
66	47	84	0.423	1506.9	-231.2	-243.0	-241.7	-250.8	-241.9	1.751	85.5
72	39	80	0.351	1265.0	-250.7	-251.9	-251.8	-250.4	-251.4	1.737	80.8
78	32	73	0.288	1013.6	-250.5	-233.4	-235.9	-218.0	-234.5	1.725	73.8
84	28	64	0.252	779.1	-218.4	-200.8	-203.7	-185.2	-202.1	1.718	64.4
90	24	54	0.216	577.0	-185.8	-159.8	-164.8	-139.3	-162.4	1.710	55.0
96	22	44	0.198	414.6	-140.3	-114.3	-120.1	-95.3	-117.4	1.707	45.4
102	21	36	0.189	297.2	-96.6	-76.7	-81.7	-62.8	-79.4	1.705	37.1
108	20	30	0.180	217.8	-64.1	-49.9	-54.0	-40.6	-52.1	1.703	30.7
114	19	25	0.171	165.7	-41.8	-28.7	-32.9	-20.8	-31.0	1.701	26.0
120	19	22	0.171	134.8	-22.1	-18.3	-19.6	-16.0	-19.0	1.701	22.7
126	18	19	0.162	115.8						1.699	20.7

[a]SSQ = 26.5 and the optimal parameters are K = 0.689, w = 0.123, a = 1.665, b = 0.574, and c = 0.383.

Table 5. Optimal Results for Continuous Exponent Parameter for the FORK Procedure in Example 3 (multi-peak hydrograph)

t	I	Q	I/Imax	S	k1	k2	k3	k4	kave	â	Estimated Q
0	166.2	118.4	0.094	90.5	47.8	72.2	58.8	86.6	66.1	1.140	118.4
1	263.6	197.4	0.148	156.5	79.0	94.0	86.6	104.2	90.7	1.148	184.6
2	365.3	214.1	0.206	247.3	100.4	173.9	141.9	216.8	158.1	1.156	264.9
3	580.5	402.1	0.327	405.4	203.7	131.1	159.5	94.9	146.7	1.170	376.8
4	594.7	518.2	0.335	552.0	104.8	105.0	104.9	106.4	105.2	1.171	489.9
5	662.6	523.9	0.373	657.2	106.2	221.1	181.8	287.9	200.0	1.175	556.4
6	920.3	603.1	0.518	857.2	276.2	575.0	491.0	768.2	529.4	1.189	644.1
7	1568.8	829.7	0.884	1386.6	749.1	693.7	706.6	670.6	703.4	1.217	819.7
8	1775.5	1124.2	1.000	2090.0	672.0	344.3	417.0	121.0	385.9	1.225	1103.5
9	1489.5	1379	0.839	2475.9	135.1	-75.0	-25.4	-213.6	-46.5	1.214	1354.4
10	1223.3	1509.3	0.689	2429.4	-203.2	-513.1	-429.4	-704.3	-465.4	1.204	1426.5
11	713.6	1379	0.402	1964.0	-682.5	-521.1	-572.0	-420.4	-548.2	1.178	1396.1
12	645.6	1050.6	0.364	1415.8	-436.0	46.1	-98.9	301.2	-40.1	1.174	1081.6
13	1166.7	1013.7	0.657	1375.7	268.7	358.0	334.7	417.6	345.3	1.201	898.0
14	1427.2	1013.7	0.804	1721.0	412.4	219.5	267.3	95.7	247.0	1.212	1014.8
15	1282.8	1013.7	0.723	1968.0	105.9	-42.4	-4.3	-136.2	-20.6	1.206	1176.9
16	1098.7	1209.1	0.619	1947.4	-127.5	-321.3	-266.5	-436.4	-289.9	1.198	1226.2
17	764.6	1248.8	0.431	1657.4	-422.1	-503.8	-476.8	-535.8	-486.5	1.181	1186.7
18	458.7	1002.4	0.258	1170.9	-528.7	-404.1	-452.1	-325.4	-427.8	1.163	987.4
19	351.1	713.6	0.198	743.2	-345.3	-240.6	-285.2	-180.1	-262.9	1.155	696.4
20	288.8	464.4	0.163	480.3	-200.1	-144.9	-170.5	-113.0	-157.3	1.150	488.9
21	228.8	325.6	0.129	323.0	-125.7	-97.4	-111.6	-80.4	-104.0	1.146	354.5
22	170.2	265.6	0.096	219.0	-88.3	-56.6	-73.6	-39.2	-64.6	1.141	258.5
23	143	222.6	0.081	154.3						1.138	192.1

[a] *SSQ* = 63,300 and the optimal parameters are $K = 0.389$, $w = 0.010$, $a = 1.125$, $b = 0.093$, and $c = 1.910$.

has multiple peaks. Examination of the results show that two of the estimated outflows of the continuous function that lie near the outflow peaks (time = 12 and 13 days) substantially contribute to *SSQ*. It appears that the continuous function is unable to adapt to the sudden variations in the observed outflow hydrograph. For such hydrographs, the non-smooth trend of the discontinuous parameter provides more flexibility in fitting the outflow data. On the other hand, the smooth trend of the continuous parameter appears to better fit the observed outflows when the hydrograph is smooth.

3.2 Guidelines for Model Selection

Based on the application results, guidelines for model selection for different types of hydrographs are presented in Fig. 6. For each of the four combinations of the exponent parameter type (continuous and discontinuous) and the routing procedure type (ME and FORK), the type of hydrograph for which the combination is best is shown in the figure. The continuous exponent parameter is the best for smooth and non-smooth hydrographs, while the discontinuous exponent parameter is the best for multi-peak hydrographs for both ME and FORK procedure. However, for each exponent parameter type the best routing procedure is shown. Note that although ME procedure is slightly better for non-smooth hydrograph, FORK procedure is also recommended as it produced substantial improvement over the discontinuous parameter.

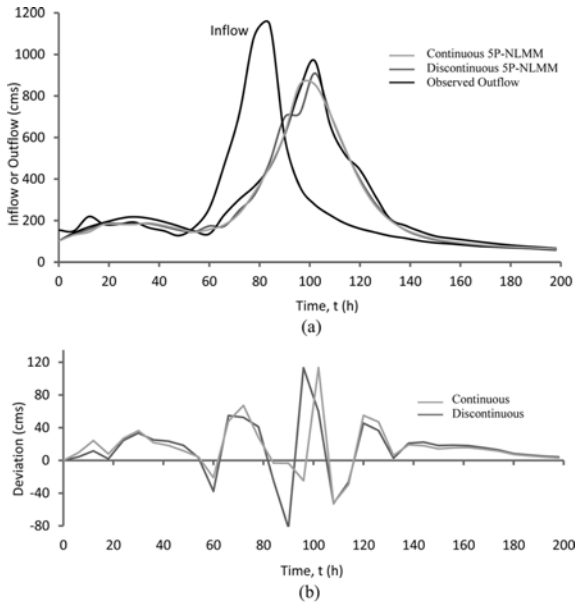

Fig. 4. Results of the Continuous and Discontinuous Exponent Parameters for the FORK Procedure in Example 2 (non-smooth hydrograph): (a) Outflow Comparison, (b) Deviation Comparison

Therefore, the final guidelines are: (1) for smooth hydrographs,

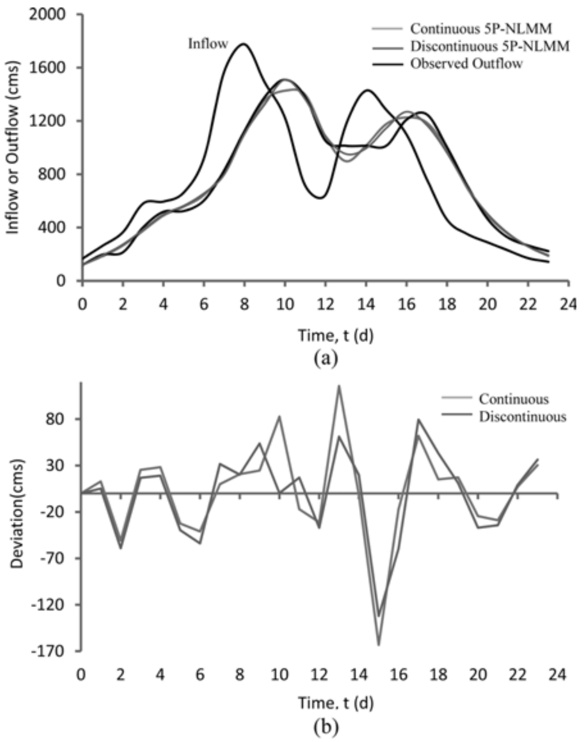

Fig. 5. Results of the Continuous and Discontinuous Exponent Parameters for the FORK Procedure in Example 3 (multi-peak hydrograph): (a) Outflow Comparisonm, (b) Deviation Comparison

Exponent Parameter	Routing Procedures	
	ME	FORK
Continuous	*Smooth* [a]	Smooth
	Non-smooth [a]	Non-smooth [b]
Discontinuous	Multi-peak	*Multi-peak* [a]

[a] The corresponding routing procedure provides better performance.
[b] Continuous parameter with FORK may also be used for non-smooth hydrographs.

Fig. 6. Guidelines for Model Selection for Different Types of Hydrographs

use a continuous exponent parameter with ME procedure, (2) For non-smooth hydrographs, use a continuous exponent parameter with either ME or FORK procedure, and (3) for multi-peak hydrographs, use a discontinuous exponent parameter with FORK procedure.

4. Conclusions

This paper has proposed a continuous exponent parameter for the nonlinear Muskingum model that accounts for the nonlinearity of flood wave in a continuous manner. The continuous parameter was expressed as a function of a dimensionless inflow variable. Two functional forms of the continuous parameter (logarithmic and exponential) combined with two different routing procedures (ME and FORK) were evaluated using three application examples. Based on this study, the following comments are offered:

1. The continuous or discontinuous parameter was expressed as a function of dimensionless inflow variable. The results show that the trend of the optimal values of these parameters resembles that of the inflow. It appears that the calibrated optimal parameter carries the signature of the inflow. This may provide support for using the inflow as a relevant flood property for defining the exponent parameter and its transferability to the prediction stage.
2. The optimization model with a continuous parameter was straightforward to calibrate and faster to run. For multi-peak hydrographs, the performance of the discontinuous parameter was substantially better and is recommended for such hydrographs. For smooth and non-smooth hydrographs, the continuous parameter provided better performance when combined with ME procedure than when combined FORK procedure. However, the FORK procedure may perform better when other criteria (e.g. storage criterion) are considered (Easa 2014b). The FORK method is a micro-simulation of the storage rate within each time interval, where four storage rates are simulated and the average rate is used to calculate the accumulated storage for the next time interval. As such, it may have beneficial effects on other modeling aspects.
3. The functional forms of the continuous parameter presented in this paper have been selected based on the analysis of numerous functions. They were used in the paper just to illustrate the concept. In fact, different flood data may require different functional forms to improve model performance. In applying the proposed model for a given flood data, it is recommended that the user should evaluate a number of continuous functional forms in addition to those presented, especially forms involving some transcendental functions (Campos 2011).
4. Early researchers recognized that the parameters of the Muskingum model vary with flood characteristics. However, no methods were developed at that time to account for this variation since they would have made the model too complex to solve using prevailing technical knowledge. This is not the case today. Encouraged by the results of the variable exponent parameter model (Easa 2013), the author has developed a new Muskingum model with four variable parameters (Easa 2014c). The model produces 15 combinations of special models. As for future research, K and w of the linear Muskingum model may be considered as variables and compared with 5P-NLMM with continuous and discontinuous exponent parameters presented in this paper. Note that considering K and w as variables in the linear model will result in a nonlinear model. If each variable is represented by continuous (three-coefficient) function or discontinuous (two-step) function, the resulting model will be 6P-NLMM or 5P-NLMM, respectively.
5. The presented continuous and discontinuous 5P-NLMM have more parameters than the traditional 3P-NLMM. However, this is not an issue given current advances in computer technology and optimization algorithms that can efficiently

handle hundreds of parameters. It is important, however, to keep the model as simple as possible to ensure model validity in prediction. Several issues related to the effect of model structure and calibration process on model prediction are addressed in Easa (2014d).

Acknowledgements

The author is grateful to two anonymous reviewers for their most helpful comments. Their thorough reviews and generous ideas are greatly appreciated.

Notations

a, b, c = coefficients (parameters) of the continuous exponent parameter $m(u_j)$
I_j = Observed inflow for time interval j
I_{max} = Observed maximum inflow during the routing period
j = Index for time interval
K = Storage parameter
$k1_j$, $k2_j$, $k3_j$, $k4_j$ = Storage rates within time interval j
M = Number of inflow levels (exponent parameters)
N = Number of time intervals
Q_j = Observed outflow for time interval j
$\overline{Q}_j$ = Estimated outflow for time interval j
S_j = Weighted storage for time interval j
SSQ = Sum of the squared deviations between the observed and estimated outflows
t = Time
u_j = Dimensionless inflow variable for time interval j
v_i = Dimensionless inflow dividing variable i
w = Weighting parameter for inflow that represents the inflow-outflow relative effects on the storage
β = Constant exponent parameter
$\beta(u_j)$ = Variable exponent parameter
Δt = Size of time interval

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