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Substructuring Technique for Dynamics Analysis of Flexible Beams with Large Deformation

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Abstract: In this paper, the substructuring technique is extended for the dynamics simulation of flexible beams with large deformation. The dynamics equation of a spatial straight beam undergoing large displacement and small deformation is deduced by using the Jourdain variation principle and the model synthesis method. The longitudinal shortening effect due to the transversal deformation is taken into consideration in the dynamics equation. In this way, the geometric stiffening effect, which is also called stress stiffening effect, is accounted for in the dynamics equation. The transfer equation of the flexible beam is obtained by assembling the dynamics equation and the kinematic relationship between the two connection points of the flexible beam. Treating a flexible beam with small deformation as a substructure, one can solve the dynamics of a flexible beam with large deformation by using the substructuring technique and the transfer matrix method. The dynamics simulation of a flexible beam with large deformation is carried out by using the proposed approach and the results are verified by comparing with those obtained from Abaqus software.
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0 Introduction

Based on the descriptions of the displacement and deformation of flexible bodies, the popular multi-flexible-body system dynamics methods can be roughly classified into two types: relative node coordinate description methods and absolute node coordinate description methods[1-2]. For the relative node coordinate description methods, the general motion of a flexible body is described by the rigid motion of a floating frame and the relative deformation of the flexible body with respect to the floating frame. This kind of description is intuitive and has various strategies to decrease the dimension of the dynamics equation of system when dealing with flexible bodies with large displacement and small deformation, such as modal synthesis methods[3]. On the other hand, for the absolute node coordinate description methods, the node coordinates are defined in the global inertial frame, which leads to a highly nonlinear expression of potential energy of deformation. This kind of description is much more appropriate to deal with large deformation problems, but the stiffening problem will arise as there is no model reduction method[4].

In this paper, the relative node coordinate description strategy is extended for the dynamics modeling of flexible beams with large deformation by introducing the substructuring technique. Firstly, the dynamics equation of a spatial straight beam undergoing large displacement and small deformation is deduced by using the Jourdain variation principle and the model synthesis method. The longitudinal shortening effect[5-8] due to the transversal deformation is taken into consideration to stabilize the dynamics equation when the system is undergoing high rotational speed[9]. Thus, the geometric stiffening effect[10-11], which is also called stress stiffening effect, is accounted for in the dynamics equation, which makes it applicable for high rotational speed problems. Secondly, the transfer equation[12] of the flexible beam is obtained by assembling the dynamics equation and the kinematic relationship between the two connection points of the flexible beam. Treating a flexible beam with small deformation as a substructure, one can solve the dynamics of a flexible beam with large deformation by using the substructuring technique and the transfer matrix method. Finally, the dynamics of a flexible beam with large deformation is simulated by

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using the proposed approach and the results are verified by comparing with those obtained from Abaqus software.

1 Dynamics Equation of a Spatial Straight Beam

In this section, the dynamics equation of a spatial straight beam with large displacement and small deformation is deduced. Such a beam undergoing small deformation will be treated as a substructure, namely a sub beam, of the flexible beam undergoing large deformation. An arbitrary sub beam, namely a spatial straight beam with small deformation, is numbered i ($i = 1, 2, \cdots, N$), where N denotes the total number of the segments of the flexible beam with large deformation.

1.1 A Spatial Straight Beam and Its Coordinate Systems

A spatial straight beam, numbered i, as shown in Fig. 1, is taken as an example to sketch the idea to deduce the dynamics equation and transfer equation of a spatial straight beam undergoing large displacement and small relative deformation by using the Jourdain variation principle and the model synthesis method.

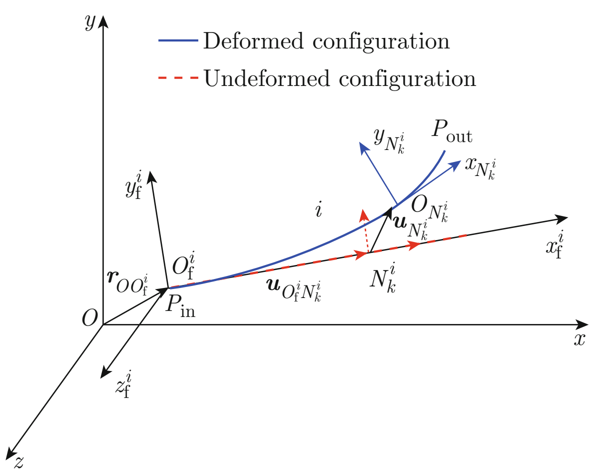


Fig. 1 A spatial straight beam and the coordinate systems

As shown in Fig. 1, $N_k^i (k = 1, 2, \cdots, n)$ is used to denote the kth node of beam i and n is the total node number. The two connection point of beam i are denoted as P_{in} (Input end) and P_{out} (Output end), respectively. $Oxyz$ is the global inertial coordinate system, $O_{\text{f}}^i x_{\text{f}}^i y_{\text{f}}^i z_{\text{f}}^i$ is the floating frame of beam i and $O_{N_k^i} x_{N_k^i} y_{N_k^i} z_{N_k^i}$ is the node coordinate system attached to node N_k^i. $\boldsymbol{r}_{OO_{\text{f}}^i}$ is the position vector of the origin of $O_{\text{f}}^i x_{\text{f}}^i y_{\text{f}}^i z_{\text{f}}^i$, decomposed in the global inertial coordinate system $Oxyz$, $\boldsymbol{u}_{O_{\text{f}}^i N_k^i}$ is used to represent the original coordinate of node k in the undeformed configuration of beam i, and $\boldsymbol{u}_{N_k^i} = [u \quad v \quad w]_{N_k^i}^{\text{T}}$ is the relative deformation vector of node N_k^i with respect to the floating frame of reference.

Additionally, the angular deformation vector of node N_k^i with respect to the floating frame of reference is denoted as $\boldsymbol{\theta}_{N_k^i} = [\theta_x \quad \theta_y \quad \theta_z]_{N_k^i}^{\text{T}}$, whose combination with $\boldsymbol{u}_{N_k^i} = [u \quad v \quad w]_{N_k^i}^{\text{T}}$ gives the complete deformation vector $\boldsymbol{\delta}_{N_k^i} = [\boldsymbol{u}_{N_k^i}^{\text{T}} \quad \boldsymbol{\theta}_{N_k^i}^{\text{T}}]^{\text{T}}$ of node N_k^i. The overall relative deformation vector of the nodes of beam i with respect to $O_{\text{f}}^i x_{\text{f}}^i y_{\text{f}}^i z_{\text{f}}^i$ is denoted as

$$\boldsymbol{\delta}^i = [\boldsymbol{\delta}_{N_1^i}^{\text{T}} \quad \boldsymbol{\delta}_{N_2^i}^{\text{T}} \quad \cdots \quad \boldsymbol{\delta}_{N_n^i}^{\text{T}}]^{\text{T}}.$$

If all the components of the overall relative deformation vector $\boldsymbol{\delta}^i$ remain relatively small enough, then $\boldsymbol{\delta}^i$ could be represented by a modal synthesis method, namely

$$\boldsymbol{\delta}^i = \boldsymbol{\Phi}^i \boldsymbol{q}_{\text{f}}^i, \tag{1}$$

$$\boldsymbol{\Phi}^i = [\boldsymbol{\Phi}_1^i \quad \boldsymbol{\Phi}_2^i \quad \cdots \quad \boldsymbol{\Phi}_M^i], \tag{2}$$

$$\boldsymbol{q}_{\text{f}}^i = [q_{\text{f},1}^i \quad q_{\text{f},2}^i \quad \cdots \quad q_{\text{f},M}^i]^{\text{T}}, \tag{3}$$

where $\boldsymbol{\Phi}_j^i$ $(j = 1, 2, \cdots, M)$ is the jth mode shape of beam i, $q_{\text{f},j}^i$ is the corresponding jth generalized deformation coordinate, and M is the total number of the mode shapes. It should be noted that no matter which kind of modal synthesis method is chosen, there should be no rigid mode shapes involved in $\boldsymbol{\Phi}^i$.

1.2 The Modified Craig-Bampton Modal Synthesis Method

The modified Craig-Bampton modal synthesis method[13] was proposed for generalizing the application of the modal shapes to a general flexible body connected to many other bodies with all kinds of constraint conditions. Taking the flexible beam i as an example, the procedure of the modified Craig-Bampton modal synthesis method is briefly introduced as follows.

The overall relative deformation vector $\boldsymbol{\delta}^i$ are partitioned into two parts: the sub deformation vector of the interface nodes, labeled as $\boldsymbol{\delta}_{\text{Interface}}^i$ and the remaining sub deformation vector of the interior nodes, labeled as $\boldsymbol{\delta}_{\text{Interior}}^i$. And then the mode shapes $\boldsymbol{\Phi}^i$ are also partitioned into two parts, namely

$$\boldsymbol{\Phi}_{\text{s-d}}^i = [\boldsymbol{\Phi}_{\text{s}}^i \quad \boldsymbol{\Phi}_{\text{d}}^i], \tag{4}$$

where $\boldsymbol{\Phi}_{\text{s}}^i$ consists of static modes obtained by setting a nonzero value to one of the degree of freedom of the interface nodes while keeping the other degrees of freedom of the interface nodes constrained; $\boldsymbol{\Phi}_{\text{d}}^i$ is obtained by solving the eigenvalues and expanding the eigenvectors under the condition that all the interface nodes are constrained. The next step is to orthogonalize $\boldsymbol{\Phi}_{\text{s-d}}^i$, which is done by solving the following eigenvalue problem

$$(\boldsymbol{\Phi}_{\text{s-d}}^i)^{\text{T}} \boldsymbol{K}_{\text{c}}^i \boldsymbol{\Phi}_{\text{s-d}}^i \boldsymbol{\Psi}^i = \lambda (\boldsymbol{\Phi}_{\text{s-d}}^i)^{\text{T}} \boldsymbol{M}_{\text{l}}^i \boldsymbol{\Phi}_{\text{s-d}}^i \boldsymbol{\Psi}^i, \tag{5}$$

where $\boldsymbol{K}_{\rm c}^i$ and $\boldsymbol{M}_{\rm l}^i$ are the consistent stiffness matrix and lumped mass matrix of flexible beam i, respectively.

Solving the above eigenvalue problem, one can obtain a group of new eigenvectors $\boldsymbol{\Psi}^i$. And then the orthogonalized modified Crig-Bampton mode shapes can be acquired as follows

$$\boldsymbol{\Phi}^i = \boldsymbol{\Phi}^i_{\text{s-d}}\boldsymbol{\Psi}^i. \tag{6}$$

1.3 Kinematics Equations of the Flexible Beam

The position vector of N_k^i decomposed in the global inertial coordinate system can be expressed as

$$\boldsymbol{r}_{N_k^i} = \boldsymbol{r}_{OO_{\rm f}^i} + \boldsymbol{A}_{OO_{\rm f}^i}\Big\{\boldsymbol{u}_{O_{\rm f}^iN_k^i} + \boldsymbol{\Phi}_{\boldsymbol{u},{\rm N}_k^i}\boldsymbol{q}_{\rm f}^i + \boldsymbol{H}_1\Big[-\frac{1}{2}\sum_{p=1}^{k-1}\int_0^{l_p}\Big[\Big(\frac{\partial v}{\partial x}\Big)^2+\Big(\frac{\partial w}{\partial x}\Big)^2\Big]{\rm d}x\Big]\Big\}, \tag{7}$$

where $\boldsymbol{A}_{OO_{\rm f}^i}$ is the direction cosine matrix of $O_{\rm f}^ix_{\rm f}^iy_{\rm f}^iz_{\rm f}^i$ with respect to $Oxyz$; $\boldsymbol{H}_1 = [1 \quad 0 \quad 0]^{\rm T}$ is a constant vector; $\boldsymbol{\Phi}_{\boldsymbol{u},N_k^i} = \Big[\boldsymbol{\Phi}^{\rm T}_{u,N_k^i} \quad \boldsymbol{\Phi}^{\rm T}_{v,{\rm N}_k^i} \quad \boldsymbol{\Phi}^{\rm T}_{w,N_k^i}\Big]^{\rm T}$ are the deformation modal shapes of N_k^i along x, y and z axes of $O_{\rm f}^ix_{\rm f}^iy_{\rm f}^iz_{\rm f}^i$, respectively; v and w are the relative deformation of the neutral axis of the beam finite element along y and z axes of $O_{\rm f}^ix_{\rm f}^iy_{\rm f}^iz_{\rm f}^i$, respectively; l_p is the initial length of beam finite element p. The last term in Eq. (7) is used to account for the longitudinal shortening effect due to the transversal deformation. By using the shape functions of the beam finite element, this term could be rewritten as

$$-\frac{1}{2}\sum_{p=1}^{k-1}\int_0^{l_p}\Big[\Big(\frac{\partial v}{\partial x}\Big)^2+\Big(\frac{\partial w}{\partial x}\Big)^2\Big]{\rm d}x = (\boldsymbol{\delta}^i)^{\rm T}\hat{\boldsymbol{\Gamma}}_{N_k^i}\boldsymbol{\delta}^i, \tag{8}$$

where

$$(\boldsymbol{\delta}^i)^{\rm T}\hat{\boldsymbol{\Gamma}}_{N_k^i}\boldsymbol{\delta}^i = \sum_{p=1}^{k-1}(\boldsymbol{\delta}^{ip})^{\rm T}\Big\{-\frac{1}{2}\int_0^{l_p}[(\boldsymbol{N}'_v)^{\rm T}\boldsymbol{N}'_v + (\boldsymbol{N}'_w)^{\rm T}\boldsymbol{N}'_w]{\rm d}x\Big\}\boldsymbol{\delta}^{ip}, \tag{9}$$

$\boldsymbol{\delta}^{ip}$ is relative deformation vector of element p of beam i. $\boldsymbol{N}_v(x)$ and $\boldsymbol{N}_w(x)$ are the shape functions corresponding to v and w, respectively. $\boldsymbol{N}'_\bullet(x)$ is the first derivation of $\boldsymbol{N}_\bullet(x)$ with respect to the local position coordinate x of the beam finite element.

Substituting Eq. (1) into Eq. (8) yields

$$-\frac{1}{2}\sum_{p=1}^{k-1}\int_0^{l_p}\Big[\Big(\frac{\partial v}{\partial x}\Big)^2+\Big(\frac{\partial w}{\partial x}\Big)^2\Big]{\rm d}x = (\boldsymbol{q}_{\rm f}^i)^{\rm T}\boldsymbol{\Gamma}_{N_k^i}\boldsymbol{q}_{\rm f}^i, \tag{10}$$

where

$$\boldsymbol{\Gamma}_{N_k^i} = (\boldsymbol{\Phi}^i)^{\rm T}\hat{\boldsymbol{\Gamma}}_{N_k^i}\boldsymbol{\Phi}^i. \tag{11}$$

Substituting Eq. (10) back into Eq. (7), one can obtain

$$\boldsymbol{r}_{N_k^i} = \boldsymbol{r}_{OO_{\rm f}^i} + \boldsymbol{A}_{OO_{\rm f}^i}[\boldsymbol{u}_{O_{\rm f}^iN_k^i} + \boldsymbol{\Phi}_{\boldsymbol{u},N_k^i}\boldsymbol{q}_{\rm f}^i + \boldsymbol{H}_1(\boldsymbol{q}_{\rm f}^i)^{\rm T}\boldsymbol{\Gamma}_{N_k^i}\boldsymbol{q}_{\rm f}^i]. \tag{12}$$

Solving the first derivation of Eq. (12) with respect to time, one can obtain the absolute velocity of N_k^i decomposed in the global inertial frame as follows

$$\dot{\boldsymbol{r}}_{N_k^i} = \dot{\boldsymbol{r}}_{OO_{\rm f}^i} + \tilde{\boldsymbol{\Omega}}_{OO_{\rm f}^i}\boldsymbol{A}_{OO_{\rm f}^i}[\boldsymbol{u}_{O_{\rm f}^iN_k^i} + \boldsymbol{\Phi}_{\boldsymbol{u},N_k^i}\boldsymbol{q}_{\rm f}^i + \boldsymbol{H}_1(\boldsymbol{q}_{\rm f}^i)^{\rm T}\boldsymbol{\Gamma}_{N_k^i}\boldsymbol{q}_{\rm f}^i] + \boldsymbol{A}_{OO_{\rm f}^i}[\boldsymbol{\Phi}_{\boldsymbol{u},N_k^i}\dot{\boldsymbol{q}}_{\rm f}^i + 2\boldsymbol{H}_1(\boldsymbol{q}_{\rm f}^i)^{\rm T}\boldsymbol{\Gamma}_{N_k^i}\dot{\boldsymbol{q}}_{\rm f}^i]. \tag{13}$$

where $\boldsymbol{\Omega}_{OO_{\rm f}^i}$ is the absolute angular velocity of $O_{\rm f}^ix_{\rm f}^iy_{\rm f}^iz_{\rm f}^i$ decomposed in $Oxyz$.

According to the superposition theorem of angular velocity, the absolute angular velocity of $O_{N_k^i}x_{N_k^i}y_{N_k^i}z_{N_k^i}$ decomposed in $Oxyz$ can be written as

$$\boldsymbol{\Omega}_{N_k^i} = \boldsymbol{\Omega}_{OO_{\rm f}^i} + \boldsymbol{A}_{OO_{\rm f}^i}\boldsymbol{\Phi}_{\boldsymbol{\theta},N_k^i}\dot{\boldsymbol{q}}_{\rm f}^i, \tag{14}$$

where $\boldsymbol{\Phi}_{\boldsymbol{\theta},N_k^i} = \Big[\boldsymbol{\Phi}^{\rm T}_{\theta_x,N_k^i} \quad \boldsymbol{\Phi}^{\rm T}_{\theta_y,N_k^i} \quad \boldsymbol{\Phi}^{\rm T}_{\theta_z,N_k^i}\Big]^{\rm T}$ are the rotational deformation modal shapes of N_k^i.

Kinematics Eqs. (13) and (14) could be collected into one formula, namely

$$\dot{\boldsymbol{R}}_{N_k^i} = \boldsymbol{L}_{N_k^i}\boldsymbol{Y}_i, \tag{15}$$

where

$$\dot{\boldsymbol{R}}_{N_k^i} = \begin{bmatrix}\dot{\boldsymbol{r}}\\ \boldsymbol{\Omega}\end{bmatrix}_{N_k^i},$$

$$\boldsymbol{L}_{N_k^i} = \left[\begin{array}{cc:c}\boldsymbol{I}_3 & \boldsymbol{A}_{OO_{\rm f}^i}\tilde{\boldsymbol{\rho}}^{\rm T}_{N_k^i}\boldsymbol{A}^{\rm T}_{OO_{\rm f}^i} & \boldsymbol{A}_{OO_{\rm f}^i}[\boldsymbol{\Phi}_{\boldsymbol{u},N_k^i} + 2\boldsymbol{H}_1(\boldsymbol{q}_{\rm f}^i)^{\rm T}\boldsymbol{\Gamma}_{N_k^i}]\\ \boldsymbol{0}_3 & \boldsymbol{I}_3 & \boldsymbol{A}_{OO_{\rm f}^i}\boldsymbol{\Phi}_{\boldsymbol{\theta},N_k^i}\end{array}\right],$$

$$\boldsymbol{Y}_i = \left[\begin{array}{c}\dot{\boldsymbol{R}}_{OO_{\rm f}^i}\\ \hdashline \dot{\boldsymbol{q}}_{\rm f}^i\end{array}\right]. \tag{16}$$

$$\boldsymbol{\rho}_{N_k^i} = \boldsymbol{u}_{O_{\rm f}^iN_k^i} + \boldsymbol{\Phi}_{\boldsymbol{u},N_k^i}\boldsymbol{q}_{\rm f}^i + \boldsymbol{H}_1(\boldsymbol{q}_{\rm f}^i)^{\rm T}\boldsymbol{\Gamma}_{N_k^i}\boldsymbol{q}_{\rm f}^i, \tag{17}$$

$\boldsymbol{I}_3$ is a 3 by 3 identity matrix and $\boldsymbol{0}_3$ is a 3 by 3 zero matrix.

The Jourdain variation of Eq. (15) reads as

$$\delta\dot{\boldsymbol{R}}_{N_k^i} = \boldsymbol{L}_{N_k^i}\delta\boldsymbol{Y}_i. \tag{18}$$

Solving the first derivation of Eq. (15) with respect to time, one can obtain the absolute accelerations and angular accelerations of N_k^i decomposed in the global inertial frame as follows

$$\ddot{\boldsymbol{R}}_{N_k^i} = \boldsymbol{L}_{N_k^i}\dot{\boldsymbol{Y}}_i + \dot{\boldsymbol{L}}_{N_k^i}\boldsymbol{Y}_i. \tag{19}$$

1.4 Dynamics Equations of the Flexible Beam

The virtual power equation of beam i could be obtained by using the Jourdain variation principle, namely

$$\begin{aligned}&\sum_{k=1}^{n}\delta\dot{\boldsymbol{r}}_{N_k^i}^{\mathrm{T}}\boldsymbol{A}_{OO_{\mathrm{f}}^i}\boldsymbol{m}_{N_k^i}\boldsymbol{A}_{OO_{\mathrm{f}}^i}^{\mathrm{T}}\ddot{\boldsymbol{r}}_{N_k^i}+\\&\quad\sum_{k=1}^{n}\delta\dot{\boldsymbol{\Omega}}_{N_k^i}^{\mathrm{T}}\boldsymbol{A}_{OO_{\mathrm{f}}^i}\boldsymbol{J}_{N_k^i}\boldsymbol{A}_{OO_{\mathrm{f}}^i}^{\mathrm{T}}\dot{\boldsymbol{\Omega}}_{N_k^i}=\\&\quad\delta\dot{\boldsymbol{r}}_{i,P_{\mathrm{in}}}^{\mathrm{T}}\boldsymbol{q}_{i,P_{\mathrm{in}}}-\delta\boldsymbol{\Omega}_{i,P_{\mathrm{in}}}^{\mathrm{T}}\boldsymbol{m}_{i,P_{\mathrm{in}}}-\delta\dot{\boldsymbol{r}}_{i,P_{\mathrm{out}}}^{\mathrm{T}}\boldsymbol{q}_{i,P_{\mathrm{out}}}+\\&\quad\delta\boldsymbol{\Omega}_{i,P_{\mathrm{out}}}^{\mathrm{T}}\boldsymbol{m}_{i,P_{\mathrm{out}}}+\sum_{k=1}^{n}\delta\dot{\boldsymbol{r}}_{N_k^i}^{\mathrm{T}}\boldsymbol{F}_{N_k^i}^{\mathrm{e}}+\sum_{k=1}^{n}\delta\boldsymbol{\Omega}_{N_k^i}^{\mathrm{T}}\boldsymbol{M}_{N_k^i}^{\mathrm{e}}-\\&\quad(\delta\dot{\boldsymbol{q}}_{\mathrm{f}}^i)^{\mathrm{T}}\boldsymbol{K}_{\boldsymbol{FF}}^i\boldsymbol{q}_{\mathrm{f}}^i-(\delta\dot{\boldsymbol{q}}_{\mathrm{f}}^i)^{\mathrm{T}}(\boldsymbol{\alpha}\boldsymbol{M}_{\boldsymbol{FF}}^i+\boldsymbol{\beta}\boldsymbol{K}_{\boldsymbol{FF}}^i)\dot{\boldsymbol{q}}_{\mathrm{f}}^i\end{aligned}\tag{20}$$

where $\boldsymbol{m}_{N_k^i}=m_{N_k^i}\boldsymbol{I}_3$ is the lumped mass matrix of N_k^i decomposed in $O_{\mathrm{f}}^i x_{\mathrm{f}}^i y_{\mathrm{f}}^i z_{\mathrm{f}}^i$, $\boldsymbol{J}_{N_k^i}$ is the corresponding moment of inertial matrix; $\boldsymbol{F}_{N_k^i}^{\mathrm{e}}$ and $\boldsymbol{M}_{N_k^i}^{\mathrm{e}}$ are the external forces and torques acting on N_k^i, respectively; $\boldsymbol{M}_{\boldsymbol{FF}}^i$ and $\boldsymbol{K}_{\boldsymbol{FF}}^i$ are the generalized deformation mass matrix and stiffness matrix of beam i, respectively; $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the corresponding proportional damping coefficients; $\ddot{\boldsymbol{r}}_{i,P_{\mathrm{in}}}(\ddot{\boldsymbol{r}}_{i,P_{\mathrm{out}}})$ and $\dot{\boldsymbol{\Omega}}_{i,P_{\mathrm{in}}}(\dot{\boldsymbol{\Omega}}_{i,P_{\mathrm{out}}})$ are the absolute accelerations and angular accelerations of the input (ouput) end of beam i; $\boldsymbol{q}_{i,P_{\mathrm{in}}}(\boldsymbol{q}_{i,P_{\mathrm{out}}})$ and $\boldsymbol{m}_{i,P_{\mathrm{in}}}(\boldsymbol{m}_{i,P_{\mathrm{out}}})$ are the internal forces and torques of the input (output) end of beam i. Positive directions of $\boldsymbol{q}_{i,P_{\mathrm{in}}}$ coincides with the positive directions of the axes of $Oxyz$, and the positive directions of $\boldsymbol{m}_{i,P_{\mathrm{in}}}$ coincides with the negative directions of the axes of $Oxyz$. The positive directions of $\boldsymbol{q}_{i,P_{\mathrm{out}}}$ and $\boldsymbol{m}_{i,P_{\mathrm{out}}}$ opposite with those of $\boldsymbol{q}_{i,P_{\mathrm{in}}}$ and $\boldsymbol{m}_{i,P_{\mathrm{in}}}$, respectively[12]. Substituting Eqs. (18) and (19) into Eq. (20), one can obtain

$$\begin{aligned}\delta\boldsymbol{Y}_i^{\mathrm{T}}\boldsymbol{M}_i\dot{\boldsymbol{Y}}_i=&\,\delta\boldsymbol{Y}_i^{\mathrm{T}}\boldsymbol{\Psi}_{i,P_{\mathrm{in}}}\boldsymbol{Q}_{i,P_{\mathrm{in}}}+\\&\delta\boldsymbol{Y}_i^{\mathrm{T}}\boldsymbol{\Psi}_{i,P_{\mathrm{out}}}\boldsymbol{Q}_{i,P_{\mathrm{out}}}+\delta\boldsymbol{Y}_i^{\mathrm{T}}\boldsymbol{Q}_i^0,\end{aligned}\tag{21}$$

where $\boldsymbol{M}_i$ is the generalized inertia matrix of beam i; $\boldsymbol{Q}_{i,P_{\mathrm{in}}}$ and $\boldsymbol{Q}_{i,P_{\mathrm{out}}}$ are the column vectors of internal forces of the input and output ends, respectively; $\boldsymbol{Q}_i^0$ is the column vector of generalized forces consisting of centrifugal inertial forces, Coriolis inertial forces, generalized external forces and elastic forces. The detailed expression of each matrix reads as follows

$$\left.\begin{aligned}\boldsymbol{M}_i&=\begin{bmatrix}\boldsymbol{M}_{\boldsymbol{rr}}&\boldsymbol{M}_{\boldsymbol{r\Omega}}&\boldsymbol{M}_{\boldsymbol{rF}}\\\boldsymbol{M}_{\boldsymbol{r\Omega}}^{\mathrm{T}}&\boldsymbol{M}_{\boldsymbol{\Omega\Omega}}&\boldsymbol{M}_{\boldsymbol{\Omega F}}\\\boldsymbol{M}_{\boldsymbol{rF}}^{\mathrm{T}}&\boldsymbol{M}_{\boldsymbol{\Omega F}}^{\mathrm{T}}&\boldsymbol{M}_{\boldsymbol{FF}}\end{bmatrix}_i\\\boldsymbol{Q}_i^0&=\begin{bmatrix}\boldsymbol{Q}_{\boldsymbol{r}}^0\\\boldsymbol{Q}_{\boldsymbol{\Omega}}^0\\\boldsymbol{Q}_{\boldsymbol{F}}^0\end{bmatrix}_i\end{aligned}\right\},\tag{22}$$

$$\left.\begin{aligned}&\boldsymbol{\Psi}_{i,P_{\mathrm{in}}}=\\&\begin{bmatrix}\boldsymbol{0}_3&\boldsymbol{I}_3\\-\boldsymbol{I}_3&\boldsymbol{A}_{OO_{\mathrm{f}}^i}\tilde{\boldsymbol{\rho}}_{i,P_{\mathrm{in}}}\boldsymbol{A}_{OO_{\mathrm{f}}^i}^{\mathrm{T}}\\-\boldsymbol{\Phi}_{\boldsymbol{\theta},i,P_{\mathrm{in}}}^{\mathrm{T}}\boldsymbol{A}_{OO_{\mathrm{f}}^i}^{\mathrm{T}}&(\boldsymbol{\Phi}_{\boldsymbol{u},i,P_{\mathrm{in}}}^{\mathrm{T}}+2\boldsymbol{\Gamma}_{i,P_{\mathrm{in}}}\boldsymbol{q}_{\mathrm{f}}^i\boldsymbol{H}_1^{\mathrm{T}})\boldsymbol{A}_{OO_{\mathrm{f}}^i}^{\mathrm{T}}\end{bmatrix}\\&\boldsymbol{\Psi}_{i,P_{\mathrm{out}}}=\\&\begin{bmatrix}\boldsymbol{0}_3&-\boldsymbol{I}_3\\\boldsymbol{I}_3&-\boldsymbol{A}_{OO_{\mathrm{f}}^i}\tilde{\boldsymbol{\rho}}_{i,P_{\mathrm{out}}}\boldsymbol{A}_{OO_{\mathrm{f}}^i}^{\mathrm{T}}\\\boldsymbol{\Phi}_{\boldsymbol{\theta},i,P_{\mathrm{out}}}^{\mathrm{T}}\boldsymbol{A}_{OO_{\mathrm{f}}^i}^{\mathrm{T}}&-(\boldsymbol{\Phi}_{\boldsymbol{u},i,P_{\mathrm{out}}}^{\mathrm{T}}+2\boldsymbol{\Gamma}_{i,P_{\mathrm{out}}}\boldsymbol{q}_{\mathrm{f}}^i\boldsymbol{H}_1^{\mathrm{T}})\boldsymbol{A}_{OO_{\mathrm{f}}^i}^{\mathrm{T}}\end{bmatrix}\end{aligned}\right\},\tag{23}$$

$$\left.\begin{aligned}\boldsymbol{Q}_{i,P_{\mathrm{in}}}&=\begin{bmatrix}\boldsymbol{m}_{i,P_{\mathrm{in}}}^{\mathrm{T}}&\boldsymbol{q}_{i,P_{\mathrm{in}}}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\\\boldsymbol{Q}_{i,P_{\mathrm{out}}}&=\begin{bmatrix}\boldsymbol{m}_{i,P_{\mathrm{out}}}^{\mathrm{T}}&\boldsymbol{q}_{i,P_{\mathrm{out}}}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\end{aligned}\right\}.\tag{24}$$

The sub matrices of $\boldsymbol{M}_i$ and $\boldsymbol{Q}_i^0$ are

$$\left.\begin{aligned}&\boldsymbol{M}_{\boldsymbol{rr}}=\boldsymbol{I}^1\\&\boldsymbol{M}_{\boldsymbol{r\Omega}}=\boldsymbol{A}_{OO_{\mathrm{f}}^i}\Big[(\tilde{\boldsymbol{I}}^2)^{\mathrm{T}}+\sum_{j=1}^{M}(\tilde{\boldsymbol{I}}_j^3)^{\mathrm{T}}q_{\mathrm{f},j}^i+\\&\qquad\tilde{\boldsymbol{H}}_1^{\mathrm{T}}(\boldsymbol{q}_{\mathrm{f}}^i)^{\mathrm{T}}\boldsymbol{I}^{10}\boldsymbol{q}_{\mathrm{f}}^i\Big]\boldsymbol{A}_{OO_{\mathrm{f}}^i}^{\mathrm{T}}\\&\boldsymbol{M}_{\boldsymbol{rF}}=\boldsymbol{A}_{OO_{\mathrm{f}}^i}[\boldsymbol{I}^3+2\boldsymbol{H}_1(\boldsymbol{q}_{\mathrm{f}}^i)^{\mathrm{T}}\boldsymbol{I}^{10}]\\&\boldsymbol{M}_{\boldsymbol{\Omega\Omega}}=\boldsymbol{A}_{OO_{\mathrm{f}}^i}\Big[\boldsymbol{I}^7+\sum_{j=1}^{M}\boldsymbol{I}_j^8q_{\mathrm{f},j}^i+\\&\qquad\sum_{j=1}^{M}(\boldsymbol{I}_j^8)^{\mathrm{T}}q_{\mathrm{f},j}^i+\sum_{j,h=1}^{M}\boldsymbol{I}_{j,h}^9q_{\mathrm{f},j}^iq_{\mathrm{f},h}^i\Big]\boldsymbol{A}_{OO_{\mathrm{f}}^i}^{\mathrm{T}}+\\&\qquad\boldsymbol{A}_{OO_{\mathrm{f}}^i}\Big\{\sum_{j,h=1}^{n}(\boldsymbol{I}_{j,h}^{11}q_{\mathrm{f},j}^iq_{\mathrm{f},h}^i)\tilde{\boldsymbol{H}}_1^{\mathrm{T}}+\\&\qquad\Big[\sum_{j,h=1}^{n}(\boldsymbol{I}_{j,h}^{11}q_{\mathrm{f},j}^iq_{\mathrm{f},h}^i)\tilde{\boldsymbol{H}}_1^{\mathrm{T}}\Big]^{\mathrm{T}}\Big\}\boldsymbol{A}_{OO_{\mathrm{f}}^i}^{\mathrm{T}}\\&\boldsymbol{M}_{\boldsymbol{\Omega F}}=\\&\qquad\boldsymbol{A}_{OO_{\mathrm{f}}^i}\Big[\boldsymbol{I}^4+\sum_{j=1}^{M}\boldsymbol{I}_j^5q_{\mathrm{f},j}^i+2\sum_{j=1}^{M}(\boldsymbol{I}_j^{12}q_{\mathrm{f},j}^i)\Big]\\&\boldsymbol{M}_{\boldsymbol{FF}}=\boldsymbol{I}^6\end{aligned}\right\},\tag{25}$$

$$\begin{aligned}\boldsymbol{Q}_{\boldsymbol{r}}^0=&\sum_{k=1}^{n}\boldsymbol{F}_{N_k^i}^{\mathrm{e}}-\\&\boldsymbol{A}_{OO_{\mathrm{f}}^i}\Big\{\tilde{\boldsymbol{\omega}}_{OO_{\mathrm{f}}^i}\Big[(\tilde{\boldsymbol{I}}^2)^{\mathrm{T}}+\sum_{j=1}^{M}(\tilde{\boldsymbol{I}}_j^3)^{\mathrm{T}}q_{\mathrm{f},j}^i+\\&\tilde{\boldsymbol{H}}_1^{\mathrm{T}}(\boldsymbol{q}_{\mathrm{f}}^i)^{\mathrm{T}}\boldsymbol{I}^{10}\boldsymbol{q}_{\mathrm{f}}^i\Big]\boldsymbol{\omega}_{OO_{\mathrm{f}}^i}\Big\}-\\&\boldsymbol{A}_{OO_{\mathrm{f}}^i}\Big\{2\Big[\sum_{j=1}^{M}(\tilde{\boldsymbol{I}}_j^3)^{\mathrm{T}}\dot{q}_{\mathrm{f},j}^i+2\tilde{\boldsymbol{H}}_1^{\mathrm{T}}(\boldsymbol{q}_{\mathrm{f}}^i)^{\mathrm{T}}\boldsymbol{I}^{10}\dot{\boldsymbol{q}}_{\mathrm{f}}^i\Big]\times\\&\boldsymbol{\omega}_{OO_{\mathrm{f}}^i}+2\boldsymbol{H}_1(\dot{\boldsymbol{q}}_{\mathrm{f}}^i)^{\mathrm{T}}\boldsymbol{I}^{10}\dot{\boldsymbol{q}}_{\mathrm{f}}^i\Big\},\end{aligned}\tag{26}$$

$$\begin{aligned}\boldsymbol{Q}_{\boldsymbol{\Omega}}^0 = &\sum_{k=1}^{n}(\boldsymbol{M}^{\mathrm{e}}_{N_k^i} + \boldsymbol{A}_{OO_\mathrm{f}^i}\tilde{\boldsymbol{\rho}}_{N_k^i}\boldsymbol{A}^{\mathrm{T}}_{OO_\mathrm{f}^i}\boldsymbol{F}^{\mathrm{e}}_{N_k^i}) - \\ &\boldsymbol{A}_{OO_\mathrm{f}^i}\Big\{\sum_{l=1}^{3}\omega_{OO_\mathrm{f}^i,l}\Big[\boldsymbol{I}_l^{14} + \sum_{j=1}^{M} q^i_{\mathrm{f},j}\boldsymbol{I}^{15}_{l,j} - \\ &\sum_{j=1}^{M} q^i_{\mathrm{f},j}(\boldsymbol{I}^{15}_{l,j})^{\mathrm{T}} + \sum_{j,h=1}^{M} q^i_{\mathrm{f},h}q^i_{\mathrm{f},j}\boldsymbol{I}^{16}_{l,j,h}\Big]\Big\}\boldsymbol{\omega}_{OO_\mathrm{f}^i} - \\ &\boldsymbol{A}_{OO_\mathrm{f}^i}\Big\{\sum_{l=1}^{3}\omega_{OO_\mathrm{f}^i,l}\Big[\sum_{j,h=1}^{M}(q^i_{\mathrm{f},h}q^i_{\mathrm{f},j}\boldsymbol{I}^{11}_{j,h})\tilde{\boldsymbol{H}}_l\tilde{\boldsymbol{H}}_1^{\mathrm{T}} + \\ &\tilde{\boldsymbol{H}}_1\tilde{\boldsymbol{H}}_l\sum_{j,h=1}^{M} q^i_{\mathrm{f},h}q^i_{\mathrm{f},j}(\boldsymbol{I}^{11}_{j,h})^{\mathrm{T}}\Big]\Big\}\boldsymbol{\omega}_{OO_\mathrm{f}^i} - \\ &\boldsymbol{A}_{OO_\mathrm{f}^i}\Big[\sum_{j=1}^{M}(\dot{q}^i_{\mathrm{f},j}\boldsymbol{I}^{13}_j) + 2\sum_{j=1}^{M}\dot{q}^i_{\mathrm{f},j}\boldsymbol{I}^8_j + \\ &2\sum_{j,h=1}^{M} q^i_{\mathrm{f},j}\dot{q}^i_{\mathrm{f},h}\boldsymbol{I}^9_{j,h} + 4\sum_{j,h=1}^{M}(q^i_{\mathrm{f},j}\dot{q}^i_{\mathrm{f},h}\boldsymbol{I}^{11}_{j,h})\tilde{\boldsymbol{H}}_1^{\mathrm{T}}\Big]\boldsymbol{\omega}_{OO_\mathrm{f}^i} - \\ &\boldsymbol{A}_{OO_\mathrm{f}^i}\Big[2\sum_{j,h=1}^{M}(\dot{q}^i_{\mathrm{f},j}\dot{q}^i_{\mathrm{f},h}\boldsymbol{I}^{11}_{j,h})\boldsymbol{H}_1\Big],\end{aligned} \tag{27}$$

$$\begin{aligned}\boldsymbol{Q}_{\boldsymbol{F}}^0 = &\sum_{k=1}^{n}(\boldsymbol{\Phi}^{\mathrm{T}}_{\boldsymbol{u},N_k^i} + 2\boldsymbol{\Gamma}_{N_k^i}\boldsymbol{q}^i_\mathrm{f}\boldsymbol{H}_1^{\mathrm{T}})\boldsymbol{A}^{\mathrm{T}}_{OO_\mathrm{f}^i}\boldsymbol{F}^{\mathrm{e}}_{N_k^i} + \\ &\sum_{k=1}^{n}\boldsymbol{\Phi}^{\mathrm{T}}_{\boldsymbol{\theta},N_k^i}\boldsymbol{A}^{\mathrm{T}}_{OO_\mathrm{f}^i}\boldsymbol{M}^{\mathrm{e}}_{N_k^i} - \\ &\boldsymbol{K}^i_{\boldsymbol{FF}}\boldsymbol{q}^i_\mathrm{f} - (\boldsymbol{\alpha}\boldsymbol{M}^i_{\boldsymbol{FF}} + \boldsymbol{\beta}\boldsymbol{K}^i_{\boldsymbol{FF}})\dot{\boldsymbol{q}}^i_\mathrm{f} - \\ &\Big\{\sum_{j=1}^{M}(\boldsymbol{I}^{17}_j\dot{q}^i_{\mathrm{f},j}) + \sum_{i=1}^{3}\omega_{OO_\mathrm{f}^i,l}\Big[\boldsymbol{I}^{18}_l + \sum_{j=1}^{M}\boldsymbol{I}^{19}_{l,j}q^i_{\mathrm{f},j} + \\ &\sum_{j,h=1}^{M}(\boldsymbol{I}^{20}_{j,h}q^i_{\mathrm{f},j}q^i_{\mathrm{f},h}\tilde{\boldsymbol{H}}_l\tilde{\boldsymbol{H}}_1^{\mathrm{T}}\Big]\Big\}\boldsymbol{\omega}_{OO_\mathrm{f}^i} - \\ &\Big[2\sum_{j=1}^{M}(\boldsymbol{I}^5_j)^{\mathrm{T}}\dot{q}^i_{\mathrm{f},j} + 4\sum_{j,h=1}^{M}(\boldsymbol{I}^{20}_{j,h}q^i_{\mathrm{f},j}\dot{q}^i_{\mathrm{f},h})\tilde{\boldsymbol{H}}_1^{\mathrm{T}}\Big]\boldsymbol{\omega}_{OO_\mathrm{f}^i} - \\ &\Big[2\sum_{j,h=1}^{M}(\boldsymbol{I}^{20}_{j,h}\dot{q}^i_{\mathrm{f},j}\dot{q}^i_{\mathrm{f},h})\Big]\boldsymbol{H}_1 - \\ &\Big\{2\sum_{l=1}^{3}\omega_{OO_\mathrm{f}^i,l}\Big[\sum_{h=1}^{M}\boldsymbol{I}^{21}_{l,h}q^i_{\mathrm{f},h} + \sum_{j,h=1}^{M}\boldsymbol{I}^{22}_{l,j,h}q^i_{\mathrm{f},j}q^i_{\mathrm{f},h}\Big] + \\ &4\sum_{j,h=1}^{M}\boldsymbol{I}^{23}_{j,h}q^i_{\mathrm{f},j}\dot{q}^i_{\mathrm{f},h}\Big\}\boldsymbol{\omega}_{OO_\mathrm{f}^i},\end{aligned} \tag{28}$$

where $\boldsymbol{\omega}_{OO_\mathrm{f}^i} = \boldsymbol{A}^{\mathrm{T}}_{OO_\mathrm{f}^i}\boldsymbol{\Omega}_{OO_\mathrm{f}^i}$ is the absolute angular velocity of $O_\mathrm{f}^i x_\mathrm{f}^i y_\mathrm{f}^i z_\mathrm{f}^i$ decomposed in $O_\mathrm{f}^i x_\mathrm{f}^i y_\mathrm{f}^i z_\mathrm{f}^i$ and $\omega_{OO_\mathrm{f}^i,l}(l = 1, 2, 3)$ denotes the lth component of $\boldsymbol{\omega}_{OO_\mathrm{f}^i}$; $\boldsymbol{I}^1, \boldsymbol{I}^2, \cdots, \boldsymbol{I}^{23}$ are constant matrices and could be calculated by a preprocessor, their detailed expressions are as follows

$$\boldsymbol{I}^1 = \boldsymbol{I}_3\sum_{k=1}^{n} m_{N_k^i}, \qquad \boldsymbol{I}^2 = \sum_{k=1}^{n} m_{N_k^i}\boldsymbol{u}_{O_\mathrm{f}^iN_k^i},$$

$$\boldsymbol{I}^3_j = \sum_{k=1}^{n} m_{N_k^i}\boldsymbol{\Phi}_{\boldsymbol{u},N_k^i,j}, \quad \boldsymbol{I}^3 = [\boldsymbol{I}^3_1 \;\; \boldsymbol{I}^3_2 \;\; \cdots \;\; \boldsymbol{I}^3_M],$$

$$\boldsymbol{I}^4 = \sum_{k=1}^{n}(m_{N_k^i}\tilde{\boldsymbol{u}}_{O_\mathrm{f}^iN_k^i}\boldsymbol{\Phi}_{\boldsymbol{u},N_k^i} + \boldsymbol{J}_{N_k^i}\boldsymbol{\Phi}_{\boldsymbol{\theta},N_k^i}),$$

$$\boldsymbol{I}^5_j = \sum_{k=1}^{n} m_{N_k^i}\tilde{\boldsymbol{\Phi}}_{\boldsymbol{u},N_k^i,j}\boldsymbol{\Phi}_{\boldsymbol{u},N_k^i},$$

$$\boldsymbol{I}^6 = \sum_{k=1}^{n}(\boldsymbol{\Phi}^{\mathrm{T}}_{\boldsymbol{u},N_k^i}m_{N_k^i}\boldsymbol{\Phi}_{\boldsymbol{u},N_k^i} + \boldsymbol{\Phi}^{\mathrm{T}}_{\boldsymbol{\theta},N_k^i}\boldsymbol{J}_{N_k^i}\boldsymbol{\Phi}_{\boldsymbol{\theta},N_k^i}),$$

$$\boldsymbol{I}^7 = \sum_{k=1}^{n}(\boldsymbol{J}_{N_k^i} + m_{N_k^i}\tilde{\boldsymbol{u}}_{O_\mathrm{f}^iN_k^i}\tilde{\boldsymbol{u}}^{\mathrm{T}}_{O_\mathrm{f}^iN_k^i}),$$

$$\boldsymbol{I}^8_j = \sum_{k=1}^{n} m_{N_k^i}\tilde{\boldsymbol{u}}_{O_\mathrm{f}^iN_k^i}\tilde{\boldsymbol{\Phi}}^{\mathrm{T}}_{\boldsymbol{u},N_k^i,j},$$

$$\boldsymbol{I}^9_{j,h} = \sum_{k=1}^{n} m_{N_k^i}\tilde{\boldsymbol{\Phi}}_{\boldsymbol{u},N_k^i,j}\tilde{\boldsymbol{\Phi}}^{\mathrm{T}}_{\boldsymbol{u},N_k^i,h},$$

$$\boldsymbol{I}^{10} = \sum_{k=1}^{n} m_{N_k^i}\boldsymbol{\Gamma}_{N_k^i},$$

$$\boldsymbol{I}^{11}_{j,h} = \sum_{k=1}^{n} m_{N_k^i}\boldsymbol{\Gamma}_{j,N_k^i,h}\tilde{\boldsymbol{u}}_{O_\mathrm{f}^iN_k^i},$$

$$\boldsymbol{I}^{12}_j = \sum_{k=1}^{n} m_{N_k^i}\tilde{\boldsymbol{u}}_{O_\mathrm{f}^iN_k^i}\boldsymbol{H}_1\boldsymbol{\Gamma}_{j,N_k^i},$$

$$\boldsymbol{I}^{13}_j = \sum_{k=1}^{n}\boldsymbol{J}_{N_k^i}\tilde{\boldsymbol{\Phi}}^{\mathrm{T}}_{\boldsymbol{\theta},N_k^i,j},$$

$$\boldsymbol{I}^{14}_l = \sum_{k=1}^{n} m_{N_k^i}\tilde{\boldsymbol{u}}_{O_\mathrm{f}^iN_k^i}\tilde{\boldsymbol{H}}_l\tilde{\boldsymbol{u}}^{\mathrm{T}}_{O_\mathrm{f}^iN_k^i},$$

$$\boldsymbol{I}^{15}_{l,j} = \sum_{k=1}^{n} m_{N_k^i}\tilde{\boldsymbol{u}}_{O_\mathrm{f}^iN_k^i}\tilde{\boldsymbol{H}}_l\tilde{\boldsymbol{\Phi}}^{\mathrm{T}}_{\boldsymbol{u},N_k^i,j},$$

$$\boldsymbol{I}^{16}_{l,j,h} = \sum_{k=1}^{n} m_{N_k^i}\tilde{\boldsymbol{\Phi}}_{\boldsymbol{u},N_k^i,j}\tilde{\boldsymbol{H}}_l\tilde{\boldsymbol{\Phi}}^{\mathrm{T}}_{\boldsymbol{u},N_k^i,h},$$

$$\boldsymbol{I}^{17}_j = \sum_{k=1}^{n}\boldsymbol{\Phi}^{\mathrm{T}}_{\boldsymbol{\theta},N_k^i}\boldsymbol{J}_{N_k^i}\tilde{\boldsymbol{\Phi}}^{\mathrm{T}}_{\boldsymbol{\theta},N_k^i,j},$$

$$\boldsymbol{I}^{18}_l = \sum_{k=1}^{n} m_{N_k^i}\boldsymbol{\Phi}^{\mathrm{T}}_{\boldsymbol{u},N_k^i}\tilde{\boldsymbol{H}}_l\tilde{\boldsymbol{u}}^{\mathrm{T}}_{O_\mathrm{f}^iN_k^i},$$

$$\boldsymbol{I}^{19}_{l,j} = \sum_{k=1}^{n} m_{N_k^i}\boldsymbol{\Phi}^{\mathrm{T}}_{\boldsymbol{u},N_k^i}\tilde{\boldsymbol{H}}_l\tilde{\boldsymbol{\Phi}}^{\mathrm{T}}_{\boldsymbol{u},N_k^i,j},$$

$$\boldsymbol{I}^{20}_{j,h} = \sum_{k=1}^{n} m_{N_k^i}\Gamma_{j,N_k^i,h}\boldsymbol{\Phi}^{\mathrm{T}}_{\boldsymbol{u},N_k^i},$$

$$\boldsymbol{I}^{21}_{l,h} = \sum_{k=1}^{n} m_{N_k^i}\boldsymbol{\Gamma}_{N_k^i,h}\boldsymbol{H}_1^{\mathrm{T}}\tilde{\boldsymbol{H}}_l\tilde{\boldsymbol{u}}^{\mathrm{T}}_{O_\mathrm{f}^iN_k^i},$$

$$\boldsymbol{I}^{22}_{l,j,h} = \sum_{k=1}^{n} m_{N^i_k} \boldsymbol{\Gamma}_{N^i_k,j} \boldsymbol{H}_1^{\mathrm{T}} \tilde{\boldsymbol{H}}_l \tilde{\boldsymbol{\Phi}}^{\mathrm{T}}_{\boldsymbol{u},N^i_k,h},$$
$$\boldsymbol{I}^{23}_{j,h} = \sum_{k=1}^{n} m_{N^i_k} \boldsymbol{\Gamma}_{N^i_k,j} \boldsymbol{H}_1^{\mathrm{T}} \tilde{\boldsymbol{\Phi}}^{\mathrm{T}}_{\boldsymbol{u},N^i_k,h},$$
$$\boldsymbol{H}_1 = [1 \;\; 0 \;\; 0]^{\mathrm{T}},$$
$$\boldsymbol{H}_2 = [0 \;\; 1 \;\; 0]^{\mathrm{T}},$$
$$\boldsymbol{H}_3 = [0 \;\; 0 \;\; 1]^{\mathrm{T}}.$$

Then, the dynamics equation of beam i can be derived from the virtual power equation (21) of beam i, namely

$$\boldsymbol{M}_i \dot{\boldsymbol{Y}}_i = \boldsymbol{\Psi}_{i,P_{\mathrm{in}}} \boldsymbol{Q}_{i,P_{\mathrm{in}}} + \boldsymbol{\Psi}_{i,P_{\mathrm{out}}} \boldsymbol{Q}_{i,P_{\mathrm{out}}} + \boldsymbol{Q}^0_i. \quad (29)$$

2 Transfer Equation of the Flexible Beam

The transfer equation of the flexible beam could be obtained by assembling the dynamics equation and the kinematic relationship between the two connection points of the flexible beam. Here, one of the connection points is treated as the input end P_{in} of the beam and the other as the output end P_{out}. The detailed procedures are as follows.

The acceleration of the input and output ends of beam i could be obtained from Eq. (19), respectively. They are

$$\ddot{\boldsymbol{R}}_{i,P_{\mathrm{in}}} = \boldsymbol{L}_{i,P_{\mathrm{in}}} \dot{\boldsymbol{Y}}_i + \dot{\boldsymbol{L}}_{i,P_{\mathrm{in}}} \boldsymbol{Y}_i, \quad (30)$$
$$\ddot{\boldsymbol{R}}_{i,P_{\mathrm{out}}} = \boldsymbol{L}_{i,P_{\mathrm{out}}} \dot{\boldsymbol{Y}}_i + \dot{\boldsymbol{L}}_{i,P_{\mathrm{out}}} \boldsymbol{Y}_i. \quad (31)$$

From Eq. (29), one can obtain

$$\dot{\boldsymbol{Y}}_i = \boldsymbol{M}_i^{-1} \boldsymbol{\Psi}_{i,P_{\mathrm{in}}} \boldsymbol{Q}_{i,P_{\mathrm{in}}} + \boldsymbol{M}_i^{-1} \boldsymbol{\Psi}_{i,P_{\mathrm{out}}} \boldsymbol{Q}_{i,P_{\mathrm{out}}} + \boldsymbol{M}_i^{-1} \boldsymbol{Q}^0_i. \quad (32)$$

Substituting Eq. (32) into Eqs. (30) and (31) leads to

$$\boldsymbol{E}_1 \begin{bmatrix} \ddot{\boldsymbol{R}}_{i,P_{\mathrm{out}}} \\ \boldsymbol{Q}_{i,P_{\mathrm{out}}} \end{bmatrix} = \boldsymbol{E}_2 \begin{bmatrix} \ddot{\boldsymbol{R}}_{i,P_{\mathrm{in}}} \\ \boldsymbol{Q}_{i,P_{\mathrm{in}}} \end{bmatrix} + \boldsymbol{E}_3, \quad (33)$$

where

$$\boldsymbol{E}_1 = \begin{bmatrix} \boldsymbol{0}_6 & \boldsymbol{L}_{i,P_{\mathrm{in}}} \boldsymbol{M}_i^{-1} \boldsymbol{\Psi}_{i,P_{\mathrm{out}}} \\ -\boldsymbol{I}_6 & \boldsymbol{L}_{i,P_{\mathrm{out}}} \boldsymbol{M}_i^{-1} \boldsymbol{\Psi}_{i,P_{\mathrm{out}}} \end{bmatrix},$$
$$\boldsymbol{E}_2 = \begin{bmatrix} \boldsymbol{I}_6 & -\boldsymbol{L}_{i,P_{\mathrm{in}}} \boldsymbol{M}_i^{-1} \boldsymbol{\Psi}_{i,P_{\mathrm{in}}} \\ \boldsymbol{0}_6 & -\boldsymbol{L}_{i,P_{\mathrm{out}}} \boldsymbol{M}_i^{-1} \boldsymbol{\Psi}_{i,P_{\mathrm{in}}} \end{bmatrix},$$
$$\boldsymbol{E}_3 = -\begin{bmatrix} \boldsymbol{L}_{i,P_{\mathrm{in}}} \boldsymbol{M}_i^{-1} \boldsymbol{Q}^0_i + \dot{\boldsymbol{L}}_{i,P_{\mathrm{in}}} \boldsymbol{Y}_i \\ \boldsymbol{L}_{i,P_{\mathrm{out}}} \boldsymbol{M}_i^{-1} \boldsymbol{Q}^0_i + \dot{\boldsymbol{L}}_{i,P_{\mathrm{out}}} \boldsymbol{Y}_i \end{bmatrix}.$$

The form of the state vectors[14] of the input and output ends is defined as

$$\boldsymbol{z}_{\aleph} = [\ddot{\boldsymbol{R}}^{\mathrm{T}} \;\; \boldsymbol{Q}^{\mathrm{T}} \;\; 1]^{\mathrm{T}}_{\aleph}, \quad (34)$$

whose detailed expression reads as

$$\boldsymbol{z}_{\aleph} = [\ddot{\boldsymbol{r}}^{\mathrm{T}} \;\; \dot{\boldsymbol{\Omega}}^{\mathrm{T}} \;\; \boldsymbol{m}^{\mathrm{T}} \;\; \boldsymbol{q}^{\mathrm{T}} \;\; 1]^{\mathrm{T}}_{\aleph}, \quad (35)$$

or

$$\boldsymbol{z}_{\aleph} = [\ddot{x} \; \ddot{y} \; \ddot{z} \; \dot{\Omega}_x \; \dot{\Omega}_y \; \dot{\Omega}_z \; m_x \; m_y \; m_z \; q_x \; q_y \; q_z \; 1]^{\mathrm{T}}_{\aleph}, \quad (36)$$

where $\aleph$ could be input end P_{in} or the output end P_{out}.

Then the transfer equation of beam i can be obtained from Eq. (33), namely

$$\boldsymbol{z}_{i,P_{\mathrm{out}}} = \boldsymbol{U}_i \boldsymbol{z}_{i,P_{\mathrm{in}}}, \quad (37)$$

where the transfer matrix reads as

$$\boldsymbol{U}_i = \begin{bmatrix} \boldsymbol{E}_1^{-1} \boldsymbol{E}_2 & \boldsymbol{E}_1^{-1} \boldsymbol{E}_3 \\ \boldsymbol{0}_{1\times 12} & 1 \end{bmatrix}. \quad (38)$$

3 Numerical Example and Discussion

In this section, the substructuring technique for the dynamics of a flexible beam with large deformation proposed in this paper is utilized to carry out the numerical simulation and analysis for the dynamics of a cantilever beam system undergoing gravity and large deformation. The parameters of the system are given as follows: the length of beam $L = 10\,\mathrm{m}$, the mass density $\rho = 2.766\,7 \times 10^3\,\mathrm{kg/m^3}$, elastic modulus $E = 68.95\,\mathrm{GPa}$, Poisson's ratio $\mu = 0.33$, shear modulus $G = \frac{E}{2(1+\mu)}$, the area of the cross section $A = 7.3 \times 10^{-5}\,\mathrm{m^2}$, moment of inertia of the cross section $I_y = I_z = 8.218 \times 10^{-9}\,\mathrm{m^4}$, the polar moment of inertia of the cross section $J_P = I_y + I_z$. The acceleration of gravity $\boldsymbol{g} = [0 \;\; -9.8 \;\; 0]^{\mathrm{T}}\,\mathrm{m/s^2}$ and there is no damping considered for this dynamics system.

As shown in Fig. 2, the cantilever beam is divided into N identical segments and each segment is treated as a substructure of the cantilever beam. In this way, each sub beam will undergo large motion but relatively small deformation, whose dynamics behavior could be modeled by the dynamics equation and transfer equation deduced in Section 1 and Section 2. The sub beams are numbered as $1, 2, \cdots, N$ in sequence from the fixed end to the free end.

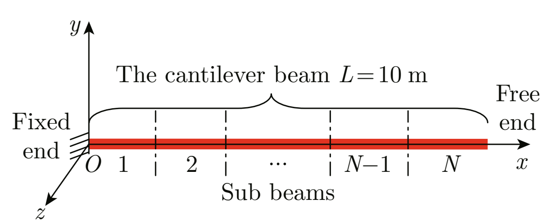


Fig. 2 Sub structures of the cantilever beam

From the transfer equation of each beam i, given in Eq. (37), and the topology given in Fig. 2, one can obtain the overall transfer equation of the cantilever beam system, namely

$$\boldsymbol{z}_{N,P_{\text{out}}} = \boldsymbol{U}_N \cdots \boldsymbol{U}_2 \boldsymbol{U}_1 \boldsymbol{z}_{1,P_{\text{in}}}. \tag{39}$$

Considering the definition of the form of the state vectors, given in Eq. (36), one can represent the boundary conditions of the cantilever beam system as

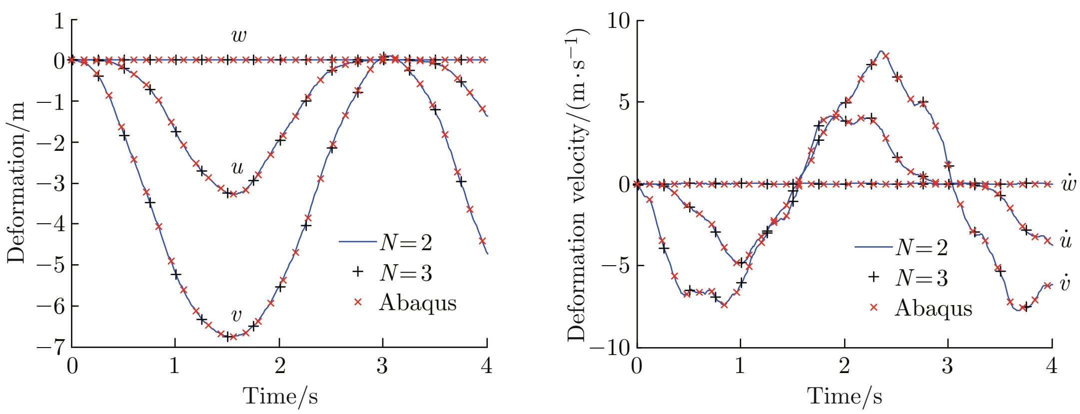
$$\boldsymbol{z}_{1,P_{\text{in}}} = [\underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ m_x\ m_y\ m_z\ q_x\ q_y\ q_z]^{\text{T}}_{1,P_{\text{in}}},$$
$$\boldsymbol{z}_{N,P_{\text{out}}} = [\ddot{x}\ \ddot{y}\ \ddot{z}\ \dot{\Omega}_x\ \dot{\Omega}_y\ \dot{\Omega}_z\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}]^{\text{T}}_{N,P_{\text{out}}}.$$

Substituting the above boundary conditions into Eq. (39), one can obtain the unknown state variables of the boundary state vectors $\boldsymbol{z}_{1,P_{\text{in}}}$ and $\boldsymbol{z}_{N,P_{\text{out}}}$. Then utilizing again Eq. (37), one can obtain the input and output state vectors $\boldsymbol{z}_{i,P_{\text{in}}}$ and $\boldsymbol{z}_{i,P_{\text{out}}}$ of each sub beam i. Further, Eq. (32) could be used to calculate the accelerations $\ddot{\boldsymbol{r}}_{OO_{\text{f}}^i}$, angular accelerations $\dot{\boldsymbol{\Omega}}_{OO_{\text{f}}^i}$ and the generalized deformation accelerations $\ddot{\boldsymbol{q}}_{\text{f}}^i$ of each sub beam i.

The response of the free end's deformation with respect to the global inertial coordinate system $Oxyz$ is shown in Fig. 3, where u, v and w represent the free end's deformation along x, y and z axes of $Oxyz$, respectively, and $\dot{u}$, $\dot{v}$ and $\dot{w}$ are the corresponding time derivatives.

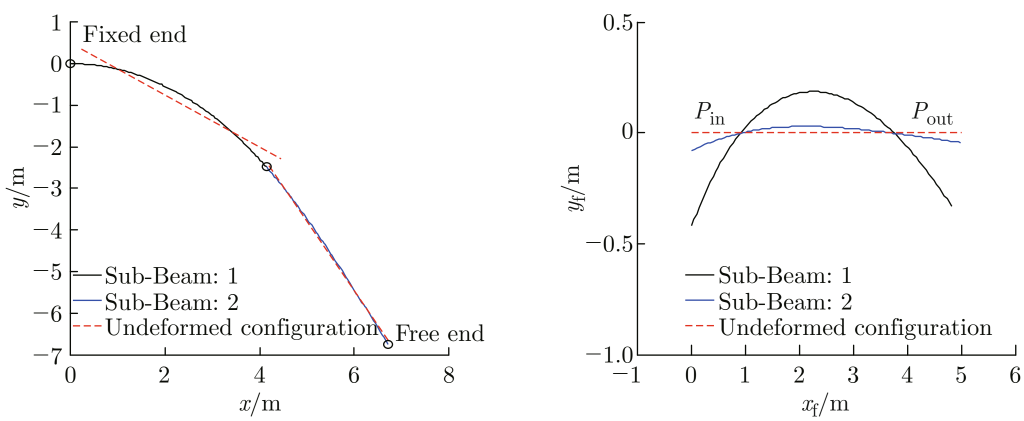
One can find that dividing the cantilever beam into two sub beams ($N = 2$) is competent to obtain acceptable results compared with those obtained from the Abaqus software. The deformations reach their maximal values at time instant $t = 1.54$ s, and the maximal deformation along x and y axes are $u = -3.270$ m and $v = -6.752$ m, respectively. When dividing the cantilever beam into two sub beams ($N = 2$), the deformed configuration of the cantilever beam at time instant $t = 1.54$ s is shown in Fig. 4(a). The deformed configuration of each sub beam with respect to their own floating frame is shown in Fig. 4(b).

From Fig. 4, one can find that the relative deformations of each sub beam with respect to their own floating frames all remain small, and the maximum value is about 0.418 m, which verify the small deformation assumption of the sub beams.



(a) Free end's deformation (b) Free end's deformation velocity

Fig. 3 Response of the free end with respect to $Oxyz$



(a) Deformed configuration in $Oxyz$ (b) Deformed configuration in $O_{\text{f}}x_{\text{f}}y_{\text{f}}z_{\text{f}}$

Fig. 4 Deformed configuration

4 Conclusion

In this paper, the relative node coordinate description strategy is extended for the dynamics modeling of flexible beams with large deformation by introducing the substructuring technique. The dynamics equation of a spatial straight beam undergoing large displacement and small deformation is deduced by using the Jourdain variation principle and the model synthesis method. The longitudinal shortening effect due to the transversal deformation is taken into consideration in the dynamics equation. The transfer equation of the flexible beam is deduced by assembling the dynamics equation and the kinematic relationship between the two connection points of the flexible beam. Treating a flexible beam with small deformation as a substructure, one can solve the dynamics of a flexible beam with large deformation by using the substructuring technique and the transfer matrix method. The dynamics of a flexible beam with large deformation is simulated by using the proposed approach and the results are verified by comparing with those obtained from Abaqus software.

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