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Optimal Preventive Maintenance Policy with Consideration of Production Wait

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Abstract: In this paper, we consider the replacement of a single unit with catastrophic failure mode. Besides replaced at a preset time, the unit is also replaced at failure time or if it encounters a production wait and its age has reached a threshold. The joint preventive maintenance interval and threshold optimization problem are formulated with the objective of minimizing the expected cost per unit time in long run. A numerical example is presented to illustrate the applicability of the model.
Key words: preventive, maintenance, production wait, age replacement, threshold
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Nomenclature

$C_{\rm f}$ — Average cost of a failure replacement
$C_{\rm p}$ — Average cost of an age replacement
$C_{\rm w}$ — Average cost of a replacement at production wait
$C(T,\tau)$ — The long run cost per unit time
$E[C]$ — The expected renewal cycle cost
$E[Z]$ — The expected renewal cycle length
$f_X(t)$— Probability density function of X, $t \in (\tau, T]$
$F_X(t)$— Cumulative distribution function (CDF) of X
$N(t)$— The number of production waits that occur during $[0,t]$
T— The planned replacement time which ranges over $[0,\infty)$
X— The random variable denoting the age of the unit with PDF $f_X(t)$ and CDF $F_X(t) \equiv P\{X \leqslant t\}$
Y— The random arrival time of the nearest production wait that occurs after time τ
Z— The random length of a renewal cycle
λ— The arrival rate of the production wait
τ— The age threshold representing the minimum age requirement for replacing the unit at production wait

0 Introduction

Failures of units are roughly classified into two modes: catastrophic failure in which a unit fails suddenly and completely, and degraded failure in which a unit fails gradually with time due to performance deterioration. In practice, a catastrophic failure may cause interruption of production and incur heavy loss. It is an important problem to determine when to replace or preventively repair a unit before failure[1]. In case of degraded failure, maintenance costs of a unit increase with its age, whereas its performance may suffer some deterioration. It is also required to measure some performance parameters and to determine when to replace or preventively repair a unit before it degrades into failure state[2]. In this paper, we consider the replacement of a single unit with catastrophic failure mode. Some electronic and electric parts or equipment are typical examples. A typical replacement policy for such a unit is "age replacement"[3]. A unit is always replaced at failure or time T if it has not failed up to time T, where T is constant. Age replacement policies have been studied by many authors. The known results were summarized and the optimum policies were studied in detail in Ref. [3]. Some chapters of the recently published books[4-6] summarized the basic results of age and the other replacement policies.

In practice, a unit may experience some production wait due to exhaustion of raw material or installation of a new mold. The unit can be replaced to avoid

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unnecessary interruption to the production process if it encounters a production wait before T and its age has exceeded a certain threshold value τ, where $\tau \in [0, T]$ is also a decision variable. The optimal preventive replacement interval and the optimal threshold are studied in this paper.

1 Model Assumptions

We propose the following assumptions for our model.

(1) The time horizon considered is infinite.

(2) The arrival of production wait observes a Poisson process.

(3) The time that each production wait lasts is negligible.

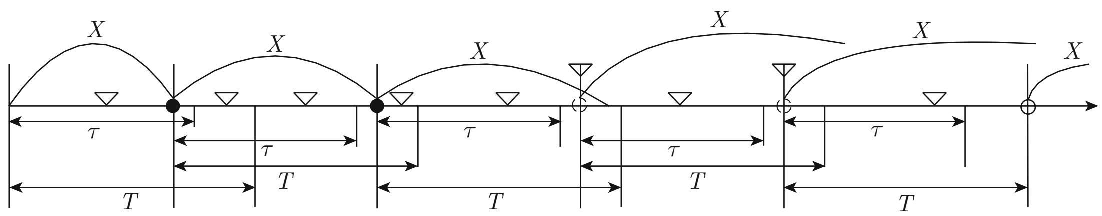
(4) The time for unit replacement is negligible. A new installed unit begins to operate instantly, and is independent of the former.

(5) There exists only a single failure mode for the single-unit system[7]. Once the unit fails, the failure can be detected immediately.

(6) Repair or replacement is regarded as renewing the system, though for a single-unit system, replacement might be the only option.

2 The Cost Model

Consider a single unit subject to catastrophic mode. The unit is replaced or preventively repaired to "as good as new" every time T to prevent it from unexpected failure, though replacement might be the only option for a single-unit system[7]. Besides replaced at T, the unit is also replaced in case it fails before T or it encounters a production wait before T given that its age has reached a threshold τ. The replacement scenarios of the unit are as shown in Fig. 1.



▽ Production wait that occurs, ● Replacement at failure

○ Planned replacement at time T, ◌ Replacement under product wait

Fig. 1 Illustration of replacement scenarios

From the model assumptions in Section 1, we can obtain the probability that the nearest production wait after time τ occurs after y ($y \in (\tau, \infty)$):

$$P\{Y > y\} = P\{N(y) - N(\tau) = 0\} = \mathrm{e}^{-\lambda(y-\tau)}. \quad (1)$$

Furthermore, the probability density function (PDF) of Y can be obtained as

$$f_Y(y) = -\mathrm{d}P\{Y > y\}/\mathrm{d}y = \lambda \mathrm{e}^{-\lambda(y-\tau)}, \quad (2)$$

and the expected value of Y is given by

$$E[Y] = \int_\tau^\infty y f_Y(y)\mathrm{d}y = \int_\tau^\infty y\lambda \mathrm{e}^{-\lambda(y-\tau)}\mathrm{d}y = \tau + \frac{1}{\lambda}. \quad (3)$$

This section formulates the cost model using renewal theorem to optimize the planned replacement T and the age threshold value τ[8-9]. There are three different kinds of renewal cycles: failure replacement, replacement at production wait, and age-based replacement. The expected renewal cycle cost $\mathrm{E}[C]$ requires the probability model for each kind of renewal cycle[9].

Firstly, a failure replacement happens in two situations: the unit fails before time τ, or the unit fails at time $x \in (\tau, T]$ and no production wait occurs during $[\tau, x]$. So the probability for a failure renewal is given by

$$P_1 = P\{X \leqslant \tau\} + P\{\tau < X \leqslant T, N(X) - N(\tau) = 0\} = F_X(\tau) + \int_\tau^T P\{N(X) - N(\tau) = 0 \mid X = x\} f_X(x)\mathrm{d}x = F_X(\tau) + \int_\tau^T \mathrm{e}^{-\lambda(x-\tau)} f_X(x)\mathrm{d}x. \quad (4)$$

Secondly, a replacement at production wait happens when the unit fails at time $x \in (\tau, T]$ and at least one production wait occurs during $[\tau, x]$, or the unit fails after T and at least one production wait occurs during $[\tau, T]$. So the probability for a preventive replacement renewal under production wait is given by

$$P_2 = P\{\tau < X \leqslant T, N(X) - N(\tau) > 0\} + P\{X > T, N(T) - N(\tau) > 0\} = \int_\tau^T [1 - P\{N(x) - N(\tau) = 0\}] f_X(x)\mathrm{d}x + \int_T^\infty [1 - P\{N(T) - N(\tau) = 0\}] f_X(x)\mathrm{d}x = \int_\tau^T [1 - \mathrm{e}^{-\lambda(x-\tau)}] f_X(x)\mathrm{d}x +$$

$$\int_T^{\infty}[1-\mathrm{e}^{-\lambda(T-\tau)}]f_X(x)\mathrm{d}x = 1-F_X(\tau)- \int_\tau^T \mathrm{e}^{-\lambda(x-\tau)}f_X(x)\mathrm{d}x - \mathrm{e}^{-\lambda(T-\tau)}[1-F_X(T)]. \quad (5)$$

Finally, an age-based replacement happens when the unit fails after T and no production wait occurs during $[\tau, T]$. So the probability for an age-based replacement renewal is given by

$$P_3 = P\{X > T, N(T)-N(\tau)=0\} = \mathrm{e}^{-\lambda(T-\tau)}[1-F_X(T)]. \quad (6)$$

The cost parameters C_f, C_w and C_p are introduced. Then the expected cost of one cycle is given by

$$E[C] = C_\mathrm{f}P_1 + C_\mathrm{w}P_2 + C_\mathrm{p}P_3 = C_\mathrm{f}\left[F_X(\tau)+\int_\tau^T \mathrm{e}^{-\lambda(x-\tau)}f_X(x)\mathrm{d}x\right]+ C_\mathrm{w}\left\{1-F_X(\tau)-\int_\tau^T \mathrm{e}^{-\lambda(x-\tau)}f_X(x)\mathrm{d}x- \mathrm{e}^{-\lambda(T-\tau)}[1-F_X(T)]\right\}+ C_\mathrm{p}\{\mathrm{e}^{-\lambda(T-\tau)}[1-F_X(T)]\}. \quad (7)$$

There are three different kinds of renewal cycles and each happens in its certain situations. So the length of a renewal cycle Z can be expressed as follows.

(1) If $X \leqslant \tau$, or if $\tau < X \leqslant T$ and $N(X)-N(\tau)=0$, there is $Z = X$.

(2) If $\tau < X \leqslant T$ and $\tau \leqslant Y \leqslant X$, $X \leqslant \tau$ or if $X > T$ and $\tau \leqslant Y \leqslant T$, there is $Z = Y$.

(3) If $X > T$ and $N(T)-N(\tau)=0$, there is $Z = T$.

Accordingly, the expected length of a renewal cycle can be obtained as

$$E[Z] = \int_0^\tau xf_X(x)\mathrm{d}x + \int_\tau^T xP\{N(X)-N(\tau)=0|X=x\}f_X(x)\mathrm{d}x+ \int_\tau^T\int_\tau^x yf_X(x)f_Y(y)\mathrm{d}y\mathrm{d}x+ \int_T^\infty\int_\tau^T yf_X(x)f_Y(y)\mathrm{d}y\mathrm{d}x+ \int_T^\infty TP\{N(T)-N(\tau)=0|X=x\}f_X(x)\mathrm{d}x = \int_0^\tau xf_X(x)\mathrm{d}x + \int_\tau^T x\mathrm{e}^{-\lambda(x-\tau)}f_X(x)\mathrm{d}x+ \int_\tau^T\int_\tau^x yf_X(x)\lambda\mathrm{e}^{-\lambda(y-\tau)}\mathrm{d}y\mathrm{d}x+ \int_T^\infty\int_\tau^T yf_X(x)\lambda\mathrm{e}^{-\lambda(y-\tau)}\mathrm{d}y\mathrm{d}x+$$

$$T\mathrm{e}^{-\lambda(T-\tau)}\int_T^\infty f_X(x)\mathrm{d}x = \int_0^\tau xf_X(x)\mathrm{d}x - \frac{1}{\lambda}\int_\tau^T \mathrm{e}^{-\lambda(x-\tau)}f_X(x)\mathrm{d}x+ \left(\tau+\frac{1}{\lambda}\right)[1-F_X(\tau)] - \frac{1}{\lambda}\mathrm{e}^{-\lambda(T-\tau)}[1-F_X(T)]. \quad (8)$$

Let $C(T,\tau)$ be the long run cost per unit time. Since each maintenance action renews the system with associated costs, the objective function is given by[9-10]

$$C(T,\tau) = E[C]/E[Z] = (C_\mathrm{f}P_1 + C_\mathrm{w}P_2 + C_\mathrm{p}P_3) \div \left\{\int_0^\tau xf_X(x)\mathrm{d}x - \frac{1}{\lambda}\int_\tau^T \mathrm{e}^{-\lambda(x-\tau)}f_X(x)\mathrm{d}x+ \left(\tau+\frac{1}{\lambda}\right)[1-F_X(\tau)] - \frac{1}{\lambda}\mathrm{e}^{-\lambda(T-\tau)}[1-F_X(T)]\right\}. \quad (9)$$

Based on the cost model, the optimal combination of the preventive replacement interval and the age threshold value, i.e. (T^*, τ^*) that minimizes the expected cost rate $C(T,\tau)$, can be solved.

3 Numerical Example

A numerical example is presented to show the application of the model to minimize the expected cost per unit time in this section. The Weibull distribution has been widely used in industrial practice, and shown to be able to provide a close approximation to the lifetime distribution of different units and systems[11]. Therefore, this paper assumes that the PDF for $f_X(x)$ follows Weibull distribution with parameters (a, b) as

$$f_X(x;a,b) = \begin{cases} \dfrac{b}{a}\left(\dfrac{x}{a}\right)^{b-1}\mathrm{e}^{-(x/a)^b}, & x \geqslant 0 \\ 0, & x < 0 \end{cases}, \quad (10)$$

where $b > 0$ is the shape parameter and $a > 0$ is the scale parameter of the distribution. In addition, the cost parameters for the example are assumed and listed in Table 1.

Besides the parameters in Table 1, it is also assumed that the preventive replacement interval T ranges from 1 to 100 and age threshold value τ ranges from 0 to T.

(1) When $C_\mathrm{w} \geqslant C_\mathrm{p}$, the optimal solution is set as $\tau^* = T^*$. In this situation, the model becomes the "age replacement model", and the replacement takes place at only failure or planned time T. This is consistent with intuition, as it is not cost effective to replace at production wait if its cost is higher than the age replacement cost.

(2) When $C_\mathrm{w} < C_\mathrm{p}$, the optimal solution is set as $\tau^* < T^*$. That is, besides replaced at T, the unit is also replaced in case it fails before T or when it encounters a production wait after a threshold τ but before T. This means that replacement at production wait may reduce costs. It can be seen that the replacements at

production wait save more costs when the production wait is more frequent (when λ increases from 0.25 to 2).

The cost parameter settings of Models 1 and 8 in Table 1 are drawn for graphical illustration, as shown in Figs. 2 and 3.

Table 1 The distribution and cost parameters

Model	λ	a	b	C_f	C_w	C_p	(T^*, τ^*)	$C(T^*, \tau^*)$
1	0.5	50	1.5	1 000	80	30	(8, 8)	11.556 2
2	2	50	1.5	1 000	300	200	(34, 34)	19.725 0
3	0.5	50	1.5	1 000	300	300	(51, 51)	21.218 7
4	2	50	1.5	1 000	300	300	(51, 51)	21.218 7
5	0.25	50	1.5	1 000	150	200	(87, 23)	18.535 2
6	0.5	50	1.5	1 000	150	200	(54, 24)	18.484 2
7	1	50	1.5	1 000	150	200	(49, 25)	18.469 4
8	2	50	1.5	1 000	150	200	(33, 26)	18.465 6

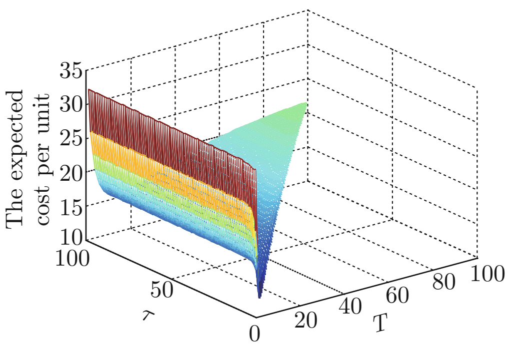


Fig. 2 Output result of Model 1

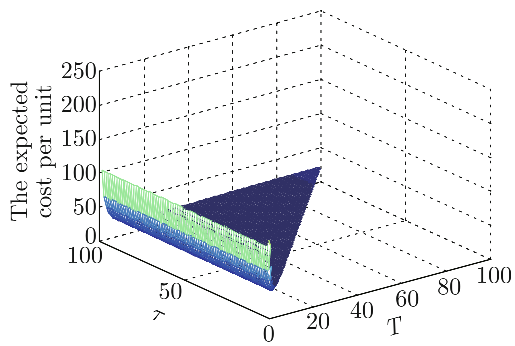


Fig. 3 Output result of Model 8

4 Conclusion

In this paper, we consider the preventive maintenance policy with consideration of the production wait. Besides replaced at T, the unit is also replaced in case it fails before T or when it encounters a production wait after a threshold τ but before T. The cost model is proposed through analysing all the possible renewal scenarios, and the optimization problem is formulated. Numerical example is presented to illustrate the application of our model. A future work is to extend the model to more complex systems considering delay time and inspection with arbitrary number of components and clustering maintenance.

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