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Boundedly Rational Agents in Sequential Posted Pricing

HUANG Wenhan[1] (黄文瀚), *DENG Xiaotie*[2*] (邓小铁)
(1. Department of Computer Science and Engineering, Shanghai Jiao Tong University, Shanghai 200240, China;
2. Center on Frontiers of Computing Studies, Peking University, Beijing 100871, China)

Abstract: We consider the well-studied sequential posted pricing scenarios. In these scenarios, an auctioneer typically learns the value distributions of all agents as prior information and then offers a take-it-or-leave-it price to each sequentially coming agent. If the value distributions are correctly learned, the dominant strategy of each agent is telling the truth. However, an agent could manipulate her value distribution to exploit the auctioneer. We study the behavior of sophisticated agents predicted by two prominent bounded rationality models: the level-k and the cognitive hierarchy models. We begin with analyzing the structure of the optimal reported distributions and then provide algorithms to compute the optimal distributions for each model. In the continuous scenarios, we show that both models are ill-defined by some examples. Moreover, we evaluate both models in discrete scenarios with different numbers of agents, different minimum units of the values, and different risk tolerances. The empirical results and a brief discussion about the Bayesian Nash equilibrium of the experimental scenarios show that both the level-k model and the equilibrium suggest the highest possible prices. In contrast, the cognitive hierarchy model suggests low prices. The level-k model and the equilibrium somehow explain the "winner's curse" in online markets. The models and the equilibrium fail to explain that the same item could have different prices in different shops. To explain the different-price phenomenon, we suggest trying other bounded rationality models for agents and/or considering the auctioneers with bounded rationality.

Keywords: bounded rationality, level-k model, cognitive hierarchy model, sequential posted pricing
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0 Introduction

In many online markets, e.g., eBay, Taobao, and Xianyu, a shopkeeper puts some items on the shelf and waits for the buyers. When a buyer comes to the shop, she might negotiate with the shopkeeper about the price of an item and decide to take it or leave. These scenarios are modeled as the sequential posted pricing auctions[1-2]. An auctioneer offers each agent a take-it-or-leave-it price for a sequential posted pricing schema. Ideally, the auctioneer can control the arrival order of agents as he might choose the order to respond if several agents come simultaneously. Since an agent only decides whether to accept the posted price, telling a lie would not increase her utility. Thus, a sequential posted pricing auction is a truthful auction. Usually, the auctioneer can learn the value distributions of the truth-telling agents from the trading history[3-4], so that the auctioneer can optimize his expected revenue under the Bayesian setting[5]. However, if an agent observes the learning behavior of the auctioneer, she might manipulate her distribution to obtain more utility[6-9]. Here is an example that a sophisticated agent might completely exploit the auctioneer.

Example 1 Consider that there is only 1 agent, and the auctioneer chooses the optimal posted price according to the reported distribution $\tilde{D}_1$. Suppose that Agent 1 knows this fact. Then she could manipulate her distribution to obtain almost all possible utility.

Actually, unless distribution D_1 is a one-point distribution with support $\{0\}$, Agent 1 could report the distribution $\tilde{D}_1$ with only support $\{\epsilon\}$. Then the auctioneer posts price ϵ and obtains revenue ϵ. Since ϵ could be any positive number, Agent 1 obtains almost all possible utility. Furthermore, the auctioneer would not detect the misreporting even in the repeated setting. The agent would always accept price ϵ unless the auctioneer tried to offer a higher price.

The most promising way to analyze the exploitation behaviors of sophisticated agents is to compute the Bayesian Nash equilibrium[10] or to apply some bounded rationality models[11]. First, notice that an agent's "pure" strategy is to report a distribution. Then the Bayesian Nash equilibrium strategies usually

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*E-mail: xiaotie@pku.edu.cn

report a distribution over distributions, which could be hard to formulate. For example, the agent's optimal strategy in Example 1 is not a well-formulated distribution. So we start with the bounded rationality models.

In this paper, we study the agents predicted by the level-k model[12-14] and the cognitive hierarchy model[15], which are the most prominent ones among the bounded rationality models. These models have succeeded in solving many games[16] and some auctions[14]. First, we analyze the behavior of a level-1 agent. We prove a structure lemma to bound the possible highest support of the reported distribution. Then we analyze the 2-agent case and a large market with enough i.i.d. agents. For each case, we calculate the maximal possible utility of the agent. After that, we turn to the general case and provide an algorithm to compute the optimal utility of the level-1 agent. The analysis of the behavior of the level-1 agents provides insight for designing the algorithms for the level-k model and the cognitive hierarchy model. For both models, we provide algorithms to compute the optimal strategies of agents with different sophistication levels. And then, we construct simple examples to show that both models could be ill-defined when the support elements of the distribution are continuous. We evaluate the models in discrete scenarios with different numbers of agents, different risk tolerances, and different minimum units of the values. A long pathological period typically occurs before the convergence for the level model. Even worse, the strategies of the sophisticated agents could form cycles. The convergence of the behavior is mainly affected by a large number of agents, the risk-averse level, and the sizeable minimum unit. When the number of agents is large enough, the level-k model suggests a price near the highest possible value of agents. For the cognitive hierarchy model, the strategies will quickly converge after a short pathological period whose length mainly depends on the belief of the sophisticated agents. The cognitive hierarchy model usually suggests a low price for the item, even with many agents. After evaluating two models, we also briefly discuss the Bayesian Nash equilibrium of the experimental scenarios. We can obtain a Bayesian Nash equilibrium for each scenario by solving the bi-matrix game induced by the strategies predicted by the level-k model. The Bayesian Nash equilibrium usually suggests the same price as the level-k model in large markets. However, in a real online market, the same item might have different prices in different shops. Neither the two models nor the equilibrium could explain this phenomenon. Our study suggests that a new bounded rationality model for the agents should be proposed for the sequential posted pricing scenarios, and/or the auctioneers might have bounded rationality.

1 Related Work

1.1 Sequential posted pricing

Sequential posted pricing[1-2,17] is a take-it-or-leave-it schema in that the auctioneer offers a price to each coming agent, and the agent decides whether to take the item with the posted price or leave the market forever. The auction is truthful because an agent cannot increase her utility by misreporting the acceptance or decline. More robustness discussions can be seen in previous work[1-2,17]. Usually, the auctioneer can optimize his revenue under the Bayesian setting[5]. Furthermore, the sequential posted pricing has a good revenue guarantee compared with other auctions[18-19].

However, an agent may manipulate her distribution and mislead the auctioneer to obtain more utility. This paper assumes that the agents can misreport (or manipulate) their value distributions while the auctioneer will optimize his revenue with the reported distributions. Then we study the behaviors of agents with bounded rationality in such scenarios.

1.2 Level-*k* Model

The level-k model[12-13] assumes that a series of sophisticated agents iteratively best respond to “naive” (non-strategic) agents who act randomly or tell the truth[14]. This model is usually used to explain the agents' behaviors in both cooperative and competitive scenarios, e.g., first-price or second-price auctions[14], betting games[20], and coordination games[21]. The model can also explain the “winner's curse” in some auctions[14,22]. Levin and Zhang[23] proposed an NLK model bridging the level-k model and the Bayesian Nash equilibrium so that an agent might assume that other agents have the same sophistication level as hers. In this paper, we will analyze the agents predicted by the level-k model when they can manipulate their value distributions.

1.3 Cognitive Hierarchy Model

The cognitive hierarchy model[15] generalizes the level-k model so that a sophisticated agent could assume that other agents have different lower sophistication levels. Usually, the belief forms a Poisson distribution, and such a model is named the Poisson cognitive hierarchy model. This model has been used to explain the experimental data of p-beauty games[15], coordination games[24], lottery games[25], and action commitment games[26]. Furthermore, Koriyama and Ozkes[27] proposed the inclusive cognitive hierarchy model to allow the agents to consider others at the same sophistication level. This paper will also analyze the agents predicted by the Poisson cognitive hierarchy model in our scenarios.

2 Background and Problem Formulation

2.1 Problem Formulation

The sequential posted pricing with a single item can be formalized into a pair $I = \{N, D\}$, where $N = \{1, 2, \cdots, n\}$ is the set of n agents, $D = \{D_1, D_2, \cdots, D_n\}$, and D_i is the true probability distribution for the item's value v_i of Agent i. We follow the Bayesian setting that each agent's value v_i is independently drawn from D_i.

During an auction, each agent comes sequentially and receives a take-it-or-leave-it price. If the received price is less than or equal to her value, she buys the item for that price, and the auction ends. Otherwise, she leaves the auction, and the next agent comes. When the auctioneer knows the true value distributions of the agents, he will decide an order (or permutation) $\pi : [n] \mapsto N$ and posted prices $\boldsymbol{p} \in \mathbb{R}^n$ to maximize his expected revenue.

We consider the scenarios where agents can manipulate their value distributions. The problem can be divided into two phases: In Phase I, each Agent i reports a value distribution $\tilde{D}_i$. In Phase II, the auctioneer decides on the order and the prices to maximize his expected revenue in his mind according to reported distributions $(\tilde{D}_i)_{i=1}^n$. We assume that both distributions D_i and $\tilde{D}_i$ for each Agent i have finite support. Let $\mathrm{supp}(D)$, $|\mathrm{supp}(D)|$ denote the support and the support size of the distribution D, respectively. Let $c_i(x), \tilde{c}_i(x)$ denote the probability masses at x in distributions D_i and $\tilde{D}_i$, respectively. Let $d_i(x), \tilde{d}_i(x)$ denote the probability masses of random variable greater than or equal to x in distributions D_i and $\tilde{D}_i$, respectively. The auctioneer calculates his expected revenue as

$$\sum_{i=1}^{n}\left[p_{\pi(i)}\tilde{d}_{\pi(i)}(p_{\pi(i)})\prod_{j=1}^{i-1}\left(1-\tilde{d}_{\pi(j)}(p_{\pi(j)})\right)\right].$$

We assume that the agents are risk-neutral and the utility of a risk-neutral Agent i is

$$\left[\sum_{v: v\in \mathrm{supp}(D_i), v>p_i} c_i(v)(v-p_i)\right]\times \prod_{j=1}^{\pi^{-1}(i)-1}\left(1-d_{\pi(j)}(p_{\pi(j)})\right),$$

where $\pi^{-1}(i)$ is the position of Agent i in order π.

2.2 Structure Lemma of Auctioneer

The previous work[5,28] shows that the optimal strategy of the auctioneer satisfies the following lemmas.

Lemma 1 (Order property[28]) In a sequential posted pricing scheme with n agents, if the auctioneer posts price p_i to Agent i, then the monotone decreasing order with p_i gives the highest expected revenue.

Lemma 2 (Price property[5]) In a sequential posted pricing scheme with n agents, given a order fixed π, the optimal posted price vector $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)$ of auctioneer should be on $\times_{i=1}^n \mathrm{supp}(D_i)$.

2.3 Nonstrategic Agent

As formulated by Crawford and Iriberri[14], a nonstrategic agent or a "naive" agent tells a distribution randomly or her truthful distribution. However, it is hard to define how to tell a value distribution randomly. So in this paper, a nonstrategic Agent i can only tell her truthful distribution D_i.

2.4 Sophisticated Agent

A sophisticated agent usually optimizes her reported distribution with the belief that other agents are nonstrategic or no more sophisticated than her. The level-k theory[12-13] and the cognitive hierarchy theory[15] are the most prominent models to characterize agents with different sophistication levels. We characterize the behaviors of agents predicted by the level-k model and the cognitive hierarchy model, respectively, in sequential posted pricing with a single item.

Let $\tilde{D}_{i,k}$ denote the reported distribution of an Agent i with level-k sophistication. Both models begin with a nonstrategic level-0 Agent i with $\tilde{D}_{i,0} = D_i$.

2.4.1 *Level-k Model Agent*

For a level-k Agent i predicted by the level-k model, she supposes that all other agents have level-$(k-1)$ sophistication. So her reported distribution $\tilde{D}_{i,k}$ should maximize her utility:

$$\left[\sum_{v: v\in \mathrm{supp}(D_i), v>\hat{p}_i} c_i(v)(v-\hat{p}_i)\right] \prod_{j=1}^{\hat{\pi}^{-1}(i)-1}(1-d_j(\hat{p}_j)),$$

where $\hat{\pi}, \hat{\boldsymbol{p}} = (\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_n)$ are the auctioneer's optimal ordering and optimal posted prices, respectively, when Agent i reports $\tilde{D}_{i,k}$ and each other Agent j $(j \neq i)$ reports $D_{j,k-1}$.

2.4.2 *Cognitive Hierarchy Model Agent*

For a level-k Agent i predicted by the cognitive hierarchy model, she usually believes that a proportion $g_{k,l}$ of other agents have a level-l $(l < k)$ sophistication. In other words, $\boldsymbol{g}_k = (g_{k,0}, g_{k,1}, \cdots, g_{k,k-1})$ denotes the belief of the level-k agent. However, defining the number of agents with level-l sophistication is difficult when $g_{k,l}n$ is not an integer. So we consider two different approaches:

(1) Individual sophistication. The level-k Agent i believes that the probability of an agent with level-l sophistication is $g_{k,l}$. Then her reported distribution

$\tilde{D}_{i,k}$ should maximize

$$\mathbb{E}_{\hat{\pi},\hat{\boldsymbol{p}}\sim\{\boldsymbol{g}_k,\tilde{D}_{i,k},\{\tilde{D}_{j,l}\}_{j\neq i,0\leqslant l<k}\}} \left\{\left[\sum_{v:v\in\text{supp}(D_i),v>\hat{p}_i} c_i(v)(v-\hat{p}_i)\right]\prod_{j=1}^{\hat{\pi}^{-1}(i)-1}(1-d_j(\hat{p}_j))\right\},$$

where $\hat{\pi}$, $\hat{\boldsymbol{p}}=(\hat{p}_1,\hat{p}_2,\cdots,\hat{p}_n)$ are the auctioneer's optimal ordering and optimal posted prices respectively for each instance of reported distributions.

(2) Global sophistication. The level-k Agent i believes that the probability of all other agents with level-l sophistication is $g_{k,l}$. Then her reported distribution $\tilde{D}_{i,k}$ should maximize

$$\sum_{l=0}^{k-1} g_{k,l}\left\{\left[\sum_{v:v\in\text{supp}(D_i),v>\hat{p}_i} c_i(v)(v-\hat{p}_i)\right]\times \prod_{j=1}^{\hat{\pi}^{-1}(i)-1}(1-d_j(\hat{p}_j))\right\},$$

where $\hat{\pi}$, $\hat{\boldsymbol{p}}=(\hat{p}_1,\hat{p}_2,\cdots,\hat{p}_n)$ are the auctioneer's optimal ordering and optimal posted prices, respectively, when other agents report distributions $\{\tilde{D}_{j,l}\}_{j\neq i}$.

3 Theoretical Analysis

This section will first analyze the level-1 agent's strategy in Subsection 3.1. A level-1 agent behaves the same in both the level-k and the cognitive hierarchy models, as she considers that all other agents are nonstrategic. After analyzing the level-1 agents' behaviors, we will individually consider the level-k model agents in Subsection 3.2 and the cognitive hierarchy model agents in Subsection 3.3.

3.1 Level-1 Agents

First of all, we consider the possible elements in optimal reported distributions. The analysis shows that an agent's optimal reported distribution has no support element greater than the highest support of her true value distribution. After that, we consider two special cases, the 2-agent case and the enough i.i.d. agents case. For the 2-agent case, we provide an algorithm to compute the maximal utility for a level-1 agent. For the enough i.i.d. agents case, we show that a level-1 could obtain little utility when all agents have i.i.d. value distributions. At last, we provide an algorithm to compute the optimal reported distribution for level-1 agents in the general case.

3.1.1 *Elements in Support*

As Example 1 shows, for Agent i, an element in support of the reported distribution $\text{supp}(\tilde{D}_i)$ might not appear in support of the true value distribution $\text{supp}(D_i)$. It seems that $\text{supp}(\tilde{D}_i)$ might contain any positive number. However, when considering the highest element in $\text{supp}(\tilde{D}_i)$, we could prove that it cannot be greater than the highest element in $\text{supp}(D_i)$.

Lemma 3 In a sequential posted pricing scheme with n agents, Agent n knows that the first $(n-1)$ agents will report distributions $(\tilde{D}_i)_{i=1}^{n-1}$, and the auctioneer will choose order π and posted prices $\boldsymbol{p}$ according to all reported distributions. Then there exists an optimal reported distribution $\tilde{D}_n$ of Agent n so that no element in its support $\text{supp}(\tilde{D}_n)$ can be greater than the highest element in $\text{supp}(D_n)$.

Proof Let $v_n^{|\text{supp}(D_n)|}$ denote the highest element of $\text{supp}(D_n)$. Let $\tilde{v}_n^{|\text{supp}(\tilde{D}_n)|}$ denote the highest element of $\text{supp}(\tilde{D}_n)$. Let $\tilde{v}_n^{|\text{supp}(\tilde{D}_n)|-1}$ denote the second highest element (if it exists) of $\text{supp}(\tilde{D}_n)$. When $\tilde{v}_n^{|\text{supp}(\tilde{D}_n)|} > v_n^{|\text{supp}(D_n)|}$, we consider the following three cases.

(1) $|\tilde{D}_n|=1$. In this case, Agent n has no chance to take the item, so her utility is 0. After moving the probability mass at $\tilde{v}_n^{|\text{supp}(\tilde{D}_n)|}$ to $v_n^{|\text{supp}(D_n)|}$, her utility keeps 0 and the new reported distribution has no element greater than $v_n^{|\text{supp}(D_n)|}$.

(2) $\tilde{v}_n^{|\text{supp}(\tilde{D}_n)|-1} > v_n^{|\text{supp}(D_n)|}$. In this case, Agent n has no chance to take the item for both prices $\tilde{v}_n^{|\text{supp}(\tilde{D}_n)|}$ and $\tilde{v}_n^{|\text{supp}(\tilde{D}_n)|-1}$. After moving the probability mass at $\tilde{v}_n^{|\text{supp}(\tilde{D}_n)|}$ to $\tilde{v}_n^{|\text{supp}(\tilde{D}_n)|-1}$, her expected utility for any possible order π and posted prices $\boldsymbol{p}$ would not change, since her utility is 0 when $p_n > v_n^{|\text{supp}(D_n)|}$ and the moving does not change the probability of Agent n accepting the posted price $p_n \leqslant v_n^{|\text{supp}(\tilde{D}_n)|}$ in the auctioneer's mind. The highest element in $\tilde{D}_n$ decreases after the moving.

(3) $\tilde{v}_n^{|\text{supp}(\tilde{D}_n)|-1} \leqslant v_n^{|\text{supp}(D_n)|}$. In this case, Agent n has no chance to take the item for price $\tilde{v}_n^{|\text{supp}(\tilde{D}_n)|}$. After moving the probability mass at $\tilde{v}_n^{|\text{supp}(\tilde{D}_n)|}$ to $v_n^{|\text{supp}(D_n)|}$, her expected utility for any possible order π and posted prices $\boldsymbol{p}$ would not change, since her utility is 0 when $p_n = \tilde{v}_n^{|\text{supp}(\tilde{D}_n)|}$ and the moving does not change the probability of Agent n accepting the posted price $p_n \leqslant v_n^{|\text{supp}(D_n)|}$ in the auctioneer's mind. The new reported distribution has no element greater than $v_n^{|\text{supp}(\tilde{D}_n)|}$.

When the first or third case occurs, we can obtain an optimal distribution satisfying the lemma. When the second case occurs, we can repeatedly maintain the reported distribution until the first or the third case occurs. This concludes the proof.

3.1.2 *2-Agent Case*

Consider a scenario with only 2 agents, Agent 1 with level-0 sophistication and Agent 2 with level-1 sophistication. To make the analysis easier, we could assume the posted prices $p_1 \neq p_2$ as Agent 2 could add a small enough ϵ_{v} to the support $\text{supp}(\tilde{D}_{2,1})$ with little loss of her expected utility. Give posted prices $\boldsymbol{p} = (p_1, p_2)$, when $p_1 > p_2$, the expected revenue in the auctioneer's mind is

$$p_1 d_1(p_1) + (1 - d_1(p_1)) p_2 \tilde{d}_{2,1}(p_2),$$

and the expected utility of Agent 2 is

$$(1 - d_1(p_1)) \sum_{v: v \in \operatorname{supp}(D_2), v \geqslant p_2} (v - p_2) c_2(v);$$

when $p_1 < p_2$, the expected revenue in the auctioneer's mind is

$$p_2 \tilde{d}_{2,1}(p_2) + (1 - \tilde{d}_{2,1}(p_2)) p_1 d_1(p_1),$$

and the expected utility of Agent 2 is

$$\sum_{v: v \in \operatorname{supp}(D_2), v \geqslant p_2} (v - p_2) c_2(v).$$

Consider the optimal posted prices $\boldsymbol{p}$. When $p_1 \geqslant p_2$, assume that $d_1(p_1) < 1$ or Agent 1 will definitely take the item and Agent 2 obtains utility 0. For arbitrary p_1' that $d_1(p_1) \neq d_1(p_1')$, we have

$$\begin{aligned} &p_1 d_1(p_1) + (1 - d_1(p_1)) p_2 \tilde{d}_{2,1}(p_2) \geqslant \\ &p_1' d_1(p_1') + (1 - d_1(p_1')) \max_{p_2' \leqslant p_1'} p_2' \tilde{d}_{2,1}(p_2'). \end{aligned} \tag{1}$$

When $p_1' > p_1$, Inequality (1) turns to

$$\begin{aligned} p_1 d_1(p_1) &- p_1' d_1(p_1') \geqslant \\ &(1 - d_1(p_1')) \max_{p_2' \leqslant p_1'} p_2' \tilde{d}_{2,1}(p_2') - \\ &(1 - d_1(p_1)) p_2 \tilde{d}_{2,1}(p_2) \Rightarrow \\ &(p_2 < p_1 < p_1') \geqslant \\ &(d_1(p_1) - d_1(p_1')) p_2 \tilde{d}_{2,1}(p_2). \end{aligned}$$

As $p_1' > p_1$, we have $d_1(p_1) - d_1(p_1') > 0$. Then we can make left side of the inequality to $p_2 \tilde{d}_{2,1}(p_2)$ as

$$p_2 \tilde{d}_{2,1}(p_2) \leqslant p_1 - (p_1' - p_1) \frac{d_1(p_1')}{d_1(p_1) - d_1(p_1')}. \tag{2}$$

When $p_2 < p_1' < p_1$, Inequality (1) turns to

$$\begin{aligned} &p_1 d_1(p_1) - p_1' d_1(p_1') \geqslant \\ &(1 - d_1(p_1')) \max_{p_2' \leqslant p_1'} p_2' \tilde{d}_{2,1}(p_2') - (1 - d_1(p_1)) p_2 \tilde{d}_{2,1}(p_2) \Rightarrow \\ &(\{p_1, p_2\} \text{ is the optimal posted prices}) = \\ &(d_1(p_1) - d_1(p_1')) p_2 \tilde{d}_{2,1}(p_2). \end{aligned}$$

We can make left side of the inequality to $p_2 \tilde{d}_{2,1}(p_2)$ as

$$p_2 \tilde{d}_{2,1}(p_2) \geqslant p_1 - (p_1 - p_1') \frac{d_1(p_1')}{d_1(p_1') - d_1(p_1)}. \tag{3}$$

When $p_1' \leqslant p_2 < p_1$, Inequality (1) turns to

$$\begin{aligned} &p_1 d_1(p_1) - p_1' d_1(p_1') \geqslant \\ &(1 - d_1(p_1')) \max_{p_2' \leqslant p_1'} p_2' \tilde{d}_{2,1}(p_2') - (1 - d_1(p_1)) p_2 \tilde{d}_{2,1}(p_2) \geqslant \\ &-(1 - d_1(p_1)) p_2 \tilde{d}_{2,1}(p_2). \end{aligned}$$

We can make left side of the inequality to $p_2 \tilde{d}_{2,1}(p_2)$ as

$$p_2 \tilde{d}_{2,1}(p_2) \geqslant p_1 - \frac{p_1 - p_1' d_1(p_1')}{1 - d_1(p_1)}. \tag{4}$$

Inequalities (2), (3), and (4) describe the constraints for the optimal posted prices $\boldsymbol{p}$ when $p_1 > p_2$.

When $p_1 < p_2$, the expected revenue of the auctioneer can be rewritten as

$$p_2 \tilde{d}_{2,1}(p_2) + (1 - \tilde{d}_{2,1}(p_2)) \max_{p_1' \leqslant p_2} p_1 d_1(p_1'),$$

due to the optimality of $\boldsymbol{p}$. Notice that D_1 is a fixed distribution known by Agent 2 so that she can calculate the value of $\max_{p_1' \leqslant p_2} p_1 d_1(p_1')$ when reporting $\tilde{D}_{2,1}$.

Suppose the optimal posted prices of auctioneer with constraint $p_1 < p_2$ are $\boldsymbol{p}' = (p_1', p_2')$ and those with constraint $p_1 > p_2$ are $\boldsymbol{p}'' = (p_1'', p_2'')$. The difference of these two posted price vectors is

$$\begin{aligned} &p_1' d_1(p_1') + (1 - d_1(p_1')) p_2' \tilde{d}_{2,1}(p_2') - \\ &p_2'' \tilde{d}_{2,1}(p_2'') - (1 - \tilde{d}_{2,1}(p_2'')) p_1'' d_1(p_1''). \end{aligned} \tag{5}$$

The sign of Formula (5) depends on which posted price vector is the global optimal one. Especially, if $p_2' = p_2''$, Formula (5) becomes

$$\begin{aligned} &p_1' d_1(p_1') - d_1(p_1') p_2' \tilde{d}_{2,1}(p_2') - \\ &p_1'' d_1(p_1' + \tilde{d}_{2,1}(p_2') p_1'' d_1(p_1''). \end{aligned} \tag{6}$$

When p_1', p_1'', and $p_2' \tilde{d}_{2,1}(p_2')$ are all fixed, Formula (6) is monotonic increasing with $\tilde{d}_{2,1}(p_2')$.

Now we consider how to compute the optimal reported distribution $\tilde{D}_{2,1}$. By Lemma 2, we have d_1 is in $\operatorname{supp}(D_1)$. When $p_1 > p_2$, the optimal p_2 for Agent 2 is affected by:

① The order of p_2 and all elements in $\operatorname{supp}(D_1)$ decides how many inequalities in the form of Inequality (1) should be considered; ② The order of p_2 and all elements in $\operatorname{supp}(D_2)$ decides how much utility Agent 2 obtains.

So if Agent 2 wants to mislead the auctioneer to post prices $p_1 > p_2$, she should: ① Enumerate all possible prices p_1; ② For each p_1, enumerate all possible intervals containing p_2 based on $\operatorname{supp}(D_1)$ and $\operatorname{supp}(D_2)$; ③ Calculate feasible regions of $p_2 \tilde{d}_2(p_2)$ by Inequalities (2), (3), (4), and Formula (5). Notice that if $\tilde{D}_{2,1}$ has non-zero support other than p_2, Inequalities (2) and (4) are loose and more inequalities generated by Formula

(5) should be considered. So we only need to consider the case that the reported distribution $\tilde{D}_{1,2}$ has at most 1 non-zero element p_2 in its support. To avoid some complex discussions, we could assume $p_2 \notin \mathrm{supp}(D_1)$ or a small enough positive ϵ_v could be added into p_2. The optimal expected utility of Agent 2 can be obtained by Algorithm 1.

Algorithm 1 Optimal expected utility of level-1 Agent 2
Input true value distributions D_1, D_2
Output optimal expected utility of Agent 2 optRev

let optRev $\leftarrow 0$
let $\mathrm{supp}_{\mathrm{all}} \leftarrow \mathrm{supp}(D_1) \cup \mathrm{supp}(D_2)$
for $p_1 \in \mathrm{supp}(D_1)$ do ▷ when $p_1 < p_2$
  for p_2 in which interval of $\mathrm{supp}_{\mathrm{all}}$ do
    obtain the feasible interval of $p_2\tilde{d}_2(p_2)$ by Inequalities (2), (3), (4), and positive Formula (6)
    if the feasible interval is not empty then
      find a minimal feasible p_2
      optRev $= \max\{\text{optRev}, (1 - d_1(p_1)) \sum\limits_{v:v\in\mathrm{supp}(D_2),v\geqslant p_2} (v-p_2)c_2(v)\}$
    end if
  end for
end for
for p_2 in which interval of $\mathrm{supp}_{\mathrm{all}}$ do ▷ when $p_1 > p_2$
  the feasible interval of $p_2\tilde{d}_2(p_2)$ by negative Formula (6)
  if the feasible interval is not empty then
    find a minimal feasible p_2
    optRev $= \max\{\text{optRev}, \sum\limits_{v:v\in\mathrm{supp}(D_2),v\geqslant p_2} (v-p_2)c_2(v)\}$
  end if
end for
return optRev

We can show that Algorithm 1 gives an optimal expected utility of Agent 2.

Lemma 4 Algorithm 1 outputs the expected utility of level-1 Agent 2.

Proof Considering Lemma 3 and the previous analysis, p_2 is the only possible non-zero support of $\tilde{D}_{2,1}$ and less than the highest element of $\mathrm{supp}(D_2)$.

When $p_1 > p_2$, Inequalities (2), (3), (4) and positive Formula (6) tell the exact feasible interval of $p_2\tilde{d}_2(p_2)$ after enumerating the interval that contains p_2 as Inequalities (2) and (4) are tight when p_2 is the only possible non-zero support. Then given a fixed p_1, the minimal possible p_2 can be obtained, which gives the maximal possible utility of Agent 2 in this case. After enumerating all p_1, the first part of Algorithm 1 tells the optimal utility if $p_1 > p_2$.

When $p_1 < p_2$, negative Formula (6) tells the exact feasible interval of $p_2\tilde{d}_2(p_2)$ after enumerating the interval that contains p_2 as p_1 is fixed for a fixed interval of p_2. Then we can obtain the minimal possible p_2, which gives the maximal possible utility of Agent 2 in this case. Thus, the second part of Algorithm 1 tells the optimal utility if $p_1 < p_2$.

Overall, the algorithm outputs the optimal utility of level-1 Agent 2. □

Remark The bottleneck of the computation of Algorithm 1 is in the part of the $p_1 > p_2$ case. The enumeration of p_1 is about $O(|\mathrm{supp}(D_1)|)$, the enumeration of the interval that contains p_2 is about $O(|\mathrm{supp}(D_1) \cup \mathrm{supp}(D_2)|)$, and the number of inequalities is about $O(|\mathrm{supp}(D_1)|)$. So the total time complexity for Algorithm 1 is $O(|\mathrm{supp}(D_1)|^2 \cdot |\mathrm{supp}(D_1) \cup \mathrm{supp}(D_2)|)$. However, if we enumerate the interval that contains p_2 in a monotonic order, either in an increasing order or a decreasing order, only $O(1)$ inequalities change. Then we might apply some data structures to improve Algorithm 1 and we believe the time complexity of the improved algorithm could be $O(|\mathrm{supp}(D_1)| \cdot |\mathrm{supp}(D_1) \cup \mathrm{supp}(D_2)| \cdot \log|\mathrm{supp}(D_1)|)$.

3.1.3 *Enough i.i.d. Agents*

Let $D = D_1 = \cdots = D_n$ denote the value distribution of all agents. Consider the highest support v of D with probability mass $c(v)$. Without loss of generality, we assume Agent n has level-1 sophistication. By Lemma 2 and Lemma 3, $p_n \leqslant v$. If the auctioneer posts $p_n < v$ to Agent n and v to other agents, the probability that the item is not taken before Agent n is $(1 - c(v))^{n-1}$ and the revenue of auctioneer is at least $[1-(1-c(v))^{n-1}]v$. When n is large enough, the probability is almost 0 and the revenue is almost v. If the posted price p_n is v, the utility of Agent n is 0 since v is the highest support of D_n. Overall, Agent n could have mere utility in this case.

If the level-1 agent waives the right to misreport due to the mere utility, all agents with higher sophistication will also waive the right. So all agents will tell the truth. However, the agent might declare that her value is almost always v or a little lower than v for a chance to take the item, and the winner's curse occurs.

Remark Actually, the analysis is still hold for some non-i.i.d. cases, e.g., the probability mass at v larger than some constant ϵ_c for the value distributions of enough agents.

3.1.4 *General Case*

Suppose there are n agents with true value distributions $\{D_i\}_{i=1}^n$. Without loss of generality, we assume that Agent n has level-1 sophistication. The reported distribution $\tilde{D}_{n,1}$ maximizes

$$\left[\prod_{i=1}^{\hat{\pi}^{-1}(n)-1} (1 - d_{\hat{\pi}(i)}(\hat{p}_{\hat{\pi}(i)}))\right] \times$$

$$\left[\sum_{v:v\in\mathrm{supp}(D_n),v\geqslant\hat{p}_n}(v-\hat{p}_n)c_n(v)\right],\tag{7}$$

with auctioneer's corresponding optimal order $\hat{\pi}$ and optimal posted prices $\hat{\boldsymbol{p}}=(\hat{p}_1,\hat{p}_2,\cdots,\hat{p}_n)$.

By Lemma 1 and Lemma 2, we can enumerate all other possible posted price vectors and obtain a series of inequalities to ensure the optimality of $\{\hat{\pi},\hat{\boldsymbol{p}}\}$. As the same analysis in the 2-agent case, if $\tilde{D}_{n,1}$ has non-zero support other than $\hat{p}_n$, the number of inequality constraints increases. Thus, $\tilde{D}_{n,1}$ has at most 1 non-zero element $\hat{p}_n$ in its support. After the enumeration of a interval that contains $\hat{p}_n$, the inequalities characterize the relationship between $\hat{p}_n$ and $\tilde{d}_{n,1}(\hat{p}_n)$ and we can obtain a minimal feasible $\hat{p}_n$. Algorithm 2 follows the above analysis. Notice that the overall enumeration in Algorithm 2 is $O\left(\left[\prod_{i=1}^{n-1}|\mathrm{supp}(D_i)|\right]^2\left[\sum_{i=1}^{n}|\mathrm{supp}(D_i)|\right]\right)$, which is exponential of the number of agents n.

Algorithm 2 Optimal expected utility of level-1 Agent n
Input true value distributions $\{D_i\}_{i=1}^{n-n}$
Output optimal expected utility optRev

```
let optRev ← 0
let supp_all ← ∪_{i=1}^{n} supp(D_i)
for p̂_1 ∈ supp(D_1) do
        ▷ enumerating all possible p̂_1, p̂_2, ···, p̂_{n−1}
  ···
  for p̂_{n−1} ∈ supp(D(n − 1) do
    for p̂_n in which interval of supp_all do
      enumerate all other possible price vectors and
      obtain inequalities about p_n d̃_n(p̂_n) and d̃_n(p̂_n)
      if the feasible interval is not empty then
        find a minimal feasible p̂_n
        calculate the utility by Formula (7) and update optRev
      end if
    end for
  end for
  ···
end for
return optRev
```

3.2 Level-k Model Analysis

In Subsection 3.1, we analyze the behavior of level-1 agents. We can apply a similar analysis to the behaviors of agents predicted by the level-k model with higher sophistication levels. We will first analyze the behavior of agents with level-k sophistication. Then we will provide a concrete example to show that the behaviors of a series of agents with different sophistication levels could be ill-defined.

3.2.1 *Level-k Agent*

By Algorithm 2, the reported distribution $\tilde{D}_{n,1}$ of level-1 Agent n has at most 1 non-zero element in its support. However, multiple possible solutions could exist for the reported distribution $\tilde{D}_{n,1}$. To simplify the analysis, we assume that the highest support is the minimal possible one with a maximum possible probability mass. Consider a level-k ($k>1$) Agent n and all $\{\tilde{D}_{j,k-1}\}_{j=1}^{n-1}$ with at most 1 non-zero element in each of their support. In Agent n's mind, by Lemma 2, the auctioneer can only offer an Agent i ($1\leqslant i<n$) the highest support of $\tilde{D}_{i,k-1}$. As the posted prices for all other agents are fixed, the order of all other agents is also fixed by Lemma 1. With a similar analysis of the number of non-zero elements in the support, the support of $\tilde{D}_{n,k}$ also has at most 1 non-zero element. Thus, Agent n needs to decide a $\hat{p}_n$ maximizing

$$\left[\prod_{i=1}^{\hat{\pi}^{-1}(n)-1}(1-d_{\hat{\pi}(i)}(\hat{p}_{\hat{\pi}(i)}))\right]\times \left[\sum_{v:v\in\mathrm{supp}(D_n),v\geqslant\hat{p}_n}(v-\hat{p}_n)c_n(v)\right],\tag{8}$$

where $\hat{p}_i$ is the highest support of $\tilde{D}_{i,k-1}$ for an Agent i ($1\leqslant i<n$). Since there is only one possible order and posted prices after the enumeration of $\hat{p}_n$, the exponential enumeration in Algorithm 2 is no more needed. We can obtain Algorithm 3 by eliminating the exponential enumeration in Algorithm 2.

Algorithm 3 Optimal expected utility of level-k Agent n predicted by the level-k model
Input true value distributions $\{D_i\}_{i=1}^{n}$, agents' reported distributions $\{\tilde{D}_{i,k-1}\}_{i=1}^{n-1}$
Output optimal expected utility optRev

```
let optRev ← 0
let supp_all ← (∪_{i=1}^{n−1} supp(D̃_i)) ∪ supp(D_n)
for i = 1 to n − 1 do let p̂_i be the maximal support of D̃_{i,k−1}
end for
for p̂_n in which interval of supp_all do
  obtain π̂ by Lemma 1
  calculate the utility by Formula (8) and update optRev
end for
return optRev
```

As Algorithm 3 shows, the only enumeration is for the interval that contains $\hat{p}_n$. Thus, Algorithm 3 runs in polynomial time. Since there is no inequality constraint, the optimal $\hat{p}_n$ equals $\hat{p}_{\hat{\pi}(\hat{\pi}^{-1}(n)+1)}+\epsilon_{\mathrm{v}}$, where ϵ_{v} could be an arbitrarily small positive number. This phenomenon indicates that the strategy of the level-k model could be ill-defined.

3.2.2 *Ill-Defined Example for Level-k Agent*

As Example 2 shows, the level-k model is ill-defined in continuous scenarios. Even in a discrete scenario, the highest support reported by agents with level-k sophistication could be ϵ larger than that reported by agents with level-$(k-1)$ sophistication, where ϵ equals the minimum unit μ of the scenario. Similar phenomena could be seen in Rasooly's work[29] for all-pay auctions and first-price auctions with cancel probability.

Example 2 Consider a sequential posted price schema with 2 agents. Both agents have the same value distribution $D = D_1 = D_2$ with probability mass 0.1 at 100 and probability mass 0.9 at 20. We first analyze the reported distributions $\tilde{D}_{\cdot,1} = \tilde{D}_{1,1} = \tilde{D}_{2,1}$ when both agents have level-1 sophistication.

Let the highest support of $\tilde{D}_{\cdot,1}$ be p'_1 with probability mass c'_1.

(1) When $p'_1 < 20$ and the optimal posted prices $\boldsymbol{p} = (100, p'_1)$, we have

$$0.1 \times 100 + 0.9 \times p'_1 c'_1 > 20.$$

The solution is $p'_1 c'_1 > \frac{100}{9}$. Thus the minimal possible p'_1 is approximately equal to $\frac{100}{9}$ with $c'_1 = 1$. In this case, the utility of a level-1 agent in her mind is approximately

$$0.1 \times 100 + 0.9 \times 20 - \frac{100}{9} = \frac{152}{9} \approx 16.89.$$

(2) When $p'_1 \geqslant 20$ and the optimal posted prices $\boldsymbol{p} = (100, p'_1)$, we have

$$0.1 \times 100 + 0.9 \times p'_1 c'_1 > p'_1 c'_1 + (1 - c'_1) \times 20 \Rightarrow$$
$$100 < (200 - p'_1) c'_1.$$

The minimal possible $p'_1 = 20$ with $c'_1 \in (\frac{5}{9}, 1]$. In this case, the utility of a level-1 agent in her mind is

$$0.1 \times 100 + 0.9 \times 20 - 20 = 8.$$

(3) When $p'_1 > 20$ and the optimal posted prices $\boldsymbol{p} = (p'_1, 20)$, we have

$$p'_1 c'_1 + (1 - c'_1) \times 20 > 0.1 \times 100 + 0.9 \times p'_1 c'_1 \Rightarrow$$
$$100 > (200 - p'_1) c'_1.$$

The minimal possible p'_1 is approximately equal to 20 with $c'_1 \in (0, \frac{5}{9})$. In this case, the utility of a level-1 agent in her mind is approximately

$$0.1 \times 100 + 0.9 \times 20 - 20 = 8.$$

According to the above discussion, the optimal reported distribution $\tilde{D}_{\cdot,1}$ of a level-1 agent is with $p'_1 = \frac{100}{9} + \epsilon_{\mathrm{v}}$ and $c'_1 = 1$, where ϵ_{v} could be an arbitrarily small positive number. Since $p'_1 < 20$, the item will be taken by a level-1 agent if the item is left when she comes to the market. The strategy of a level-2 agent is quite simple. She should trade before the level-1 agent with a minimal possible price $p'_2 = p'_1 + \epsilon_{\mathrm{v}}$ and probability mass $c'_2 = 1$. Similarly, for a level-k $(k > 2)$ agent, $p'_k = p'_{k-1} + \epsilon_{\mathrm{v}}$ and $c'_k = 1$. For the discrete scenarios with a minimum unit of the value, ϵ_{v} could equal the minimum unit. Then we could obtain a series of pathological strategies of agents predicted by the level-k model when the minimum unit is small enough.

Notice that the agents are i.i.d. in Example 2. So in many restricted or extended sequential posted pricing scenarios, e.g., constrained sequential posted pricing scenarios described by Xiao et al.[5], the agents predicted by the level-k model might be ill-defined or have a series of pathological strategies.

3.3 Cognitive Hierarchy Model Analysis

In the previous subsection, we analyze the agents predicted by the level-k model. In this subsection, we turn to the agents predicted by the cognitive hierarchy model. As only level-0 agents in a level-1 agent's mind, the behavior of a level-1 agent has been analyzed in Subsection 3.1. We can start with the agents with level-2 sophistication.

3.3.1 *Cognitive Hierarchy Agent*

Recall that a level-k agent predicted by the cognitive hierarchy model believes that other agents follow a proportion of different sophistication levels, from level-0 to level-$(k-1)$. Suppose that there are n agents with true value distribution $\{D_i\}_{i=1}^n$. In general, we assume Agent n has level-k sophistication. Her belief of different sophistication levels is $\{g_{k,l}\}_{l=0}^{k-1}$ and the reported distribution is $\tilde{D}_{i,l}$ for an Agent i $(1 \leqslant i < n)$ with level-l $(0 \leqslant l < k)$ sophistication with $\tilde{D}_{i,0} = Di$.

For the individual sophistication case, the optimal reported distribution $\tilde{D}_{n,k}$ should maximize

$$\mathbb{E}_{D'_i \sim \{\tilde{D}_{i,l}\}_{l=0}^{k-1}, \{g_{k,l}\}_{l=0}^{k-1}} \left\{ \left[\prod_{i=1}^{\hat{\pi}^{-1}(n)-1} (1 - d_{\hat{\pi}(i)}(\hat{p}_{\hat{\pi}(i)})) \right] \times \left[\sum_{v: v \in \mathrm{supp}(D_n), v \geqslant \hat{p}_n} (v - \hat{p}_n) c_n(v) \right] \right\},$$

where D'_i is drawn independently from $\{\tilde{D}_{i,l}\}_{l=0}^{k-1}$ with probability $g_{k,l}$ for the distribution $\tilde{D}_{i,l}$, and $\{\hat{\pi}, \hat{\boldsymbol{p}}\}$ are the auctioneer's optimal strategy when Agent n reports $\tilde{D}_{n,k}$ as well as other agents reporting $\{D'_i\}_{i=1}^{n-1}$. Notice that for different $\{D'_i\}_{i=1}^{n-1}$, the posted price $\hat{p}_n$ can have different values. Thus, the maximal number of possible non-zero elements in support of $\tilde{D}_{n,k}$ could be k^{n-1}. We can obtain $\tilde{D}_{n,k}$ by solving inequality constraints as Algorithm 2 with the following steps: ① Enumerate all possible $\{D'_i\}_{i=1}^{n-1}$; ② For each $\{D'_i\}_{i=1}^{n}$, enumerate a posted price vector $\{\hat{p}_i^{\{D'_i\}_{i=1}^n}\}_{i=1}^{n-1}$ as Algorithm 2; ③ Enumerate the order of all posted prices $\hat{p}_n$ and obtain all inequality constraints. Algorithm 4 shows a possible way to calculate $\tilde{D}_{n,k}$.

Algorithm 4 Optimal expected utility of level-k agent n predicted by the individual-sophistication cognitive hierarchy model

Input true value distribution D_n, agents' reported distributions $\{\{\tilde{D}_{i,l}\}_{l=0}^{k}\}_{i=1}^{n-1}$

Output optimal expected utility optRev

let optRev $\leftarrow 0$
let $\mathrm{supp}_{\mathrm{all}} \leftarrow \left(\cup_{i=1}^{n-1} \cup_{l=0}^{k-1} \mathrm{supp}(\tilde{D}_{i,l})\right) \cup \mathrm{supp}(D_n)$
for each possible $\{\hat{p}_1^{\{D'_i\}_{i=1}^{n-1}}, \hat{p}_2^{\{D'_i\}_{i=1}^{n-1}}, \cdots, \hat{p}_{n-1}^{\{D'_i\}_{i=1}^{n-1}}\}$ of all $\{D'_i\}_{i=1}^{n-1}$ do
  for the intervals of $\mathrm{supp}_{\mathrm{all}}$ containing each $\hat{p}_n^{\{D'_i\}_{i=1}^{n-1}}$ and the order of all $\hat{p}_n^{\{D'_i\}_{i=1}^{n-1}}$ do
    obtain a series of inequality constraints about $\hat{p}_n^{\{D'_i\}_{i=1}^{n-1}}$ and $\tilde{c}_{n,k}(\hat{p}_n^{\{D'_i\}_{i=1}^{n-1}})\hat{p}_n^{\{D'_i\}_{i=1}^{n-1}}$
    calculate the optimal utility under inequality constraints and update optRev
  end for
end for
return optRev

For the global sophistication case, the optimal reported distribution $\tilde{D}_{n,k}$ should maximize

$$\sum_{l=0}^{k-1}\left\{g_{k,l}\left[\prod_{i=1}^{\hat{\pi}^{-1}(n)-1}(1-d_{\hat{\pi}(i)}(\hat{p}_{\hat{\pi}(i)}))\right]\times \left[\sum_{v:v\in\mathrm{supp}(D_n),v\geqslant\hat{p}_n}(v-\hat{p}_n)c_n(v)\right]\right\},$$

where $(\hat{\pi}, \hat{\boldsymbol{p}} = (\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_n))$ are the auctioneer's optimal strategy when Agent n reports $\tilde{D}_{n,k}$ as well as other agents reporting $\{\tilde{D}_{i,l}\}_{i=1}^{n-1}$. Notice that for different l, the posted price $\hat{p}_n$ can have different values. Thus, the maximal number of possible non-zero elements in support of $\tilde{D}_{n,k}$ could be k. We can obtain $\tilde{D}_{n,k}$ by solving inequality constraints as Algorithm 2 with the following steps: ① For each l, enumerate a posted price vector $\{\hat{p}_i^l\}_{i=1}^{n-1}$ as Algorithm 2; ② Enumerate the order of all posted prices $\hat{p}_n^l$ and obtain inequality constraints. Algorithm 5 shows a possible way to calculate $\tilde{D}_{n,k}$.

Algorithm 5 Optimal expected utility of the level-k Agent n predicted by the individual-sophistication cognitive hierarchy model

Input True value distribution D_n, agents' reported distributions $\{\{\tilde{D}_{i,l}\}_{l=0}^{k}\}_{i=1}^{n-1}$

Output optimal expected utility optRev

let optRev $\leftarrow 0$
let $\mathrm{supp}_{\mathrm{all}} \leftarrow \left(\cup_{i=1}^{n-1} \cup_{l=0}^{k-1} \mathrm{supp}(\tilde{D}_{i,l})\right) \cup \mathrm{supp}(D_n)$
for each possible $\{\hat{p}_1^l, \hat{p}_2^l, \cdots, \hat{p}_{n-1}^l\}$ of all l do
  for the intervals of $\mathrm{supp}_{\mathrm{all}}$ containing each $\hat{p}_n^l$ and the order of all $\hat{p}_n^l$ do
    obtain a series of inequality constraints about $\hat{p}_n^l$ and $\tilde{c}_{n,k}(p_n^l)\hat{p}_n^l$
    calculate the optimal utility under inequality constraints and update optRev
  end for
end for
return optRev

3.3.2 *Ill-defined Example for Cognitive Hierarchy Agent*

As Example 3 shows, the cognitive hierarchy model with either sophistication case is ill-defined in continuous scenarios.

Example 3 Consider a similar scenario in Example 2 that 2 agents have the same value distribution $D = D_1 = D_2$ with probability mass 0.1 at 100 and probability mass 0.9 at 20. For the 2-agent case, the individual and global sophistication cases are the same ones.

As the analysis in Example 2, the optimal reported distribution of agents with level-1 sophistication is with probability mass 1 at $\frac{100}{9} + \epsilon_{\mathrm{v}}$ for an arbitrarily small positive number ϵ_{v}. Since ϵ_{v} is an arbitrarily small positive number, the distribution with probability mass 1 at $\frac{100}{9} + 2\epsilon_{\mathrm{v}}$ is a near-optimal strategy for level-1 agents. Moreover, the agent reported the new distribution could take the item before a level-1 agent. So the optimal reported distribution of agents with level-1 sophistication is probability mass 1 at $\frac{100}{9} + 2\epsilon_{\mathrm{v}}$ if $g_{2,1}$. We can apply the same analysis to each sophistication level. Thus, the reported distribution of a level-k ($k > 0$) is with probability mass 1 at $\frac{100}{9} + k\epsilon_{\mathrm{v}}$.

It should be pointed out that for a discrete scenario, the $\{g_{k,l}\}$ parameters could ease the pathological phenomenon. For example, all $g_{k,0}$ are $1 - 10^{-5}$, and the minimum units of the values in the discrete scenario are large enough, e.g., 1, so that all level-k agents only need to care about the non-strategic agents.

4 Experiments

4.1 Experimental Setup

4.1.1 *Environment*

Like Example 2 and Example 3, we consider discrete sequential posted pricing scenarios with n ($n \geqslant 2$) agents. In the scenarios, the minimum unit of the values is μ, and the true value distribution D_i of each Agent i is with probability mass 0.9 at 100 and 0.1 at 20. In our experiments, μ could be 1, 0.1, 0.01. We assume that if the auctioneer posts the same price to more than one agent, the order of these agents is randomly selected.

4.1.2 *Belief in Cognitive Hierarchy Model*

For the cognitive hierarchy model, we assume that the sophistication is Poisson distributed following the setting in the cognitive hierarchy paper[15]. For a level-k agent, she believes that an Agent i has a level-l sophistication with probability:

$$g_{k,l} = \frac{\tau^l/l!}{\sum_{i=0}^{k-1} \tau^i/i!},$$

where τ is parameter of the Poisson distribution and we set $\tau = 1.5$ as the cognitive hierarchy paper suggests[15].

4.1.3 *Risk Tolerance*

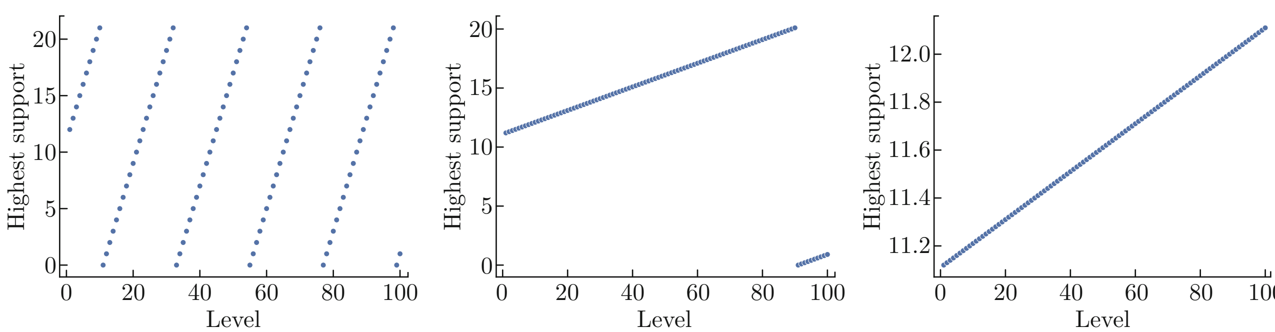
In the previous sections, we assume that agents are risk-neutral. However, an agent with bounded rationality could be risk-averse or risk-seeking. For an Agent i with risk-tolerant degree α, her utility is $u_i = (p_i - v_i)^\alpha$ when she takes the item with her value v_i and a posted price $p_i \leqslant v_i$.

4.2 Main Results and Analysis

4.2.1 *Level-k Model Results*

For the level-k model, we have predicted the existence of pathological strategies by the analysis in Subsection 3.2. We first validate the pathological phenomenon with minimum unit $\mu = 1, 0.1, 0.01$ and analyze the impact of the precision on the behaviors of agents with different sophistication levels.

As Fig. 1 shows, the pathological and cycling phenomena occur. In Fig. 1(a), the highest support increases with 1 per sophistication level. When the agent has a level-10 sophistication, the highest support of her reported distribution is 21, which is the largest possible highest support. When the agent has a level-11 sophistication, the highest support of her reported distribution drops to 0, and the highest support re-begins to increase with 1 per sophistication level. The cycling phenomenon occurs per 22 sophistication levels. In Fig. 1(b), the highest support increases with 0.1 per sophistication level. When the agent has a level-90 sophistication, the highest support of her reported distribution is 20.1, which is the largest possible highest support. When the agent has a level-91 sophistication, the highest support of her reported distribution drops to 0, and the highest support re-begins to increase with 0.1 per sophistication level. We can predict that the cycling phenomenon occurs per 202 sophistication levels. In Fig. 1(c), the highest support increases with 0.01 per sophistication level. With the previous two figures, we can predict that the highest support of her reported distribution is 20.01 when the agent has a level-890 sophistication, and it drops to 0 when the agent has a level-891 sophistication. After the dropping, the highest support will re-begin to increase with 0.01 per sophistication level. We can predict that the cycling phenomenon occurs per 2002 sophistication levels.

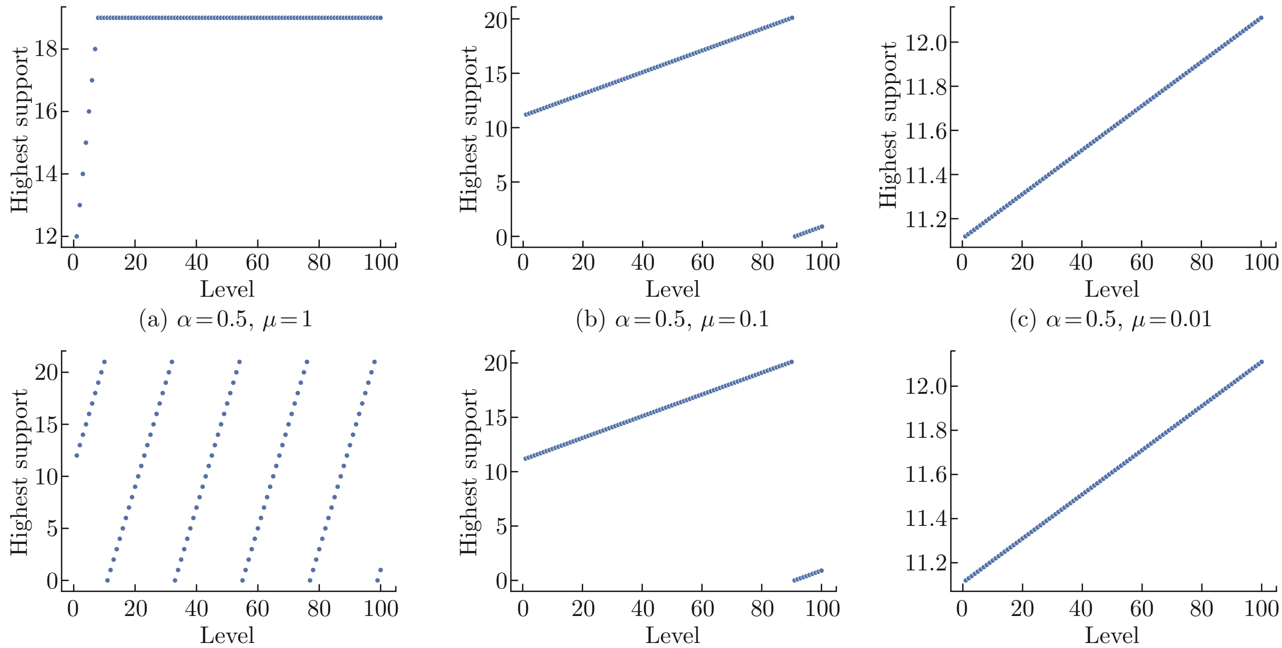


(a) $\mu = 1$ (b) $\mu = 0.1$ (c) $\mu = 0.01$

Fig. 1 Highest support of risk-neutral agents predicted by the level-k model with $\mu = 1, 0.1, 0.01$. All figures show the pathological phenomena of agents' strategies, and (b) and (c) show the cycling phenomena

Then we consider the agents with different risk tolerances. Figure 2 shows that risk aversion could help the convergence. Different from the result in Fig. 1(a), Fig. 2(a) shows that the strategies of sophisticated risk-averse agents converge after sophistication level-8. However, for $\mu = 0.1$, 0.01, the strategies remain to form cycles with each 22 levels, which indicates that the precision of μ could prevent the convergence of the level-k model. The following experiments will provide more pieces of evidence for the impact of α.

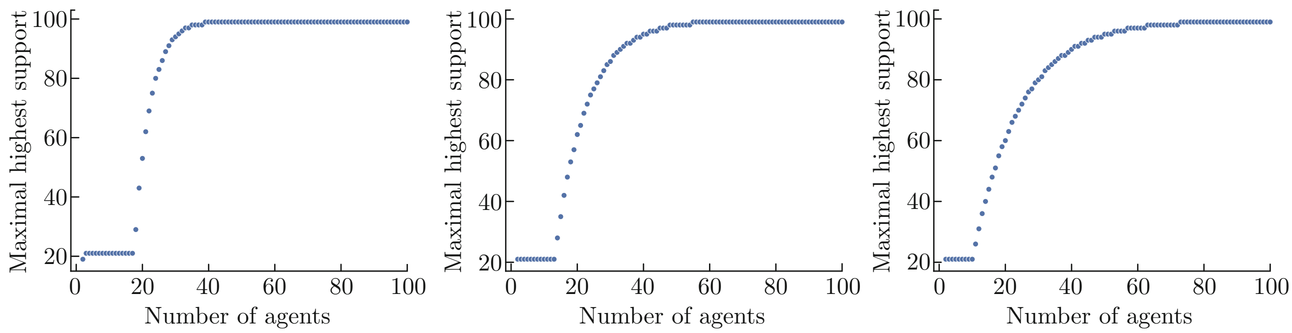
It still remains a problem whether the strategies converged or formed cycles with an increasing number of agents. In Subsection 3.1.3, we show that the utility of a sophisticated agent could be reduced with enough agents. We are curious about the relationship between convergence and the number of agents. Now we set up an experiment with the different number of agents from 2 to 100 to analyze the strategies of the agents with minimum unit $u = 1$. Figure 3 shows the relationship between the maximal highest support and the number



(a) $\alpha = 0.5$, $\mu = 1$ (b) $\alpha = 0.5$, $\mu = 0.1$ (c) $\alpha = 0.5$, $\mu = 0.01$

(d) $\alpha = 1.5$, $\mu = 1$ (e) $\alpha = 1.5$, $\mu = 0.1$ (f) $\alpha = 1.5$, $\mu = 0.01$

Fig. 2 Highest support of risk-averse ($\alpha = 0.5$) and risk-seeking ($\alpha = 1.5$) agents predicted by the level-k model with $\mu = 1, 0.1, 0.01$. Only (a) shows a different result from the risk-neutral case

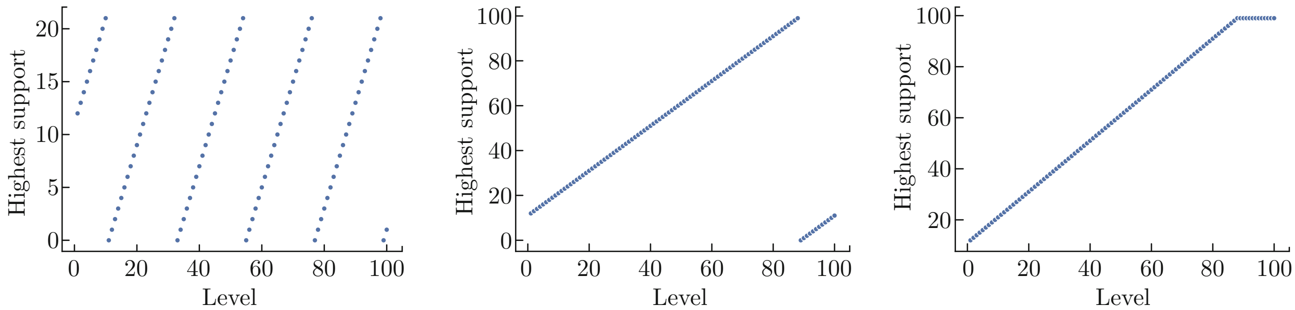


(a) $\alpha = 0.5$ (b) $\alpha = 1.0$ (c) $\alpha = 1.5$

Fig. 3 Maximal highest support of risk-averse ($\alpha = 0.5$), risk-neutral ($\alpha = 1$), and risk-seeking ($\alpha = 1.5$) agents predicted by the level-k model with the number of agents from 2 to 100. The minimum unit u is set to 1. In all three figures, the maximal highest support converges to 99

of agents with different risk tolerances. In Fig. 3(a), the maximal highest support converges to 99 with 39 risk-averse agents. In Fig. 3(b), the maximal highest support converges to 99 with 55 risk-neutral agents. In Fig. 3(c), the maximal highest support converges to 99 with 73 risk-seeking agents. All these three figures show that the maximal highest support increases with the increment of the number of agents and finally all converge to 99. Comparing Figs. 3(a)—3(c), we find that the risk-averse level promotes the agents reporting high values.

Notice that the strategies can form cycles even the maximal highest support converge. The strategies need about extra 20 agents to converge since the maximal highest support converges to 99 for the agents with different risk tolerances. Figure 4 illustrates the thresholds for the number of risk-averse agents between the converged strategies and the cycling strategies. Figures 4(a) and 4(b) show that the strategies of sophisticated agents form cycles with 3 to 54 risk-averse agents. Figure 4(c) shows that the strategies of sophisticated agents converge with 55 or more risk-averse agents. Figure 5 illustrates the thresholds for the number of risk-neutral agents between the converged strategies and the cycling strategies. Figure 5(a) shows that the strategies of sophisticated agents form cycles with 2 to 73 risk-neutral agents. Figure 5(b) shows that the strategies of sophisticated agents converge with 74 or more risk-neutral agents. Figure 6 illustrates the thresholds for the number of risk-seeking agents between the converged strategies and the cycling strategies. Figure 6(a) shows that the strategies of sophisticated agents form

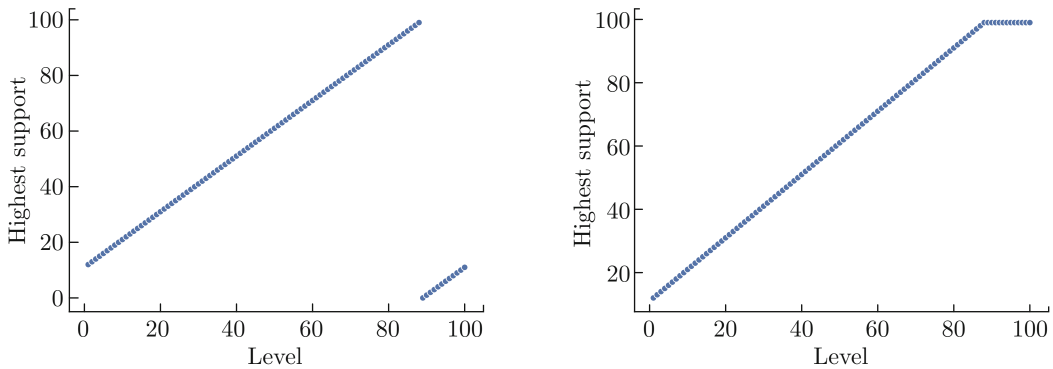


(a) Strategies still form cycles when $n=3$

(b) Strategies still form cycles when $n=54$

(c) Strategies converge when $n=55$

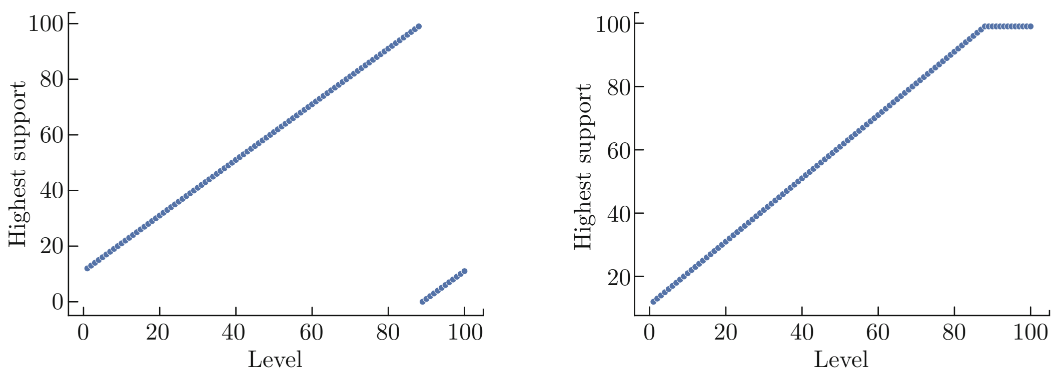
Fig. 4 Highest support of risk-averse ($\alpha = 0.5$) agents predicted by the level-k model with the number of agents $n = 3, 54, 55$



(a) Strategies still form cycles when $n=73$

(b) strategies converge when $n=74$

Fig. 5 Highest support of risk-neutral ($\alpha = 1$) agents predicted by the level-k model with the number of agents $n = 73, 74$



(a) Strategies still form cycles when $n=93$

(b) strategies converge when $n=94$

Fig. 6 Highest support of risk-seeking ($\alpha = 1.5$) agents predicted by the level-k model with the number of agents $n = 93, 94$

cycles with 2 to 93 risk-seeking agents. Figure 6(b) shows that the strategies of sophisticated agents converge with 94 or more risk-seeking agents. Table 1 shows the thresholds of the final convergence move to a larger number of agents when the agents increase their risk-seeking level. The table points out the starting point of different convergences on the number of agents. The second column illustrates the least numbers of sophisticated agents with maximal highest support 99 in Fig. 3. The third column illustrates the least numbers of sophisticated agents with the converging reported distribution of 99 with probability mass 1 in Figs. 4—6. The fourth column illustrates the length of the cycling-strategy periods with maximal highest support 99. Recall that the maximal highest support starts to converge at 39, 55, and 73 with risk-averse, risk-neutral, and risk-seeking agents, respectively. The reported distributions of sophisticated agents start to converge to the distribution of 99 with probability mass 1 at 55, 74, and 94 for three different risk-tolerant types, respectively. So the cycling-strategy periods after the convergence of the maximal highest support are 16, 19, and 21. These results indicate that the risk-averse level would urge the convergence of the strategies of sophisticated agents and reduce the length of the cycling-strategy period.

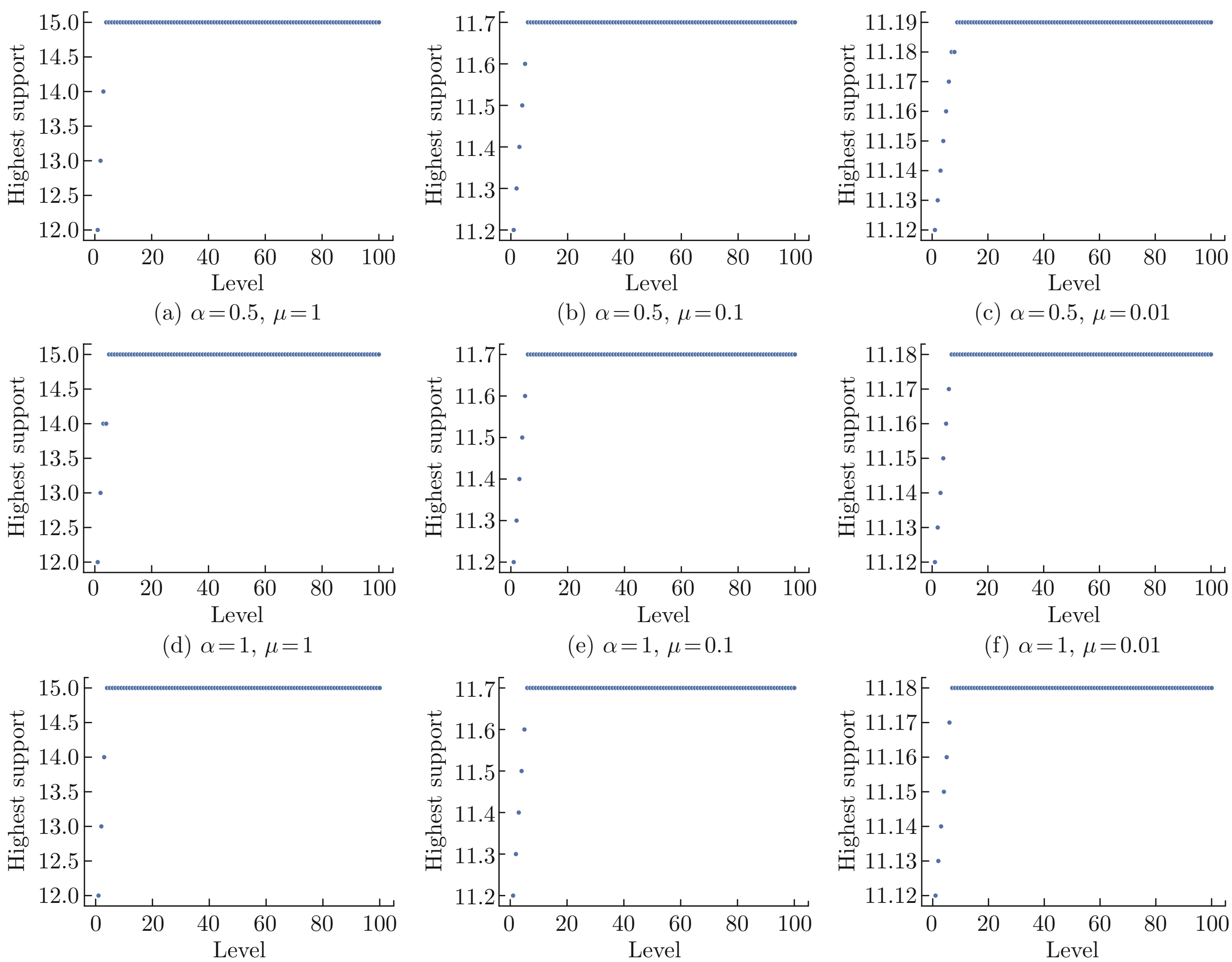
Table 1 Convergence on the number of sophisticated agents predicted the level-k model with $\mu = 1$

Risk-tolerant type	Support convergence	Strategies' final convergence	Differences
Risk-averse ($\alpha = 0.5$)	39	55	16
Risk-neutral ($\alpha = 1$)	55	74	19
Risk-seeking ($\alpha = 1.5$)	73	94	21

4.2.2 *Cognitive Hierarchy Model Results*

Since the behavior of agents predicted by the cognitive hierarchy model with Poisson distribution always converges in discrete scenarios, we focus on the length of the pathological period.

Figure 7 shows the empirical results of the cognitive hierarchy model with different risk tolerances $\alpha = 0.5, 1, 1.5$ and different minimum units $\mu = 1, 0.1, 0.01$. We observe that the risk tolerances have almost no effect in each column of the figures. The reason could be the early exponential decay of $g_{k,l}$ with $\tau = 1.5$. An agent with level-k sophistication predicted by the cognitive hierarchy model would merely consider the agents with a high sophistication level l with a small $g_{k,l}$. Although the risk-tolerant level α affects the utility of an agent, the extremely small $g_{k,l}$ prevents strategies from changing with an increment of the sophistication level. Each row illustrates the length of the pathological periods increases with high precision μ. This fact indicates that the pathological periods could be arbitrarily long for a high enough precision μ, and the model becomes ill-defined in the continuous scenarios as the analysis in Subsection 3.3.



(a) $\alpha = 0.5$, $\mu = 1$ (b) $\alpha = 0.5$, $\mu = 0.1$ (c) $\alpha = 0.5$, $\mu = 0.01$

(d) $\alpha = 1$, $\mu = 1$ (e) $\alpha = 1$, $\mu = 0.1$ (f) $\alpha = 1$, $\mu = 0.01$

(g) $\alpha = 1.5$, $\mu = 1$ (h) $\alpha = 1.5$, $\mu = 0.1$ (i) $\alpha = 1.5$, $\mu = 0.01$

Fig. 7 Highest support of risk-averse ($\alpha = 0.5$), risk-neutral ($\alpha = 1$), and risk-seeking ($\alpha = 1.5$) agents predicted by the cognitive hierarchy model with $\mu = 1, 0.1, 0.01$. The figures in the same column (with the same μ) show the similar results

Notice that the strategies could keep unchanged before the convergence as the occurrence in Figs. 7(c) and 7(d). The convergence of the agents' behaviors predicted by the cognitive hierarchy model needs a careful check. This is the reason why we still run the experiments to level-100 for the cognitive hierarchy model.

4.2.3 *Discussion about Bayesian Nash Equilibrium*

The main rival of the level-k model and the cognitive hierarchy model is the Bayesian Nash equilibrium. The Bayesian Nash equilibrium in both discrete and continuous scenarios is not easy to compute as the agents report a distribution. We might obtain the equilibrium by a bi-matrix equilibrium solver and Algorithm 2 in the discrete scenarios. For the converged scenarios in the previous experiments, the convergence of the level-k model with enough number of agents to the distribution of 99 with probability mass 1 indicates that all agents reporting this distribution form a symmetric Bayesian Nash equilibrium. For the non-converged scenarios, solving the bi-matrix game induced by cycling strategies will also obtain a symmetric Bayesian Nash equilibrium.

5 Conclusion and Discussion

We analyze the behavior of boundedly rational agents predicted by the level-k model and the cognitive hierarchy model in sequential posted pricing when the value distributions could be manipulated. We show that sophisticated agents could exploit the auctioneer by misreporting the distribution. Then we propose the algorithms to compute the optimal strategies of the sophisticated agents predicted by both models. However, we find that both models are ill-defined when agents' values are continuous and could have pathological strategies when the values are discrete. After the theoretical analysis, we evaluate the behavior of agents in different settings, including different minimum units of the values, different risk tolerances, and different numbers of agents. The results show that a large minimum unit, the risk-averse level, and a large number of agents would help the behavior of agents predicted by the level-k model converge and prevent the occurrence of cycling strategies. For the cognitive hierarchy model, the belief distributions of the sophisticated agents and the precision of the values have the greatest impact on the convergence.

In stock markets or online exchange markets, there are usually enough agents with similar highest values, and the posted prices of the auctioneer could be accurate to the penny. The behavior of agents predicted by the level-k model could somehow explain the winner's curse in these markets, which could be modeled as a sequential posted pricing auction. However, there are still some unexplained behaviors. By the level-k model, all the auctioneers would post near the highest prices that the agents could accept. Even more, each agent reporting a distribution with the highest support near her highest value could be a Bayesian Nash equilibrium, and the equilibrium would predict the same behavior of the auctioneers. However, the same item might have different posted prices in some online markets. The highest price could be about 4 or 5 times as much as the lowest. This fact illustrates that both the Bayesian Nash equilibrium and the level-k model have difficulties explaining human behaviors in sequential posted pricing. Meanwhile, the cognitive hierarchy model would suggest small prices, except τ is large enough, and the agents with extremely high sophistication levels are considered. Thus, all these three models have difficulties in explaining the behavior of the optimal auctioneer.

Although we briefly discuss the Bayesian Nash equilibrium in the discrete scenarios in Subsection 4.2.3, how to compute the Bayesian Nash equilibrium in the continuous scenarios remains unknown even with i.i.d. value distributions. Another issue is characterizing auctioneers with bounded rationality to explain the behavior of real auctioneers, which is an interesting future work. For example, the huge gap between different prices might be caused by the different estimations of the agents' values. At last, as we mentioned before, both the level-k model and cognitive hierarchy model could be ill-defined or have pathological strategies. Improving two models to avoid or ease these phenomena could be a challenge.

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