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ORIGINAL RESEARCH

Durrmeyer variant of certain approximation operators

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Abstract
In the present article, we introduce a Durrmeyer variant of certain approximation operators. We estimate the moment-generating function and moments of these operators employing the Lambert W function and establish some direct results. We further provide a composition of these operators with Szász–Mirakjan operators and estimate direct results for the composition operator. Additionally, we provide a graphical comparison of the approximation properties of the operators.

Keywords Durrmeyer variant · Integral-type operators · Composition · Szász–Mirakjan operator · Modulus of continuity

Mathematics Subject Classification 41A35

1 Introduction

Theory of positive linear operators is a very active topic of research due to its significance in computer-aided graphics design, mathematical finance, differential equations, etc. In recent years, several new operators have been constructed by combining the existing approximation operators. In [2], Abel and Gupta gave some operators by combining certain integral-type operators with discrete operators. In [7], Govil et al. studied some new classes of Durrmeyer variants of certain operators. In [8], Gupta et al. discussed Baskakov type Pólya-Durrmeyer operators.

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For any function $f : [0, \infty) \to \mathrm{R}$, the Szász–Mirakjan operators [12] are defined as

$$(S_\lambda f)(x) = \sum_{v=0}^{\infty} s_{\lambda,v}(x) f\left(\frac{v}{\lambda}\right), \tag{1}$$

and the Szász–Mirakjan–Durrmeyer operators are defined as

$$\left(\overline{S}_\lambda f\right)(x) = \lambda \sum_{j=0}^{\infty} s_{\lambda,j}(x) \int_0^\infty s_{\lambda,j}(t) f(t)dt, \tag{2}$$

where $s_{\lambda,j}(x) = e^{-\lambda x}\frac{(\lambda x)^j}{j!},\ x \in [0,\infty)$ and $\lambda \in \mathrm{N}$.

If we take $R_\lambda f := \left(S_\lambda \circ \overline{S}_\lambda f\right)$, then we get integral operators of Durrmeyer-type, unlike the reverse order composition of $\overline{S}_\lambda \circ S_\lambda$, which is a discrete operator [1] and the new Durrmeyer-type operators are given by

$$(R_\lambda f)(x) = e^{-\lambda x} \sum_{j=0}^{\infty} \frac{1}{(j!)^2} \sum_{v=0}^{\infty} \frac{e^{-v}(\lambda x)^v v^j}{v!} \int_0^\infty e^{-u} u^j f\left(\frac{u}{\lambda}\right) du. \tag{3}$$

By simple computation with $\exp_A(u) = e^{Au}$, we have

$$\begin{aligned}
\left(S_\lambda \exp_A\right)(x) &= \exp\left(\lambda x\left(e^{A/\lambda} - 1\right)\right), \\
(\overline{S}_\lambda \exp_A)(x) &= \frac{\lambda}{\lambda - A} \exp\left(\frac{\lambda A x}{\lambda - A}\right) \qquad (\lambda > A),
\end{aligned}$$

and

$$(R_\lambda \exp_A)(x) = \frac{\lambda}{\lambda - A} \exp\left(\lambda x\left(e^{\frac{A}{\lambda - A}} - 1\right)\right) \qquad (\lambda > A).$$

Very recently, Gupta-Sharma [9] introduced a new discretely defined approximation operator (4), by combining the two exponential operators, namely the Ismail-May operator [11] and the Szász–Mirakjan operators, which are respectively connected to $x(1+x)^2$ and x.

$$(\mathcal{L}_\lambda f)(x) = \sum_{j=0}^{\infty} \phi_{\lambda,j}(x) f\left(\frac{j}{\lambda}\right), \tag{4}$$

where

$$\phi_{\lambda,j}(x) = e^{\frac{-\lambda x}{1+x}} \sum_{v=0}^{\infty} \frac{\lambda(\lambda+v)^{v-1}}{v!} \left(\frac{x}{1+x}\right)^v e^{-v\left(\frac{1+2x}{1+x}\right)} \frac{v^j}{j!}.$$

The MGF of this operator is given by

$$\left(\mathcal{L}_\lambda \exp_A\right)(x) = \exp\left(-\lambda \left\{ W\left(\frac{-x}{1+x} \exp\left(e^{A/\lambda} + \frac{1}{1+x} - 2 \right)\right) + \frac{x}{1+x} \right\}\right),$$

where W stands for the Lambert W function. These discrete operators $\mathcal{L}_\lambda$ are not suitable enough to approximate Lebesgue integrable functions. We overcome this issue by presenting the Durrmeyer variant of these operators, by taking Szász–Mirakjan weight function, in the following form:

$$\left(\mathcal{D}_\lambda f\right)(x) = \lambda \sum_{j=0}^{\infty} \phi_{\lambda,j}(x) \int_0^\infty s_{\lambda,j}(t) f(t) dt, \tag{5}$$

where $\phi_{\lambda,j}(x)$ and $s_{\lambda,j}(t)$ are as defined above.

This article deals with the convergence properties of the operators $\mathcal{D}_\lambda$. We estimate moment-generating function and moments of these operators via the Lambert W function and establish some direct results. In the next sections, we further consider composition of these operators with Szász–Mirakjan operators and estimate direct results. Finally, we provide a graphical comparison of their approximation properties.

2 Estimation of moments

Lemma 1 *For $\lambda \in \mathrm{N}$, the MGF of the operators $\mathcal{D}_\lambda$ is given by*

$$\left(\mathcal{D}_\lambda \exp_A\right)(x) = \frac{\lambda}{\lambda - A} \exp\left(\frac{-\lambda x}{1+x} - \lambda W\left(\frac{-x}{1+x} e^{\frac{\lambda}{\lambda-A} - \frac{(1+2x)}{1+x}} \right)\right), \; \lambda > A,$$

where W denotes the Lambert W function and $\exp_A(q) = e^{Aq}$.

Proof From the definition of $\mathcal{D}_\lambda$, we have

$$\begin{aligned}
&\left(\mathcal{D}_\lambda \exp_A\right)(x) \\
&= \lambda \sum_{j=0}^{\infty} e^{\frac{-\lambda x}{1+x}} \sum_{v=0}^{\infty} \frac{\lambda(\lambda+v)^{v-1}}{v!} \left(x(1+x)^{-1} \right)^v e^{\frac{-v(1+2x)}{1+x}} \frac{v^j}{j!} \int_0^\infty \frac{e^{-(\lambda-A)t} (\lambda t)^j}{j!} dt \\
&= \frac{\lambda}{\lambda - A} \sum_{j=0}^{\infty} e^{\frac{-\lambda x}{1+x}} \sum_{v=0}^{\infty} \frac{\lambda(\lambda+v)^{v-1}}{v!} \left(x(1+x)^{-1} \right)^v e^{\frac{-v(1+2x)}{1+x}} \frac{1}{j!} \left(\frac{\lambda v}{\lambda - A} \right)^j \\
&= \frac{\lambda}{\lambda - A} e^{\frac{-\lambda x}{1+x}} \sum_{v=0}^{\infty} \frac{\lambda(\lambda+v)^{v-1}}{v!} \left(x(1+x)^{-1} \right)^v \left(e^{\frac{-(1+2x)}{1+x}} \right)^v \left(e^{\frac{\lambda}{\lambda-A}} \right)^v .
\end{aligned}$$

Since for $x \geq 0$, we have $\frac{-x}{1+x} e^{\frac{-(1+2x)}{1+x}} e^{\frac{\lambda}{\lambda-A}} > \frac{-1}{e}$, therefore there exists s with $|s| < 1$, such that $\frac{-x}{1+x} e^{\frac{-(1+2x)}{1+x}} e^{\frac{\lambda}{\lambda-A}} = -se^{-s}$. By the definition of Lambert W function, $W\left(\frac{-x}{1+x}e^{\frac{-(1+2x)}{1+x}}e^{\frac{\lambda}{\lambda-A}}\right)=-s$. Using the following inversion formula, given by Lagrange

$$e^{\alpha z}=\sum_{k=0}^{\infty}\frac{\alpha(\alpha+k)^{k-1}}{k!}(ze^{-z})^k,$$

with $0<\alpha<\infty$ and $|z|<1$, we get

$$\begin{aligned}&\left(\mathcal{D}_\lambda \exp_A\right)(x)\\&=\frac{\lambda}{\lambda-A}e^{\frac{-\lambda x}{1+x}}\sum_{v=0}^{\infty}\frac{\lambda(\lambda+v)^{v-1}}{v!}(se^{-s})^v\\&=\frac{\lambda}{\lambda-A}e^{\frac{-\lambda x}{1+x}}e^{\lambda s}\\&=\frac{\lambda}{\lambda-A}\exp\left(\frac{-\lambda x}{1+x}-\lambda W\left(\frac{-x}{1+x}e^{\frac{-(1+2x)}{1+x}}e^{\frac{\lambda}{\lambda-A}}\right)\right)\qquad(\lambda>A),\end{aligned}$$

hence the lemma follows. □

Remark 1 Let us denote the q-th order moments for the operators $\mathcal{D}_\lambda$ by $\left(\mathcal{D}_\lambda e_q\right)(x)$, then these can be obtained by the following relation between them and moment-generating function:

$$\left(\mathcal{D}_\lambda e_q\right)(x)=\left[\frac{\partial^q}{\partial A^q}\left\{\frac{\lambda}{\lambda-A}\exp\left(\frac{-\lambda x}{1+x}-\lambda W\left(\frac{-x}{1+x}e^{\frac{\lambda}{\lambda-A}-\frac{(1+2x)}{1+x}}\right)\right)\right\}\right]_{A=0},$$

where $e_q(t)=t^q,\ q=0,1,2,\cdots$. Similarly, the central moments, denoted by $\mu_{\lambda,q}(x)=(\mathcal{D}_\lambda(e_1-xe_0)^q)(x)$, may be obtained using the following relation:

$$\mu_{\lambda,q}(x)=\left[\frac{\partial^q}{\partial A^q}\left\{\frac{\lambda}{\lambda-A}\exp\left(\frac{-\lambda x}{1+x}-\lambda W\left(\frac{-x}{1+x}e^{\frac{\lambda}{\lambda-A}-\frac{(1+2x)}{1+x}}\right)-Ax\right)\right\}\right]_{A=0},$$

where $q=0,1,2,\cdots$.

Lemma 2 *The moments for $\mathcal{D}_\lambda$ follow this linear combination:*

$$\sum_{q\geq 0} c_q \left(\mathcal{D}_\lambda e_q\right)(x)$$
$$= c_0 + \left(x + \frac{1}{\lambda}\right) c_1 + \left(x^2 + \frac{x(5+2x+x^2)}{\lambda} + \frac{2}{\lambda^2}\right) c_2$$
$$+ \left(x^3 + \frac{3x^2(x^2+2x+4)}{\lambda} + \frac{x(3x^4+10x^3+21x^2+24x+28)}{\lambda^2} + \frac{6}{\lambda^3}\right) c_3$$
$$+ \left(x^4 + \frac{6x^5+12x^4+22x^3}{\lambda} + \frac{15x^6+52x^5+114x^4+132x^3+127x^2}{\lambda^2}\right.$$
$$\left. + \frac{15x^7+70x^6+179x^5+284x^4+325x^3+254x^2+185x}{\lambda^3} + \frac{24}{\lambda^2}\right) c_4 + ...,$$

where c_q's are arbitrary constants and $q \in \mathrm{N} \cup \{0\}$.

Proof The proof follows by the application of Lemma 1 and Remark 1. □

Lemma 3 *The central moments for $\mathcal{D}_\lambda$ follow the linear combination as follows:*

$$\sum_{q\geq 0} c_q \mu_{\lambda,q}(x) = c_0 + c_1\frac{1}{\lambda} + c_2\left(\frac{x(x^2+2x+3)}{\lambda} + \frac{2}{\lambda^2}\right)$$
$$+ c_3\left(\frac{x(3x^4+10x^3+21x^2+24x+22)}{\lambda^2} + \frac{6}{\lambda^3}\right)$$
$$+ c_4\left(\frac{15x^7+70x^6+179x^5+284x^4+325x^3+254x^2+161x}{\lambda^3}\right.$$
$$\left. + \frac{3x^6+12x^5+30x^4+36x^3+27x^2}{\lambda^2} + \frac{24}{\lambda^4}\right) + ...,$$

where c_q's are arbitrary constants and $q \in \mathrm{N} \cup \{0\}$.

Proof The proof follows by the application of Lemma 1 and Remark 1. □

3 Approximation

Let us denote $C_B[0,\infty) = \{f \mid f : [0,\infty) \to \mathrm{R},\ f \text{ is continuous and bounded}\}$ and let $C^*[0,\infty) = \{f \mid f : [0,\infty) \to \mathrm{R},\ f \text{ is continuous and } \lim_{x\to\infty} f(x) < \infty\}$.

Theorem 1 *If $f \in C_B[0,\infty)$, then*

(i) The operator $\mathcal{D}_\lambda$ satisfies the following property with operator R_λ defined in Eq. (3)

$$\lim_{\lambda\to\infty} \left(\mathcal{D}_{n\lambda} f(\lambda t)\right)\left(\frac{x}{\lambda}\right) = \left(R_n f(t)\right)(x).$$

(ii) For operator $\mathcal{L}_\lambda$ defined in Eq. (4), we have

$$\lim_{\lambda\to\infty}\left(\mathcal{L}_{n\lambda} f(\lambda t)\right)\left(\frac{x}{\lambda}\right)=\left(S_n\circ S_n f(t)\right)(x),$$

where $n\geq 1$ and $x\geq 0$.

Proof For $\lambda\in\mathrm{N}$, we have

(i) By simple calculations,

$$\begin{aligned}&\lim_{\lambda\to\infty}\left(\mathcal{D}_{n\lambda}\exp_{is\lambda}\right)\left(\frac{x}{\lambda}\right)\\&=\frac{n}{n-is}\lim_{\lambda\to\infty}\exp\left(\frac{-n\lambda x}{\lambda+x}-n\lambda W\left(\frac{-x}{\lambda+x}e^{\frac{n}{n-is}-1-\frac{x}{\lambda+x}}\right)\right)\\&=\frac{n}{n-is}\exp\left(nx\left(e^{\frac{is}{n-is}}-1\right)\right)=\left(R_n\exp_{is}\right)(x),\end{aligned}$$

where $\exp_{is\lambda}(u)=\cos(s\lambda u)+i\sin(s\lambda u)$ and $s\in\mathrm{R}$.

(ii) In a similar manner, we have

$$\begin{aligned}&\lim_{\lambda\to\infty}\left(\mathcal{L}_{n\lambda}\exp_{is\lambda}\right)\left(\frac{x}{\lambda}\right)\\&=\lim_{\lambda\to\infty}\exp\left(-n\lambda W\left(\frac{-x}{\lambda+x}\exp\left(e^{is/n}-1-\frac{x}{\lambda+x}\right)\right)-\frac{n\lambda x}{\lambda+x}\right)\\&=\exp\left(nx\left(e^{e^{\frac{is}{n}}-1}-1\right)\right)=\left(S_n\circ S_n\exp_{is}\right)(x).\end{aligned}$$

Now, the proof concludes from [4, Theorem 1] and [5, Theorem 2.1]. □

Now, we establish Korovkin–type theorem, similar to the one given in [6, 10], as follows:

Theorem 2 [10] *Let $A_\lambda : C^*[0,\infty)\to C^*[0,\infty)$ be endowed with uniform norm $\left\|A_\lambda\exp_{-q}-\exp_{-q}\right\|_{[0,\infty)}=C_\lambda^q$, $q\in\{0,1,2\}$ and $C_\lambda^q\to 0$ as $\lambda\to\infty$, then*

$$\|A_\lambda f-f\|_{[0,\infty)}\leq C_\lambda^0\|f\|_{[0,\infty)}+\left(C_\lambda^0+2\right)\omega^*\left(f;\sqrt{C_\lambda^0+2C_\lambda^1+C_\lambda^2}\right),$$

where $\omega^*(f;\sigma)=\sup\limits_{\substack{x_1,x_2\geq 0\\ \left|e^{-x_1}-e^{-x_2}\right|\leq\sigma}}|f(x_1)-f(x_2)|$ is the modulus of continuity.

Theorem 3 *For $f\in C^*[0,\infty)$ and $\lambda\in\mathrm{N}$, let $\left\|\mathcal{D}_\lambda\exp_{-q}-\exp_{-q}\right\|_{[0,\infty)}=B_\lambda^q$, where $q\in\{0,1,2\}$ and $\lim_{\lambda\to\infty}B_\lambda^q=0$, then*

$$\|\mathcal{D}_\lambda f-f\|_{[0,\infty)}\leq 2\omega^*\left(f;\sqrt{2B_\lambda^1+B_\lambda^2}\right).$$

Proof Since $\mathcal{D}_\lambda$ preserves constants, therefore $B_\lambda^0 = 0$. With the help of software Mathematica, we get

$$\begin{aligned}&\left(\mathcal{D}_\lambda \exp_{-1}\right)(x) - \exp_{-1}(x)\\&= \frac{1}{2\lambda}e^{-x}\left(x^3+2x^2+3x-2\right) + \frac{1}{24\lambda^2}e^{-x}\left(3x^6-10x^4-48x^3\right.\\&\left.-69x^2-88x+24\right) + O\left(\lambda^{-3}\right).\end{aligned}$$

Next, we have

$$\sup_{x\geq 0} e^{-x} = 1 \text{ and } \sup_{x\geq 0} x^m e^{-x} = m^m e^{-m},\ m = 1, 2, 3, \cdots,$$

whence, we get

$$\begin{aligned}B_\lambda^1(x) &= \sup_{x\geq 0}\left|\left(\mathcal{D}_\lambda \exp_{-1}\right)(x) - \exp_{-1}(x)\right|\\&\leq \frac{1}{\lambda}\left(\frac{27e^{-3}}{2} + 4e^{-2} + \frac{3}{2}e^{-1} + 1\right)\\&+ \frac{1}{\lambda^2}\left(5832e^{-6} + \frac{320e^{-4}}{3} + 54e^{-3} + \frac{69}{6}e^{-2} + \frac{11}{3}e^{-1} + 1\right)\\&+ O\left(\lambda^{-3}\right) \to 0 \text{ as } \lambda \to \infty.\end{aligned}$$

In similar manner, we have

$$\begin{aligned}&\left(\mathcal{D}_\lambda \exp_{-2}\right)(x) - \exp_{-2}(x)\\&= \frac{2}{\lambda}e^{-2x}\left(x^3+2x^2+3x-1\right) + \frac{2}{3\lambda^2}e^{-2x}\left(3x^6+6x^5+10x^4-6x^3\right.\\&\left.-21x^2-44x+6\right) + O\left(\lambda^{-3}\right).\end{aligned}$$

Also,

$$\sup_{x\geq 0} e^{-2x} = 1 \text{ and } \sup_{x\geq 0} x^m e^{-2x} = \left(\frac{m}{2}\right)^m e^{-m},\ m = 1, 2, 3, \cdots,$$

whence, we get

$$\begin{aligned} B_{\lambda}^{2}(x) &= \sup_{x\geq 0}\left|\left(\mathcal{D}_{\lambda}\exp_{-2}\right)(x)-\exp_{-2}(x)\right| \\ &\leq \frac{1}{\lambda}\left(\frac{27e^{-3}}{4}+4e^{-2}+3e^{-1}+2\right) \\ &\quad+\frac{1}{\lambda^{2}}\left(1458e^{-6}+\frac{3125e^{-5}}{8}+\frac{320e^{-4}}{3}+\frac{27e^{-3}}{2}+14e^{-2}+\frac{44e^{-1}}{3}+4\right) \\ &\quad+O\left(\lambda^{-3}\right)\rightarrow 0 \text{ as } \lambda\rightarrow\infty. \end{aligned}$$

The proof readily follows from Theorem 2. □

Theorem 4 *Let $f, f', f'' \in C^{*}[0,\infty)$, then*

$$\begin{aligned} &\left|\lambda\left[(\mathcal{D}_{\lambda}f)(x)-f(x)\right]-f'(x)-\frac{1}{2}x\left(x^{2}+2x+3\right)f''(x)\right| \\ &\leq 2\Bigg[\frac{1}{2\lambda}\left|f''(x)\right|+\frac{2}{\lambda}+x(x^{2}+2x+3) \\ &\quad+\lambda^{2}\left(\left(\mathcal{D}_{\lambda}\left(\exp_{-1}(x)-\exp_{-1}(l)\right)^{4}\right)(x)\cdot\mu_{\lambda,4}(x)\right)^{\frac{1}{2}}\Bigg]\omega^{*}\left(f'';\frac{1}{\sqrt{\lambda}}\right). \end{aligned}$$

Proof Applying Taylor's formula on f, we have for $x, l \in [0,\infty)$,

$$f(l)=f(x)+(l-x)f'(x)+\frac{1}{2}(l-x)^{2}f''(x)+(l-x)^{2}\zeta(l;x),$$

where $\lim_{l\to x}\zeta(l;x)=0$. Operating $\mathcal{D}_{\lambda}$ and using Lemma 3, we have

$$\begin{aligned} &\left|\lambda\left[(\mathcal{D}_{\lambda}f)(x)-f(x)\right]-f'(x)-\frac{1}{2}x\left(x^{2}+2x+3\right)f''(x)\right| \\ &\leq\frac{1}{\lambda}\left|f''(x)\right|+\lambda\left|\left(\mathcal{D}_{\lambda}\zeta(l;x)(l-x)^{2}\right)(x)\right|. \end{aligned} \tag{6}$$

For $\delta>0$, the modulus of continuity satisfies the following property [3]

$$\zeta(l;x)\leq 2\left(1+\frac{\left(\exp_{-1}(x)-\exp_{-1}(l)\right)^{2}}{\delta^{2}}\right)\omega^{*}\left(f'';\delta\right).$$

Applying Cauchy–Schwarz inequality on the last term in R.H.S. of (6) gives

$$
\begin{aligned}
&\lambda \left| \left(\mathcal{D}_\lambda \zeta(l; x)\,(l-x)^2 \right)(x) \right| \\
&\quad \le 2\lambda \omega^* \left(f''; \delta \right) \mu_{\lambda,2}(x) \\
&\quad + \frac{2\lambda}{\delta^2} \omega^* \left(f''; \delta \right) \left(\mathcal{D}_\lambda \left(\exp_{-1}(x) - \exp_{-1}(l) \right)^4 \right)^{1/2} \sqrt{\mu_{\lambda,4}(x)}.
\end{aligned}
$$

The proof follows by selecting $\delta = \frac{1}{\sqrt{\lambda}}$. □

4 Further composition with Szász–Mirakyan operator

Combining the operators $\mathcal{D}_\lambda$ and Szász–Mirakjan operators yields a new operator, denoted by E_λ and represented as

$$
(E_\lambda f)(x) := (\mathcal{D}_\lambda \circ S_\lambda f)(x).
$$

Lemma 4 *The MGF of the operators E_λ is*

$$
(E_\lambda \exp_A)(x) = \frac{1}{2 - e^{A/\lambda}} \exp\left(\frac{-\lambda x}{1+x} - \lambda W \left((-x)(1+x)^{-1} e^{\frac{1}{2-e^{A/\lambda}} - \frac{(1+2x)}{1+x}} \right) \right).
$$

Furthermore, let us denote the moments of q-th order by $(E_\lambda e_q)(x)$, where $e_q(x) = x^q$ and $q = 0, 1, 2, \cdots$, then

$$
\begin{aligned}
&\sum_{q \ge 0} d_q (E_\lambda e_q)(x) \\
&= d_0 + d_1 \left(x + \frac{1}{\lambda} \right) + d_2 \left(x^2 + \frac{x^3 + 2x^2 + 6x}{\lambda} + \frac{3}{\lambda^2} \right) \\
&+ d_3 \left(x^3 + \frac{3x^4 + 6x^3 + 15x^2}{\lambda} + \frac{3x^5 + 10x^4 + 24x^3 + 30x^2 + 44x}{\lambda^2} + \frac{13}{\lambda^3} \right) \\
&+ d_4 \left(x^4 + \frac{6x^5 + 12x^4 + 28x^3}{\lambda} + \frac{15x^6 + 52x^5 + 132x^4 + 168x^3 + 206x^2}{\lambda^2} \right. \\
&\left. + \frac{15x^7 + 70x^6 + 197x^5 + 344x^4 + 458x^3 + 412x^2 + 389x}{\lambda^3} + \frac{75}{\lambda^4} \right) + \cdots,
\end{aligned}
$$

where d_q's, $q = 0, 1, 2, \cdots$ are certain constants.

Lemma 5 *For the central moments of q-th order, which are denoted by* $\tilde{\mu}_{\lambda,q}(x) = (E_\lambda(e_1 - xe_0)^q)(x)$, *we have*

$$\sum_{q\geq 0} d_q \tilde{\mu}_{\lambda,q}(x)$$
$$= d_0 + \frac{d_1}{\lambda} + d_2\left(\frac{x^3+2x^2+4x}{\lambda} + \frac{3}{\lambda^2}\right)$$
$$+ d_3\left(\frac{3x^5+10x^4+24x^3+30x^2+35x}{\lambda^2} + \frac{13}{\lambda^3}\right)$$
$$+ d_4\left(\frac{3x^6+12x^5+36x^4+48x^3+48x^2}{\lambda^2}\right.$$
$$\left.+\frac{15x^7+70x^6+197x^5+344x^4+458x^3+412x^2+337x}{\lambda^3} + \frac{75}{\lambda^4}\right) + \cdots,$$

where d_q*'s,* $q = 0, 1, 2, \cdots$ *are certain constants.*

Now, we present some theorems analogous to those for the operator $\mathcal{D}_\lambda$.

Theorem 5 *If* $f \in C_B[0,\infty)$ *and* $n \geq 1$, *then*

$$\lim_{\lambda\to\infty} (E_{n\lambda} f(\lambda t))\left(\frac{x}{\lambda}\right) = (V_n f(t))(x),$$

where $V_\lambda f := (R_\lambda \circ S_\lambda f)$ *and* $x \geq 0$.

Proof For $\lambda \in \mathrm{N}$ and $s \in \mathrm{R}$, we have

$$\lim_{\lambda\to\infty} \left(E_{n\lambda} \exp_{is\lambda}\right)\left(\frac{x}{\lambda}\right)$$
$$= \lim_{\lambda\to\infty} \frac{1}{2-e^{is/n}} \exp\left(\frac{-n\lambda x}{\lambda+x} - n\lambda W\left(\frac{-x}{\lambda+x} e^{\frac{1}{2-e^{is/n}}-1-\frac{x}{\lambda+x}}\right)\right)$$
$$= \frac{1}{2-e^{is/n}} \exp\left(nx\left(e^{\frac{1}{2-e^{is/n}}-1} - 1\right)\right) = \left(V_n \exp_{is}\right)(x).$$

Now, the conclusion follows from [4, Theorem 1] and [5, Theorem 2.1]. □

Theorem 6 *For* $f \in C^*[0,\infty)$ *and* $\lambda \in \mathrm{N}$, *let* $\left\| E_\lambda \exp_{-q} - \exp_{-q} \right\|_{[0,\infty)} = M_\lambda^q$, *where* $q \in \{0, 1, 2\}$ *and* $\lim_{\lambda\to\infty} M_\lambda^q = 0$, *then*

$$\|E_\lambda f - f\|_{[0,\infty)} \leq 2\omega^*\left(f; \sqrt{2M_\lambda^1 + M_\lambda^2}\right).$$

Proof Since $(E_\lambda 1)(x) = 1$, therefore $M_\lambda^0 = 0$. Next, we have

$$\begin{aligned}&\left(E_\lambda \exp_{-1}\right)(x) - \exp_{-1}(x)\\&= \frac{1}{2\lambda} e^{-x}\left(x^3 + 2x^2 + 4x - 2\right) + \frac{1}{24\lambda^2} e^{-x}\left(3x^6 - 4x^4 - 48x^3\right.\\&\left.-72x^2 - 140x + 36\right) + O\left(\lambda^{-3}\right),\end{aligned}$$

and

$$\begin{aligned}&\left(E_\lambda \exp_{-2}\right)(x) - \exp_{-2}(x)\\&= \frac{2}{\lambda} e^{-2x}\left(x^3 + 2x^2 + 4x - 1\right) + \frac{2}{3\lambda^2} e^{-2x}\left(3x^6 + 6x^5 + 16x^4\right.\\&\left.-12x^2 - 70x + 9\right) + O\left(\lambda^{-3}\right).\end{aligned}$$

Using

$$\begin{aligned}&\sup_{x\geq 0} e^{-mx} = 1,\ \sup_{x\geq 0} x^m e^{-x} = m^m e^{-m},\ \text{and}\\&\sup_{x\geq 0} x^m e^{-2x} = \left(\frac{m}{2}\right)^m e^{-m},\ m = 1, 2, 3, \cdots,\end{aligned}$$

we get

$$\begin{aligned}M_\lambda^1(x) &= \sup_{x\geq 0}\left|\left(E_\lambda \exp_{-1}\right)(x) - \exp_{-1}(x)\right|\\&\leq \frac{1}{\lambda}\left(\frac{27e^{-3}}{2} + 4e^{-2} + 2e^{-1} + 1\right)\\&+ \frac{1}{\lambda^2}\left(5832e^{-6} + \frac{128e^{-4}}{3} + 54e^{-3} + 12e^{-2} + \frac{35}{6}e^{-1} + 1.5\right)\\&+ O\left(\lambda^{-3}\right) \to 0 \text{ as } \lambda \to \infty.\end{aligned}$$

and

$$\begin{aligned}M_\lambda^2(x) &= \sup_{x\geq 0}\left|\left(E_\lambda \exp_{-2}\right)(x) - \exp_{-2}(x)\right|\\&\leq \frac{1}{\lambda}\left(\frac{27e^{-3}}{4} + 4e^{-2} + 4e^{-1} + 2\right)\\&+ \frac{1}{\lambda^2}\left(1458e^{-6} + \frac{3125e^{-5}}{8} + \frac{512e^{-4}}{3} + 8e^{-2} + \frac{70e^{-1}}{3} + 6\right)\\&+ O\left(\lambda^{-3}\right) \to 0 \text{ as } \lambda \to \infty.\end{aligned}$$

The proof readily follows from Theorem 2. □

Theorem 7 *Let* $f, f', f'' \in C^*[0,\infty)$, *then*

$$\left|\lambda\left[\left(E_\lambda f\right)(x)-f(x)\right]-f'(x)-\frac{1}{2}x\left(x^2+2x+4\right)f''(x)\right|$$
$$\leq 2\left[\frac{3}{4\lambda}\left|f''(x)\right|+\frac{3}{\lambda}+x(x^2+2x+4)\right.$$
$$\left.+\lambda^2\left(\left(E_\lambda\left(\exp_{-1}(x)-\exp_{-1}(l)\right)^4\right)(x)\cdot\tilde{\mu}_{\lambda,4}(x)\right)^{\frac{1}{2}}\right]\omega^*\left(f'';\frac{1}{\sqrt{\lambda}}\right).$$

Proof Applying Taylor's formula to the operator E_λ and using Lemma 5, we have

$$\left|\lambda\left[\left(E_\lambda f\right)(x)-f(x)\right]-f'(x)-\frac{1}{2}x\left(x^2+2x+4\right)f''(x)\right|$$
$$\leq \frac{3}{2\lambda}\left|f''(x)\right|+\lambda\left|\left(E_\lambda\zeta(l;x)(l-x)^2\right)(x)\right|.$$

Now, for $\delta > 0$, applying Cauchy–Schwarz inequality on the last term from above and using the property $\zeta(l;x)\leq 2\left(1+\frac{\left(\exp_{-1}(x)-\exp_{-1}(l)\right)^2}{\delta^2}\right)\omega^*\left(f'';\delta\right)$, we have

$$\lambda\left|\left(E_\lambda\zeta(l;x)(l-x)^2\right)(x)\right|$$
$$\leq 2\lambda\omega^*\left(f'';\delta\right)\tilde{\mu}_{\lambda,2}(x)$$
$$+\frac{2\lambda}{\delta^2}\omega^*\left(f'';\delta\right)\left(E_\lambda\left(\exp_{-1}(x)-\exp_{-1}(l)\right)^4\right)^{1/2}\sqrt{\tilde{\mu}_{\lambda,4}(x)}.$$

The proof follows by selecting $\delta=\frac{1}{\sqrt{\lambda}}$. □

5 Graphical representation

We present following graphs to give a comparison among the rate of approximations of the operators $\mathcal{D}_\lambda$, E_λ and R_λ.
In Fig. 1, the approximations of exponential function $f(x)=e^{-4x}$, by these operators are compared (see Fig.a and Fig.b).

Likewise, in Fig. 2, the graphs (Fig.c, Fig.d) compare the approximations of cubic polynomial $f(x)=x^3+2x^2+6x+2$.

We observe that R_λ yields the best approximation, followed by $\mathcal{D}_\lambda$, with E_λ being the least precise; which indicates that higher order compositions produce less precise approximations. Moreover, as λ increases, the approximations become more precise.

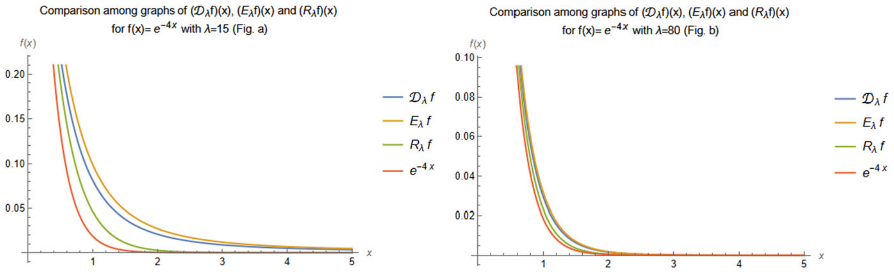

Fig. 1 Comparison among graphs of $\mathcal{D}_\lambda$, E_λ and R_λ for $f(x) = e^{-4x}$

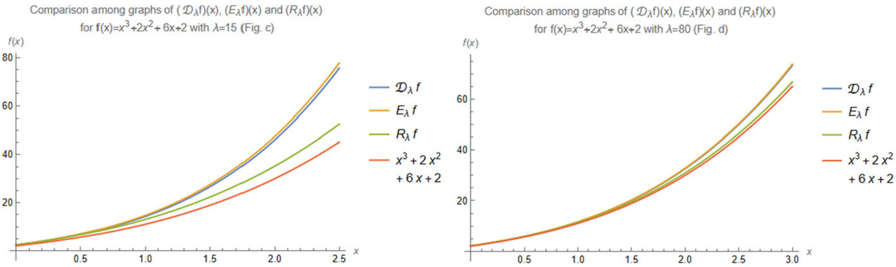

Fig. 2 Comparison among graphs of $\mathcal{D}_\lambda$, E_λ and R_λ for $f(x) = x^3 + 2x^2 + 6x + 2$

Data availibility The authors confirm that there is no associated data.

Declarations

Conflict of interest The authors declare that they do not have any conflict of interest.

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