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ORIGINAL RESEARCH

Further norm inequalities for positive semidefinite matrices

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Abstract

Let A and B be $n \times n$ positive semidefinite matrices, and let $||\cdot||_2$ be the Hilbert-Schmidt norm. Bhatia and then Hayajneh and Kittaneh, using different techniques, proved that

$$||A^v B^{1-v} + B^v A^{1-v}||2 \leq ||A + B||2$$

for $v \in \left[\frac{1}{4}, \frac{3}{4}\right]$, which gives an affirmative answer to an open problem posed by Bourin for the special case of the Hilbert–Schmidt norm. In this paper, we prove a general unitarily invariant norm inequality from which we obtain a new proof of the above Hilbert–Schmidt norm inequality. We also prove that if $r \geq 1$, then

$$|||A^v B^{1-v} + B^v A^{1-v}||| \leq |||(A^{1/r} + B^{1/r})^r|||$$

for $\frac{1}{2r} \leq v \leq \frac{2r-1}{2r}$, where $|||\cdot|||$ denotes any unitarily invariant norm.

Keywords Trace · Positive semidefinite matrix · Unitarily invariant norm · Bourin's question · Inequality

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1 Introduction

For any $n \times n$ complex matrix X, the eigenvalues and the singular values of X are denoted by $\lambda_i(X)$ and $\sigma_i(X)$ for $i = 1, 2, \ldots, n$, and arranged in such a way that

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$|\lambda_1(X)| \geq |\lambda_2(X)| \geq \cdots \geq |\lambda_n(X)|$ and $\sigma_1(X) \geq \sigma_2(X) \geq \cdots \geq \sigma_n(X)$. Thus, $\sigma_i(X) = \lambda_i(|X|)$ for $i = 1, 2, ..., n$, where $|X| = (X^*X)^{1/2}$ is the absolute value of X.

Recall that for any two $n \times n$ complex matrices X and Y, we have $\lambda_i(XY) = \lambda_i(YX)$ for $i = 1, 2, \ldots, n$, and $\|X\|_2 = \left(\sum_{i=1}^{n} \sigma_i^2(X)\right)^{1/2} = \left(\operatorname{tr}|X|^2\right)^{1/2}$ and $\|X\|_1 = \sum_{i=1}^{n} \sigma_i(X) = \operatorname{tr}|X|$ are the Hilbert–Schmidt norm and the trace norm of X, respectively. Moreover, any unitarily invariant norm is an increasing function of singular values.

Let A and B be positive semidefinite matrices, and let $|||\cdot|||$ be any unitarily invariant norm. Bourin [5], in his paper on the subadditivity of concave functions of positive semidefinite matrices, asked whether the inequality

$$|||A^v B^{1-v} + B^v A^{1-v}||| \leq |||A + B|||, \ v \in [0, 1], \tag{1.1}$$

is true.

In their works on the aforesaid conjecture, Bhatia [4] and Hayajneh and Kittaneh [9] proved that

$$||A^v B^{1-v} + B^v A^{1-v}||_2 \leq ||A + B||_2$$

is true whenever $v \in \left[\frac{1}{4}, \frac{3}{4}\right]$.

A complete answer to Bourin's question for the trace norm $\|\cdot\|_1$ has been given by Hayajneh and Kittaneh [7], that is,

$$\left\|A^v B^{1-v} + B^v A^{1-v}\right\|_1 \leq ||A + B||_1$$

is true for $v \in [0, 1]$. Several partial solutions to Bourin's problem have been given in [10] and references therein.

In this paper, we prove that if f is a nonnegative concave function on $[0, \infty)$, then

$$\left|\left|\left|f\left(\left|A^{\frac{v}{2}} B^{1-\frac{v}{2}} + B^{\frac{v}{2}} A^{1-\frac{v}{2}}\right|\right)\right|\right|\right| \leq \left|\left|\left|f\left(\frac{A+B}{2}\right)\right|\right|\right| + \left|\left|\left|f\left(\frac{B^{\frac{v}{2}} A^{1-v} B^{\frac{v}{2}} + A^{\frac{v}{2}} B^{1-v} A^{\frac{v}{2}}}{2}\right)\right|\right|\right|$$

for $v \in [0, 1]$. We also prove the following inequality related to the inequality (1.1)

$$|||A^v B^{1-v} + B^v A^{1-v}||| \leq |||(A^{1/r} + B^{1/r})^r|||$$

for $\frac{1}{2r} \leq v \leq \frac{2r-1}{2r}$, $r \geq 1$.

2 Main results

The following lemmas are required in order to support the main results.

Lemma 2.1 *[12] Given any positive semidefinite block matrix* $\begin{bmatrix} M & K \\ K^* & N \end{bmatrix}$, *where* M *and* N *are* $m \times m$ *and* $n \times n$ *complex matrices, respectively, we have*

$$2\sigma_i(K) \le \sigma_i\left(\begin{bmatrix} M & K \\ K^* & N \end{bmatrix}\right)$$

for $i = 1, \ldots, r$ *and* $r = \min\{m, n\}$.

Lemma 2.2 *[3, p. 291] Let* X *be an* $n \times n$ *complex matrix, and let* f *be a nonnegative increasing function on* $[0, \infty)$ *. Then*

$$f(\sigma_i(X)) = \sigma_i(f(|X|))$$

for $i = 1, 2, ..., n$.

Lemma 2.3 *[6] Let* X, Y, *and* Z *be* $n \times n$ *complex matrices such that the block matrix* $\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}$ *is positive semidefinite, and let* f *be a nonnegative concave function on* $[0, \infty)$ *. Then*

$$\left|\left|\left| f\left(\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}\right)\right|\right|\right| \le |||f(X)||| + |||f(Z)|||.$$

In particular,

$$\left|\left|\left|\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}\right|\right|\right| \le |||X||| + |||Z|||.$$

Using Lemma 2.1, Lemma 2.2, and Lemma 2.3, we prove our first main result.

Theorem 2.4 *Let* A *and* B *be positive semidefinite matrices, and let* f *be a nonnegative concave function on* $[0, \infty)$ *. Then*

$$\left|\left|\left| f\left(\left|A^{\frac{v}{2}} B^{1-\frac{v}{2}} + B^{\frac{v}{2}} A^{1-\frac{v}{2}}\right|\right)\right|\right|\right| \le \left|\left|\left| f\left(\frac{A+B}{2}\right)\right|\right|\right| + \left|\left|\left| f\left(\frac{B^{\frac{v}{2}} A^{1-v} B^{\frac{v}{2}} + A^{\frac{v}{2}} B^{1-v} A^{\frac{v}{2}}}{2}\right)\right|\right|\right|$$

for $v \in [0, 1]$ *.*

Proof Let $r \geq 0$, and let $X = \begin{bmatrix} A^{\frac{r}{2}} & B^{\frac{r}{2}} \\ B^{\frac{r}{2}} & A^{\frac{r}{2}} \end{bmatrix}$ and $Y = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Then

$$X^*YX = \begin{bmatrix} A^{r+1} + B^{r+1} & A^{\frac{r}{2}+1}B^{\frac{r}{2}} + B^{\frac{r}{2}+1}A^{\frac{r}{2}} \\ \left(A^{\frac{r}{2}+1}B^{\frac{r}{2}} + B^{\frac{r}{2}+1}A^{\frac{r}{2}}\right)^* & B^{\frac{r}{2}}AB^{\frac{r}{2}} + A^{\frac{r}{2}}BA^{\frac{r}{2}} \end{bmatrix}$$

is positive semidefinite, and hence by using Lemma 2.1, we have

$$\begin{aligned} &\sigma_i\left(A^{\frac{r}{2}+1}B^{\frac{r}{2}} + B^{\frac{r}{2}+1}A^{\frac{r}{2}}\right) \\ &\leq \frac{1}{2}\sigma_i\left(\begin{bmatrix} A^{r+1} + B^{r+1} & A^{\frac{r}{2}+1}B^{\frac{r}{2}} + B^{\frac{r}{2}+1}A^{\frac{r}{2}} \\ \left(A^{\frac{r}{2}+1}B^{\frac{r}{2}} + B^{\frac{r}{2}+1}A^{\frac{r}{2}}\right)^* & B^{\frac{r}{2}}AB^{\frac{r}{2}} + A^{\frac{r}{2}}BA^{\frac{r}{2}} \end{bmatrix}\right) \end{aligned} \tag{2.1}$$

for $i = 1, 2, ..., n$.

Now, for $i = 1, 2, ..., n$, we have

$$\begin{aligned} &\sigma_i\left(f\left(\left|A^{\frac{r}{2}+1}B^{\frac{r}{2}} + B^{\frac{r}{2}+1}A^{\frac{r}{2}}\right|\right)\right) \\ &= f\left(\sigma_i\left(A^{\frac{r}{2}+1}B^{\frac{r}{2}} + B^{\frac{r}{2}+1}A^{\frac{r}{2}}\right)\right) \text{ (by Lemma 2.2)} \\ &\leq f\left(\frac{1}{2}\sigma_i\left(\begin{bmatrix} A^{r+1} + B^{r+1} & A^{\frac{r}{2}+1}B^{\frac{r}{2}} + B^{\frac{r}{2}+1}A^{\frac{r}{2}} \\ \left(A^{\frac{r}{2}+1}B^{\frac{r}{2}} + B^{\frac{r}{2}+1}A^{\frac{r}{2}}\right)^* & B^{\frac{r}{2}}AB^{\frac{r}{2}} + A^{\frac{r}{2}}BA^{\frac{r}{2}} \end{bmatrix}\right)\right) \\ &= \sigma_i\left(f\left(\frac{1}{2}\begin{bmatrix} A^{r+1} + B^{r+1} & A^{\frac{r}{2}+1}B^{\frac{r}{2}} + B^{\frac{r}{2}+1}A^{\frac{r}{2}} \\ \left(A^{\frac{r}{2}+1}B^{\frac{r}{2}} + B^{\frac{r}{2}+1}A^{\frac{r}{2}}\right)^* & B^{\frac{r}{2}}AB^{\frac{r}{2}} + A^{\frac{r}{2}}BA^{\frac{r}{2}} \end{bmatrix}\right)\right) \\ &\qquad (by\ Lemma\ 2.2). \end{aligned}$$

So,

$$\begin{aligned} &\left|\left|\left|f\left(\left|A^{\frac{r}{2}+1}B^{\frac{r}{2}} + B^{\frac{r}{2}+1}A^{\frac{r}{2}}\right|\right)\right|\right|\right| \\ &\leq \left|\left|\left|f\left(\frac{1}{2}\begin{bmatrix} A^{r+1} + B^{r+1} & A^{\frac{r}{2}+1}B^{\frac{r}{2}} + B^{\frac{r}{2}+1}A^{\frac{r}{2}} \\ \left(A^{\frac{r}{2}+1}B^{\frac{r}{2}} + B^{\frac{r}{2}+1}A^{\frac{r}{2}}\right)^* & B^{\frac{r}{2}}AB^{\frac{r}{2}} + A^{\frac{r}{2}}BA^{\frac{r}{2}} \end{bmatrix}\right)\right|\right|\right| \\ &\leq \left|\left|\left|f\left(\frac{A^{r+1} + B^{r+1}}{2}\right)\right|\right|\right| + \left|\left|\left|f\left(\frac{B^{\frac{r}{2}}AB^{\frac{r}{2}} + A^{\frac{r}{2}}BA^{\frac{r}{2}}}{2}\right)\right|\right|\right| \\ &\qquad (by\ Lemma\ 2.3). \end{aligned}$$

Replacing A, B by $A^{\frac{1}{r+1}}, B^{\frac{1}{r+1}}$, respectively, and taking $v = \frac{r}{r+1}$, we obtain

$$\left|\left|\left|f\left(\left|A^{\frac{v}{2}}B^{1-\frac{v}{2}}+B^{\frac{v}{2}}A^{1-\frac{v}{2}}\right|\right)\right|\right|\right| \le \left|\left|\left|f\left(\frac{A+B}{2}\right)\right|\right|\right| + \left|\left|\left|f\left(\frac{B^{\frac{v}{2}}A^{1-v}B^{\frac{v}{2}}+A^{\frac{v}{2}}B^{1-v}A^{\frac{v}{2}}}{2}\right)\right|\right|\right|$$

for $v \in [0, 1]$. Here, we have used the fact that $|||f(|X|)||| = |||f(|X^*|)|||$ for any complex matrix X. This completes the proof of the theorem.

Taking $v = 1$ in the aforementioned Theorem 2.4, we have the following corollary.

Corollary 2.5 *Let A and B be positive semidefinite matrices, and let f be a nonnegative concave function on* $[0, \infty)$ *. Then*

$$\left|\left|\left|f\left(\left|A^{\frac{1}{2}}B^{\frac{1}{2}}+B^{\frac{1}{2}}A^{\frac{1}{2}}\right|\right)\right|\right|\right| \le 2\left|\left|\left|f\left(\frac{A+B}{2}\right)\right|\right|\right|.$$

As an application of Theorem 2.4, we give a different solution to Bourin's question for the Hilbert–Schmidt norm. To achieve this, we need the following lemmas.

Lemma 2.6 *[9] Let A and B be positive semidefinite matrices, and let* $v \in [0, 1]$*. Then*

$$\operatorname{tr}\left(\left(B^{\frac{v}{2}}A^{1-v}B^{\frac{v}{2}}\right)^2+\left(A^{\frac{v}{2}}B^{1-v}A^{\frac{v}{2}}\right)^2\right) \le \operatorname{tr}\left(A^2+B^2\right).$$

Lemma 2.7 *[9] Let A and B be positive semidefinite matrices, and let* $v \in \left[\frac{1}{2}, 1\right]$*. Then*

$$\operatorname{tr} B^{\frac{v}{2}}A^{1-v}B^{\frac{v}{2}}A^{\frac{v}{2}}B^{1-v}A^{\frac{v}{2}} \le \operatorname{tr} AB.$$

An equivalent form of the following lemma has been given in [9]. For the reader's convenience, we give a short proof of this lemma based on Lemma 2.6 and Lemma 2.7.

Lemma 2.8 *Let A and B be positive semidefinite matrices, and let* $v \in \left[\frac{1}{2}, 1\right]$*. Then*

$$\left\|B^{\frac{v}{2}}A^{1-v}B^{\frac{v}{2}}+A^{\frac{v}{2}}B^{1-v}A^{\frac{v}{2}}\right\|_2 \le \|A+B\|_2. \tag{2.2}$$

Proof We can easily check that the square of the left hand side of the inequality (2.2) is equal to

$$\operatorname{tr}\left(\left(B^{\frac{v}{2}}A^{1-v}B^{\frac{v}{2}}\right)^2+\left(A^{\frac{v}{2}}B^{1-v}A^{\frac{v}{2}}\right)^2+2B^{\frac{v}{2}}A^{1-v}B^{\frac{v}{2}}A^{\frac{v}{2}}B^{1-v}A^{\frac{v}{2}}\right).$$

Hence, the inequality (2.2) is equivalent to the inequality

$$\begin{aligned} &\operatorname{tr}\left(\left(B^{\frac{v}{2}}A^{1-v}B^{\frac{v}{2}}\right)^2+\left(A^{\frac{v}{2}}B^{1-v}A^{\frac{v}{2}}\right)^2+2B^{\frac{v}{2}}A^{1-v}B^{\frac{v}{2}}A^{\frac{v}{2}}B^{1-v}A^{\frac{v}{2}}\right)\\ &\quad\le \operatorname{tr}\left(A^2+B^2+2AB\right). \end{aligned}$$

In view of Lemma 2.8 and Theorem 2.4, applied to the Hilbert–Schmidt norm and the case $f(t)=t$, we have the following corollary.

Corollary 2.9 *Let A and B be positive semidefinite matrices, and let $v\in\left[\frac{1}{2},1\right]$. Then*

$$\begin{aligned} \left|\left|A^{\frac{v}{2}}B^{1-\frac{v}{2}}+B^{\frac{v}{2}}A^{1-\frac{v}{2}}\right|\right|_2 &\le \frac{1}{2}\left(||A+B||_2+\left|\left|B^{\frac{v}{2}}A^{1-v}B^{\frac{v}{2}}+A^{\frac{v}{2}}B^{1-v}A^{\frac{v}{2}}\right|\right|_2\right)\\ &\le ||A+B||_2. \end{aligned}$$

It should be mentioned here that Corollary 2.9 can be concluded from Theorem 2.7 in [8], using a completely different analysis.

Corollary 2.10 *Let A and B be positive semidefinite matrices, and let $v\in\left[\frac{1}{4},\frac{3}{4}\right]$. Then*

$$\left|\left|A^{v}B^{1-v}+B^{v}A^{1-v}\right|\right|_2\le ||A+B||_2. \tag{2.3}$$

Proof Using Corollary 2.9, we have

$$\left|\left|A^{\frac{v}{2}}B^{1-\frac{v}{2}}+B^{\frac{v}{2}}A^{1-\frac{v}{2}}\right|\right|_2\le ||A+B||_2$$

for $v\in\left[\frac{1}{2},1\right]$. Hence, the inequality (2.3) is valid for $v\in\left[\frac{1}{4},\frac{1}{2}\right]$. Therefore,

$$\begin{aligned} ||A^{v}B^{1-v}+B^{v}A^{1-v}||_2 &= ||(A^{v}B^{1-v}+B^{v}A^{1-v})^*||_2\\ &= ||A^{1-v}B^{v}+B^{1-v}A^{v}||_2\\ &\le ||A+B||_2 \end{aligned}$$

is also valid for $1-v\in\left[\frac{1}{4},\frac{1}{2}\right]$, i.e., $v\in\left[\frac{1}{2},\frac{3}{4}\right]$. Hence, the inequality (2.3) is valid for $v\in\left[\frac{1}{4},\frac{3}{4}\right]$.

To prove our second main result in this paper, we need the following lemmas.

Lemma 2.11 (Matrix Young Inequality) *[1] Let A and B be $n\times n$ complex matrices. Then*

$$\sigma_i\left(AB^*\right)\le\sigma_i\left(\frac{1}{p}\left|A\right|^p+\frac{1}{q}\left|B\right|^q\right)$$

for $i=1,2,...,n$, and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Lemma 2.12 *[2] Let A and B be positive semidefinite matrices. Then*

$$|||A^r + B^r||| \leq |||(A+B)^r||| \text{ for } r \geq 1$$

and

$$|||(A+B)^r||| \leq |||A^r + B^r||| \text{ for } 0 < r \leq 1.$$

Theorem 2.13 *Let A and B be positive semidefinite matrices, and let $r \geq 1$. Then*

$$|||A^v B^{1-v} + B^v A^{1-v}||| \leq |||(A^{1/r} + B^{1/r})^r|||$$

for $\frac{1}{2r} \leq v \leq \frac{2r-1}{2r}$.

Proof Let $X = \begin{bmatrix} A^v & B^v \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} B^{1-v} & A^{1-v} \\ 0 & 0 \end{bmatrix}$. Then

$$XY^* = \begin{bmatrix} A^v B^{1-v} + B^v A^{1-v} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{aligned} |||A^v B^{1-v} + B^v A^{1-v}||| = |||XY^*||| &= |||\, |X||Y| \,||| \\ &\leq |||v|X|^{\frac{1}{v}} + (1-v)|Y|^{\frac{1}{(1-v)}}||| \\ &\leq v|||\, |X|^{\frac{1}{v}} ||| + (1-v)|||\, |Y|^{\frac{1}{(1-v)}} ||| \\ &= v|||\, |X^*|^{\frac{1}{v}} ||| + (1-v)|||\, |Y^*|^{\frac{1}{(1-v)}} |||, \end{aligned}$$

where the first inequality follows from Lemma 2.11, the second inequality follows from the triangle inequality, and the last equality follows using the fact that $|||\, |X|^r \,||| = |||\, |X^*|^r \,|||$ for any complex matrix X and for $r > 0$. Hence,

$$\begin{aligned} |||A^v B^{1-v} + B^v A^{1-v}||| \leq v|||(A^{2v} + B^{2v})^{\frac{1}{2v}}||| \\ +(1-v)|||(A^{2(1-v)} + B^{2(1-v)})^{\frac{1}{2(1-v)}}|||. \end{aligned}$$

Assume that $\frac{1}{2} \leq v \leq \frac{2r-1}{2r}$. Since $\frac{1}{2v} \leq 1$, it follows, by Lemma 2.12, that

$$|||(A^{2v} + B^{2v})^{1/2v}||| \leq |||A + B|||.$$

It is known [3, p. 95] that $|||X|||_{(r)} := |||\, |X|^r \,|||^{1/r}$ is a unitarily invariant norm for $r \geq 1$. Hence, again using Lemma 2.12, we have

$$\begin{aligned} |||(A^{2(1-v)} + B^{2(1-v)})^{\frac{1}{2(1-v)}}||| &= |||\, |(A^{2(1-v)} + B^{2(1-v)})^{\frac{1}{2r(1-v)}}| \,|||_{(r)}^r \\ &\leq |||A^{1/r} + B^{1/r}|||_{(r)}^r = |||(A^{1/r} + B^{1/r})^r|||. \end{aligned}$$

Hence, when $\frac{1}{2} \leq v \leq \frac{2r-1}{2r}$, we obtain

$$|||A^v B^{1-v} + B^v A^{1-v}||| \leq v|||A + B||| + (1 - v)|||(A^{1/r} + B^{1/r})^r|||.$$

Moreover, by Lemma 2.12, we have $|||A + B||| \leq |||(A^{1/r} + B^{1/r})^r|||$ and this gives the following inequality

$$|||A^v B^{1-v} + B^v A^{1-v}||| \leq |||(A^{1/r} + B^{1/r})^r|||. \tag{2.4}$$

Similarly, when $\frac{1}{2r} \leq v \leq \frac{1}{2}$, i.e., $\frac{1}{2} \leq 1 - v \leq \frac{2r-1}{2r}$, we again have the inequality (2.4). Therefore,

$$|||A^v B^{1-v} + B^v A^{1-v}||| \leq |||(A^{1/r} + B^{1/r})^r|||$$

for $\frac{1}{2r} \leq v \leq \frac{2r-1}{2r}$.

We conclude the paper with the following remark.

Remark 2.14 The case $v = \frac{1}{2}$ and $r = 1$ in Theorem 2.13 is the inequality

$$|||A^{1/2}B^{1/2} + B^{1/2}A^{1/2}||| \leq |||A + B|||,$$

which can also be concluded from the triangle inequality and the arithmetic–geometric mean inequality for unitarily invariant norms (see, e.g., [11]).

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