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ORIGINAL RESEARCH

Single-machine common due-window assignment and scheduling with position-dependent weights, delivery time, learning effect and resource allocations

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Abstract

Under common due window assignment, a single machine scheduling problem with learning effect, delivery time and convex resource allocation is considered. Actual processing time is related to normal processing time, job dependent learning effect and allocated resources. There are three objective functions are considered. They involve earliness, tardiness, due window costs and resource costs with position dependent weights. The first objective function is to minimize the total costs of earliness, tardiness, start time of window, window size and resource allocation; the second objective function is to minimize the total costs of earliness, tardiness, start time of window and window size under resource-limited conditions; the third objective function is to minimize the cost of resource allocation under the scheduling function constraint. The goal is to determine the optimal sequence and resource allocation. All three problems are proved that they can be solved in polynomial time and polynomial time algorithms are given separately.

Keywords Scheduling · Common due window · Learning effect · Delivery time · Convex resource allocation · Position-dependent weight

Mathematics Subject Classification 90B35 · 90-08

1 Introduction

With the innovation of technology, the processing efficiency is improving. The processing time is getting shorter and shorter, this is the learning effect (Wang and Xia

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[1], Wang et al. [2], Azzouz et al. [3], Liang et al. [4], Wang et al. [5]). Qian and Zhan [6] and Qian [7] studied a single machine scheduling problem with learning effect and group technique. In 2022, Wang et al. [8] studied a single machine scheduling problem with general truncated learning effects. In 2022, Gao et al. [9] studied a single machine scheduling problem with DeJong's learning effect and maintenance activity. In 2023, Ferraro et al. [10] studied a flowshop scheduling problem with learning effect.

When a job is processed, the additional time that it is delivered to the customer is called delivery time (Koulamas and Kyparisis [11]). In some scheduling environments, delivery time is used to eliminate adverse effects on the job, which does not occupy any machine. In 2021, Sun et al. [12] studied a parallel machine scheduling problem with maintenance activity, delivery times and resource allocation. Qian and Zhan [13] studied a single machine scheduling problem with learning effect, delivery time and due date. In 2022, Wang et al. [14] studied the single machine scheduling problem with delivery times and variable processing times. Qian and Han [15] studied a single machine scheduling problem with deteriorating jobs and delivery time. Zhang et al. [16] studied the parallel machine scheduling problem with delivery time and due date. Qian and Zhan [17], Qian and Han [18], and Qian and Chang [19] studied the due window assignment problems with delivery time. In 2023, Wang et al. [20], Ren et al. [21] and Ren et al. [22] considered the single machine delivery times scheduling problems with learning effects. Pan et al. [23] considered single-machine delivery times scheduling with deteriorating jobs.

In some practical scheduling environments, the processing time is related to resources. For example, a steel production process needs to be preheated, the more air resources are given, the shorter the preheating time (Shabtay and Steiner [24], Yedidsiona and Shabtay [25]). In 2014, Wang and Wang [26] studied the single machine scheduling problems with learning effect and resource allocation. Under common due-window, they proved that some problems can be solved in polynomial time. In 2019, Wang and Liang [27] studied a single machine scheduling problem with deteriorating jobs, group technology and resource allocation. Sun at al. [28] studied a no-wait flow-shop scheduling problem with learning effect and resource allocation. Geng at al. [29] studied a no-wait flowshop with learning effect and resource allocation. In 2020, Liu and Jiang [30] studied a single machine scheduling problem with learning effects and resource allocation. Shi and Wang [31] studied a flowshop scheduling problem with learning effect and resource allocation. Sun et al. [32] studied a single machine scheduling problem with group technology, resource allocation and learning effect. In 2021, Lv and Wang [33] studied no-wait flow shop scheduling with resource allocation and learning effect. In 2022, Yan et al. [34] studied a single machine scheduling problem with group technology, resource allocation and deteriorating effect. In 2023, Zhang et al. [35] and Wang et al. [36] considered single-machine scheduling with resource allocation and deteriorating jobs. Wang and Wang [37] considered single-machine scheduling with resource allocation and time-dependent learning effect. Shioura et al. [38] considered parallel machine scheduling with resource allocation.

This paper studied a single machine scheduling problem with learning effect, delivery time and convex resource allocation. The motivation comes from references Koulamas and Kyparisis [11] and Wang and Wang [26]. For three objective functions problems, the polynomial-time algorithms are proposed. The problem is described in Sect. 2. The proofs of the polynomial time algorithm are given from Sects. 3–5. Three examples are presented to illustrate the process of each algorithm in Sect. 6. The summary is given in Sect. 7.

2 Notation and problem statement

There are n independent jobs $J = \{J_1, \ldots J_n\}$ continuously processed. The normal processing time of J_j is $\overline{p}_j$. As in Wang and Wang [26], if J_i is at the kth position, the actual processing time is

$$p_{[k]} = \left(\frac{\overline{p}_i k^{\beta_i}}{u_i}\right)^{\theta}, \tag{1}$$

where u_i represents the resources allocated to J_i, β_i is the learning rate, $\beta_i < 0, \theta > 0$. The job at the kth position is represented by the subscript $[k]$. The waiting time of $J_{[k]}$ is

$$w_{[k]} = \sum\nolimits_{i=1}^{k-1} p_{[i]}. \tag{2}$$

As in Koulamas and Kyparisis [11], the delivery time $q_{[k]}$ of $J_{[k]}$ is

$$q_{[k]} = \alpha w_{[k]} = \alpha \sum_{i=1}^{k-1} p_{[i]}, \tag{3}$$

where α is the delivery rate, $\alpha > 0$. The completion time of $J_{[k]}$ is

$$C_{[k]} = w_{[k]} + p_{[k]} + q_{[k]}. \tag{4}$$

The makespan is

$$C_{\max} = \max_{1 \le k \le n} C_{[k]}. \tag{5}$$

In this paper, the common due window (CONW) is considered. For the CONW, each job has the same window, that is, the same start time $\overline{d}$ and end time $\overline{\overline{d}}$. The size of window is

$$D = \overline{\overline{d}} - \overline{d}. \tag{6}$$

The earliness of $J_{[k]}$ is

$$E_{[k]} = \max\left\{0, \overline{d} - C_{[k]}\right\}. \tag{7}$$

The tardiness of $J_{[k]}$ is

$$T_{[k]} = \max\{0, C_{[k]} - \overline{\overline{d}}\}. \tag{8}$$

There are three objective functions are considered.
(1) The first objective function is to minimize the total costs of earliness, tardiness, start time of window, window size and resource allocation (see Graham [39]), i.e.,

$$1|p_{[k]} = \left(\frac{\overline{p}_i k^{\beta_i}}{u_i}\right)^{\theta}, q_{psd}, CONW|\sum_{k=1}^{n}\Big[a_k E_{[k]} + b_k T_{[k]} + c_k \overline{d} + d_k D\Big] + e\sum_{k=1}^{n} g_{[k]}u_{[k]}, \tag{9}$$

where q_{psd} represents the past-sequence-dependent delivery times, a_k, b_k, c_k, d_k and e are given positive constants, $1 \le k \le n$ (i.e., position-dependent weights, see Wang et al. [40–42]).
(2) The second objective function is to minimize the total costs of earliness, tardiness, start time of window and window size subject to $\sum_{k=1}^{n} g_{[k]}u_{[k]} \le W$, where W is the total number of resources, i.e.,

$$1|p_{[k]} = \left(\frac{\overline{p}_i k^{\beta_i}}{u_i}\right)^{\theta}, q_{psd}, CONW, \sum_{k=1}^{n} g_{[k]}u_{[k]} \le W|\sum_{k=1}^{n}[a_k E_{[k]} + b_k T_{[k]} + c_k \overline{d} + d_k D]. \tag{10}$$

(3) The third objective function is to minimize the cost of resource allocation subject to $\sum_{k=1}^{n}[a_k E_{[k]} + b_k T_{[k]} + c_k \overline{d} + d_k D] \le M$, where M is a given constant, i.e.,

$$1|p_{[k]} = \left(\frac{\overline{p}_i k^{\beta_i}}{u_i}\right)^{\theta}, q_{psd}, CONW, \sum_{k=1}^{n}[a_k E_{[k]} + b_k T_{[k]} + c_k \overline{d} + d_k D] \le M|\sum_{k=1}^{n} g_{[k]}u_{[k]}. \tag{11}$$

3 The problem $1|p_{[k]} = (\frac{\overline{p}_i k^{\beta_i}}{u_i})^{\theta}, q_{psd}, CONW|\sum_{k=1}^{n}[a_k E_{[k]} + b_k T_{[k]} + c_k \overline{d} + d_k D] + e\sum_{k=1}^{n} g_{[k]}u_{[k]}$

Lemma 3.1 *For any job sequence, $\overline{d}$ of the optimal scheduling is the completion time of some job.*

Proof (1) Suppose that $\overline{d}$ isn't the completion time of some job and $\overline{\overline{d}}$ is the completion time of some job, i.e., $C_{[j_1-1]} < \overline{d} < C_{[j_1]}, \overline{\overline{d}} = C_{[j_2]}, 1 \leq j_1 \leq j_2 \leq n$. The objective function is

$$\begin{aligned} Z &= \sum_{k=1}^{j_1-1} a_k\left(\overline{d} - C_{[k]}\right) + \sum_{k=j_2+1}^{n} b_k\left(C_{[k]} - C_{[j_2]}\right) \\ &\quad + \sum_{k=1}^{n} c_k\overline{d} + \sum_{k=1}^{n} d_k\left(C_{[j_2]} - \overline{d}\right) + e\sum_{k=1}^{n} g_{[k]}u_{[k]} \\ &= -\sum_{k=1}^{j_1-1} a_kC_{[k]} + \sum_{k=j_2+1}^{n} b_kC_{[k]} + \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k\right. \\ &\quad \left. - \sum_{k=1}^{n} d_k\right)\overline{d} + \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2+1}^{n} b_k\right)C_{[j_2]} + e\sum_{k=1}^{n} g_{[k]}u_{[k]}. \end{aligned} \tag{12}$$

When $\overline{d} = C_{[j_1-1]}$, the objective function is

$$\begin{aligned} Z_1 &= -\sum_{k=1}^{j_1-2} a_kC_{[k]} + \sum_{k=j_2+1}^{n} b_kC_{[k]} + \left(\sum_{k=1}^{j_1-2} a_k + \sum_{k=1}^{n} c_k\right. \\ &\quad \left. - \sum_{k=1}^{n} d_k\right)C_{[j_1-1]} + \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2+1}^{n} b_k\right)C_{[j_2]} + e\sum_{k=1}^{n} g_{[k]}u_{[k]}. \end{aligned} \tag{13}$$

When $\overline{d} = C_{[j_1]}$, the objective function is

$$\begin{aligned} Z_2 &= -\sum_{k=1}^{j_1-1} a_kC_{[k]} + \sum_{k=j_2+1}^{n} b_kC_{[k]} + \left(\sum_{k=1}^{j_1-1} a_k\right. \\ &\quad \left. + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right)C_{[j_1]} + \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2+1}^{n} b_k\right)C_{[j_2]} + e\sum_{k=1}^{n} g_{[k]}u_{[k]}. \end{aligned} \tag{14}$$

$$Z - Z_1 = \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right)\left(\overline{d} - C_{[j_1-1]}\right), \tag{15}$$

$$Z - Z_2 = \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right)\left(\overline{d} - C_{[j_1]}\right). \tag{16}$$

When $\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k \geq 0, Z_2 \geq Z \geq Z_1$; otherwise, $Z_1 > Z > Z_2$. So d_1 is the completion time of some job.

(2) Suppose that $\overline{d}$ and $\overline{\overline{d}}$ aren't the completion time of some job, i.e., $C_{[j_1-1]} < \overline{d} < C_{[j_1]}$, $C_{[j_2-1]} < \overline{\overline{d}} < C_{[j_2]}$, $1 \leq j_1 \leq j_2 \leq n$. The objective function is

$$Z = -\sum_{k=1}^{j_1-1} a_k C_{[k]} + \sum_{k=j_2}^{n} b_k C_{[k]} + \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k \right) \overline{d} + \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2}^{n} b_k \right) \overline{\overline{d}} + e \sum_{k=1}^{n} g_{[k]} u_{[k]}. \tag{17}$$

When $\overline{d} = C_{[j_1-1]}$, the objective function is

$$Z_3 = -\sum_{k=1}^{j_1-2} a_k C_{[k]} + \sum_{k=j_2}^{n} b_k C_{[k]} + \left(\sum_{k=1}^{j_1-2} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k \right) C_{[j_1-1]} + \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2}^{n} b_k \right) \overline{\overline{d}} + e \sum_{k=1}^{n} g_{[k]} u_{[k]}. \tag{18}$$

When $\overline{d} = C_{[j_1]}$, the objective function is

$$Z_4 = -\sum_{k=1}^{j_1-1} a_k C_{[k]} + \sum_{k=j_2}^{n} b_k C_{[k]} + \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k \right) C_{[j_1]} + \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2}^{n} b_k \right) \overline{\overline{d}} + e \sum_{k=1}^{n} g_{[k]} u_{[k]}. \tag{19}$$

$$Z - Z_3 7 \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k \right) \left(\overline{d} - C_{[j_1-1]} \right), \tag{20}$$

$$Z - Z_4 = \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k \right) \left(\overline{d} - C_{[j_1]} \right). \tag{21}$$

When $\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k \geq 0$, $Z_4 \geq Z \geq Z_3$; otherwise, $Z_3 > Z > Z_4$. So d_1 is the completion time of some job. □

Lemma 3.2 *For any job sequence, $\overline{\overline{d}}$ of the optimal scheduling is the completion time of some job.*

Proof (1) Suppose that $\overline{d}$ is the completion time of some job and $\overline{\overline{d}}$ isn't the completion time of some job, i.e., $\overline{d} = C_{[j_1]}$, $C_{[j_2-1]} < \overline{\overline{d}} < C_{[j_2]}$, $1 \leq j_1 < j_2 \leq n$. The objective function is

$$Z = -\sum_{k=1}^{j_1-1} a_k C_{[k]} + \sum_{k=j_2}^{n} b_k C_{[k]} + \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right) C_{[j_1]} + \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2}^{n} b_k\right) \overline{\overline{d}} + e \sum_{k=1}^{n} g_{[k]} u_{[k]}. \tag{22}$$

When $\overline{\overline{d}} = C_{[j_2-1]}$, the objective function is

$$Z_1 = -\sum_{k=1}^{j_1-1} a_k C_{[k]} + \sum_{k=j_2}^{n} b_k C_{[k]} + \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right) C_{[j_1]} + \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2}^{n} b_k\right) C_{[j_2-1]} + e \sum_{k=1}^{n} g_{[k]} u_{[k]}. \tag{23}$$

When $\overline{\overline{d}} = C_{[j_2]}$, the objective function is

$$Z_2 = -\sum_{k=1}^{j_1-1} a_k C_{[k]} + \sum_{k=j_2+1}^{n} b_k C_{[k]} + \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right) C_{[j_1]} + \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2+1}^{n} b_k\right) C_{[j_2]} + e \sum_{k=1}^{n} g_{[k]} u_{[k]}. \tag{24}$$

$$Z - Z_1 = \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2}^{n} b_k\right) \left(\overline{\overline{d}} - C_{[j_2-1]}\right), \tag{25}$$

$$Z - Z_2 = \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2}^{n} b_k\right) \left(\overline{\overline{d}} - C_{[j_2]}\right). \tag{26}$$

When $\sum_{k=1}^{n} d_k - \sum_{k=j_2}^{n} b_k \geq 0$, $Z_2 \geq Z \geq Z_1$; otherwise, $Z_1 > Z > Z_2$. So $\overline{\overline{d}}$ is the completion time of some job.

(2) Suppose that $\overline{d}$ and $\overline{\overline{d}}$ aren't the completion time of some job, i.e., $C_{[j_1-1]} < \overline{d} < C_{[j_1]}$, $C_{[j_2-1]} < \overline{\overline{d}} < C_{[j_2]}$, $1 \leq j_1 \leq j_2 \leq n$. The objective function is

$$Z = -\sum_{k=1}^{j_1-1} a_k C_{[k]} + \sum_{k=j_2}^{n} b_k C_{[k]} + \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right) \overline{d}$$

$$+\left(\sum_{k=1}^{n} d_k - \sum_{k=j_2}^{n} b_k\right)\overline{\overline{d}} + e\sum_{k=1}^{n} g_{[k]}u_{[k]}. \tag{27}$$

When $\overline{\overline{d}} = C_{[j_2-1]}$, the objective function is

$$Z_3 = -\sum_{k=1}^{j_1-1} a_k C_{[k]} + \sum_{k=j_2}^{n} b_k C_{[k]} + \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right)\overline{d}$$
$$+\left(\sum_{k=1}^{n} d_k - \sum_{k=j_2}^{n} b_k\right) C_{[j_2-1]} + e\sum_{k=1}^{n} g_{[k]}u_{[k]}. \tag{28}$$

When $\overline{\overline{d}} = C_{[j_2]}$, the objective function is

$$Z_4 = -\sum_{k=1}^{j_1-1} a_k C_{[k]} + \sum_{k=j_2+1}^{n} b_k C_{[k]} + \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right)\overline{d}$$
$$+\left(\sum_{k=1}^{n} d_k - \sum_{k=j_2+1}^{n} b_k\right) C_{[j_2]} + e\sum_{k=1}^{n} g_{[k]}u_{[k]}. \tag{29}$$

$$Z - Z_3 = \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2}^{n} b_k\right)\left(\overline{\overline{d}} - C_{[j_2-1]}\right), \tag{30}$$

$$Z - Z_4 = \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2}^{n} b_k\right)\left(\overline{\overline{d}} - C_{[j_2]}\right). \tag{31}$$

When $\sum_{k=1}^{n} d_k - \sum_{k=j_2}^{n} b_k \geq 0$, $Z_4 \geq Z \geq Z_3$; otherwise, $Z_3 > Z > Z_4$. So $\overline{\overline{d}}$ is the completion time of some job. □

Lemma 3.3 *For the optimal scheduling, $\overline{d}$ is equal to the j_1th job completion time $C_{[j_1]}$, j_1 satisfies $\sum_{k=1}^{j_1-1} a_k \leq \sum_{k=1}^{n} d_k - \sum_{k=1}^{n} c_k \leq \sum_{k=1}^{j_1} a_k$; $\overline{\overline{d}}$ is equal to the j_2th job completion time $C_{[j_2]}$, j_2 satisfies $\sum_{k=j_2+1}^{n} b_k \leq \sum_{k=1}^{n} d_k \leq \sum_{k=j_2}^{n} b_k$.*

Proof When $\overline{d} = C_{[j_1]}$ and $\overline{\overline{d}} = C_{[j_2]}$, the objective function is

$$Z = -\sum_{k=1}^{j_1-1} a_k C_{[k]} + \sum_{k=j_2+1}^{n} b_k C_{[k]} + \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right) C_{[j_1]}$$
$$+\left(\sum_{k=1}^{n} d_k - \sum_{k=j_2+1}^{n} b_k\right) C_{[j_2]} + e\sum_{k=1}^{n} g_{[k]}u_{[k]}. \tag{32}$$

(1) When $\overline{d} = C_{[j_1-1]}$ and $\overline{\overline{d}} = C_{[j_2]}$, the objective function is

$$Z_1 = -\sum_{k=1}^{j_1-2} a_k C_{[k]} + \sum_{k=j_2+1}^{n} b_k C_{[k]} + \left(\sum_{k=1}^{j_1-2} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right) C_{[j_1-1]} + \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2+1}^{n} b_k\right) C_{[j_2]} + e\sum_{k=1}^{n} g_{[k]} u_{[k]}. \tag{33}$$

Because the optimal position is j_1,

$$Z - Z_1 = \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right)(C_{[j_1]} - C_{[j_1-1]}) \le 0, \tag{34}$$

$$\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k \le 0. \tag{35}$$

When $\overline{d} = C_{[j_1+1]}$ and $\overline{\overline{d}} = C_{[j_2]}$, the objective function is

$$Z_2 = -\sum_{k=1}^{j_1} a_k C_{[k]} + \sum_{k=j_2+1}^{n} b_k C_{[k]} + \left(\sum_{k=1}^{j_1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right) C_{[j_1+1]} + \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2+1}^{n} b_k\right) C_{[j_2]} + e\sum_{k=1}^{n} g_{[k]} u_{[k]}. \tag{36}$$

Because the optimal position is j_1,

$$Z - Z_2 = \left(\sum_{k=1}^{j_1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right)(C_{[j_1]} - C_{[j_1+1]}) \le 0, \tag{37}$$

$$\sum_{k=1}^{j_1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k \ge 0. \tag{38}$$

So the best position j_1 satisfies $\sum_{k=1}^{j_1-1} a_k \le \sum_{k=1}^{n} d_k - \sum_{k=1}^{n} c_k \le \sum_{k=1}^{j_1} a_k$.
(2) When $\overline{d} = C_{[j_1]}$ and $\overline{\overline{d}} = C_{[j_2-1]}$, the objective function is

$$Z_3 = -\sum_{k=1}^{j_1-1} a_k C_{[k]} + \sum_{k=j_2}^{n} b_k C_{[k]} + \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right) C_{[j_1]}$$
$$+ \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2}^{n} b_k\right) C_{[j_2-1]} + e\sum_{k=1}^{n} g_{[k]} u_{[k]}. \tag{39}$$

Because the optimal position is j_2,

$$Z - Z_3 = \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2}^{n} b_k\right)(C_{[j_2]} - C_{[j_2-1]}) \le 0, \tag{40}$$
$$\sum_{k=1}^{n} d_k \le \sum_{k=j_2}^{n} b_k. \tag{41}$$

When $\overline{d} = C_{[j_1]}$ and $\overline{\overline{d}} = C_{[j_2+1]}$, the objective function is

$$Z_4 = -\sum_{k=1}^{j_1-1} a_k C_{[k]} + \sum_{k=j_2+2}^{n} b_k C_{[k]} + \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right) C_{[j_1]}$$
$$+ \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2+2}^{n} b_k\right) C_{[j_2+1]} + e\sum_{k=1}^{n} g_{[k]} u_{[k]}. \tag{42}$$

Because the optimal position is j_2,

$$Z - Z_4 = \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2+1}^{n} b_k\right)(C_{[j_2]} - C_{[j_2+1]}) \le 0, \tag{43}$$
$$\sum_{k=j_2+1}^{n} b_k \le \sum_{k=1}^{n} d_k. \tag{44}$$

So the best position j_2 satisfies $\sum_{k=j_2+1}^{n} b_k \le \sum_{k=1}^{n} d_k \le \sum_{k=j_2}^{n} b_k$. □

Lemma 3.4 *For the problem* $1|p_{[k]} = (\frac{\overline{p}_i k^{\beta_i}}{u_i})^\theta, q_{psd}, CONW|\sum_{k=1}^{n}[a_k E_{[k]} + b_k T_{[k]} + c_k\overline{d} + d_k D] + e\sum_{k=1}^{n} g_{[k]} u_{[k]}$, *the optimal resource allocation is*

$$u^*_{[k]} = \frac{\theta^{\frac{1}{\theta+1}} \left[(1+\alpha)\left(\sum_{j=1}^{k} a_j + \sum_{j=1}^{n} c_j\right) - a_k\right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{(e g_{[k]})^{\frac{1}{\theta+1}}}, k = 1, \ldots, j_1 - 1;$$

$$u^*_{[j_1]} = \frac{\theta^{\frac{1}{\theta+1}} \left(\alpha \sum_{j=1}^{n} d_j + \sum_{j=1}^{j_1-1} a_j + \sum_{j=1}^{n} c_j\right)^{\frac{1}{\theta+1}} (\overline{p}_{[j_1]} {j_1}^{\beta_{[j_1]}})^{\frac{\theta}{\theta+1}}}{(eg_{[j_1]})^{\frac{1}{\theta+1}}}, k = j_1;$$

$$u^*_{[k]} = \frac{\theta^{\frac{1}{\theta+1}} \left[(1+\alpha) \sum_{j=1}^{n} d_j\right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{(eg_{[k]})^{\frac{1}{\theta+1}}}, k = j_1+1, \ldots, j_2-1;$$

$$u^*_{[j_2]} = \frac{\theta^{\frac{1}{\theta+1}} \left(\alpha \sum_{j=j_2+1}^{n} b_j + \sum_{j=1}^{n} d_j\right)^{\frac{1}{\theta+1}} (\overline{p}_{[j_2]} {j_2}^{\beta_{[j_2]}})^{\frac{\theta}{\theta+1}}}{(eg_{[j_2]})^{\frac{1}{\theta+1}}}, k = j_2;$$

$$u^*_{[k]} = \frac{\theta^{\frac{1}{\theta+1}} \left[(1+\alpha) \sum_{j=k+1}^{n} b_j + b_k\right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{(eg_{[k]})^{\frac{1}{\theta+1}}}, k = j_2+1, \ldots, n-1;$$

$$u^*_{[n]} = \frac{\theta^{\frac{1}{\theta+1}} b_n^{\frac{1}{\theta+1}} \left(\overline{p}_{[n]} n^{\beta_{[n]}}\right)^{\frac{\theta}{\theta+1}}}{(eg_{[n]})^{\frac{1}{\theta+1}}}, k = n. \tag{45}$$

Proof

$$C_{[k]} = w_{[k]} + p_{[k]} + \alpha w_{[k]} = (1+\alpha) \sum_{i=1}^{k-1} p_{[i]} + p_{[k]}. \tag{46}$$

When $\overline{d} = C_{[j_1]}$ and $\overline{\overline{d}} = C_{[j_2]}$, the objective function is

$$\begin{aligned}
Z = & -\sum_{k=1}^{j_1-1} a_k C_{[k]} + \sum_{k=j_2+1}^{n} b_k C_{[k]} + \left(\sum_{k=1}^{j_1-1} a_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right) C_{[j_1]} \\
& + \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2+1}^{n} b_k\right) C_{[j_2]} + e \sum_{k=1}^{n} g_{[k]} u_{[k]} \\
= & \sum_{k=1}^{j_1-1} \left[(1+\alpha)\left(\sum_{j=1}^{k} a_j + \sum_{j=1}^{n} c_j\right) - a_k\right] \left(\frac{\overline{p}_{[k]} k^{\beta_{[k]}}}{u_{[k]}}\right)^{\theta} \\
& + \left(\alpha \sum_{j=1}^{n} d_j + \sum_{j=1}^{j_1-1} a_j + \sum_{j=1}^{n} c_j\right) \left(\frac{\overline{p}_{[j_1]} {j_1}^{\beta_{[j_1]}}}{u_{[j_1]}}\right)^{\theta} \\
& + \sum_{k=j_1+1}^{j_2-1} (1+\alpha) \sum_{j=1}^{n} d_j \left(\frac{\overline{p}_{[k]} k^{\beta_{[k]}}}{u_{[k]}}\right)^{\theta} + \left(\alpha \sum_{j=j_2+1}^{n} b_j + \sum_{j=1}^{n} d_j\right) \left(\frac{\overline{p}_{[j_2]} {j_2}^{\beta_{[j_2]}}}{u_{[j_2]}}\right)^{\theta} \\
& + \sum_{k=j_2+1}^{n-1} \left[(1+\alpha) \sum_{j=k+1}^{n} b_j + b_k\right] \left(\frac{\overline{p}_{[k]} k^{\beta_{[k]}}}{u_{[k]}}\right)^{\theta} + b_n \left(\frac{\overline{p}_{[n]} n^{\beta_{[n]}}}{u_{[n]}}\right)^{\theta} + e \sum_{k=1}^{n} g_{[k]} u_{[k]}.
\end{aligned} \tag{47}$$

$$\frac{\partial Z}{\partial u_{[k]}} = eg_{[k]} - \theta \left[(1+\alpha)(\sum_{j=1}^{k} a_j + \sum_{j=1}^{n} c_j) - a_k\right] \frac{(\overline{p}_{[k]} k^{\beta_{[k]}})^{\theta}}{u_{[k]}^{\theta+1}} = 0, k = 1, \ldots, j_1-1;$$

$$\frac{\partial Z}{\partial u_{[j_1]}} = eg_{[j_1]} - \theta\left(\alpha\sum_{j=1}^{n} d_j + \sum_{j=1}^{j_1-1} a_j + \sum_{j=1}^{n} c_j\right)\frac{(\overline{p}_{[j_1]} {j_1}^{\beta_{[j_1]}})^{\theta}}{u_{[j_1]}^{\theta+1}} = 0, k = j_1;$$
$$\frac{\partial Z}{\partial u_{[k]}} = eg_{[k]} - \theta(1+\alpha)\sum_{j=1}^{n} d_j \frac{(\overline{p}_{[k]} k^{\beta_{[k]}})^{\theta}}{u_{[k]}^{\theta+1}} = 0, k = j_1+1, \ldots, j_2-1;$$
$$\frac{\partial Z}{\partial u_{[j_2]}} = eg_{[j_2]} - \theta\left(\alpha\sum_{j=j_2+1}^{n} b_j + \sum_{j=1}^{n} d_j\right)\frac{(\overline{p}_{[j_2]} {j_2}^{\beta_{[j_2]}})^{\theta}}{u_{[j_2]}^{\theta+1}} = 0, k = j_2;$$
$$\frac{\partial Z}{\partial u_{[k]}} = eg_{[k]} - \theta\left[(1+\alpha)\sum_{j=k+1}^{n} b_j + b_k\right]\frac{(\overline{p}_{[k]} k^{\beta_{[k]}})^{\theta}}{u_{[k]}^{\theta+1}} = 0, k = j_2+1, \ldots, n-1;$$
$$\frac{\partial Z}{\partial u_{[n]}} = eg_{[n]} - \theta b_n \frac{(\overline{p}_{[n]} n^{\beta_{[n]}})^{\theta}}{u_{[n]}^{\theta+1}} = 0, k = n; \tag{48}$$

We have

$$u_{[k]}^{*} = \frac{\theta^{\frac{1}{\theta+1}}\left[(1+\alpha)\left(\sum_{j=1}^{k} a_j + \sum_{j=1}^{n} c_j\right) - a_k\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{(eg_{[k]})^{\frac{1}{\theta+1}}}, k = 1, \ldots, j_1-1;$$
$$u_{[j_1]}^{*} = \frac{\theta^{\frac{1}{\theta+1}}\left(\alpha\sum_{j=1}^{n} d_j + \sum_{j=1}^{j_1-1} a_j + \sum_{j=1}^{n} c_j\right)^{\frac{1}{\theta+1}}(\overline{p}_{[j_1]} {j_1}^{\beta_{[j_1]}})^{\frac{\theta}{\theta+1}}}{(eg_{[j_1]})^{\frac{1}{\theta+1}}}, k = j_1;$$
$$u_{[k]}^{*} = \frac{\theta^{\frac{1}{\theta+1}}\left[(1+\alpha)\sum_{j=1}^{n} d_j\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{(eg_{[k]})^{\frac{1}{\theta+1}}}, k = j_1+1, \ldots, j_2-1;$$
$$u_{[j_2]}^{*} = \frac{\theta^{\frac{1}{\theta+1}}\left(\alpha\sum_{j=j_2+1}^{n} b_j + \sum_{j=1}^{n} d_j\right)^{\frac{1}{\theta+1}}(\overline{p}_{[j_2]} {j_2}^{\beta_{[j_2]}})^{\frac{\theta}{\theta+1}}}{(eg_{[j_2]})^{\frac{1}{\theta+1}}}, k = j_2; \tag{49}$$
$$u_{[k]}^{*} = \frac{\theta^{\frac{1}{\theta+1}}\left[(1+\alpha)\sum_{j=k+1}^{n} b_j + b_k\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{(eg_{[k]})^{\frac{1}{\theta+1}}}, k = j_2+1, \ldots, n-1;$$
$$u_{[n]}^{*} = \frac{\theta^{\frac{1}{\theta+1}} b_n^{\frac{1}{\theta+1}}\left(\overline{p}_{[n]} n^{\beta_{[n]}}\right)^{\frac{\theta}{\theta+1}}}{(eg_{[n]})^{\frac{1}{\theta+1}}}, k = n.$$

□

Substitute the optimal resource allocation into the objective function

$$Z = e^{\frac{\theta}{\theta+1}}\left(\theta^{\frac{1}{\theta+1}} + \theta^{-\frac{\theta}{\theta+1}}\right)\left\{\sum_{k=1}^{j_1-1}\left[(1+\alpha)\left(\sum_{j=1}^{k} a_j + \sum_{j=1}^{n} c_j\right) - a_k\right]^{\frac{1}{\theta+1}}\left(\overline{p}_{[k]} g_{[k]} k^{\beta_{[k]}}\right)^{\frac{\theta}{\theta+1}}\right.$$
$$+\left(\alpha\sum_{j=1}^{n} d_j + \sum_{j=1}^{j_1-1} a_j + \sum_{j=1}^{n} c_j\right)^{\frac{1}{\theta+1}}(\overline{p}_{[j_1]} g_{[j_1]} {j_1}^{\beta_{[j_1]}})^{\frac{\theta}{\theta+1}}$$

$$
+\sum_{k=j_1+1}^{j_2-1}\left[(1+\alpha)\sum_{j=1}^{n}d_j\right]^{\frac{1}{\theta+1}}\left(\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}}\right)^{\frac{\theta}{\theta+1}}
+\left(\alpha\sum_{j=j_2+1}^{n}b_j+\sum_{j=1}^{n}d_j\right)^{\frac{1}{\theta+1}}\left(\overline{p}_{[j_2]}g_{[j_2]}j_2{}^{\beta_{[j_2]}}\right)^{\frac{\theta}{\theta+1}}
+\sum_{k=j_2+1}^{n-1}\left[(1+\alpha)\sum_{j=k+1}^{n}b_j+b_k\right]^{\frac{1}{\theta+1}}\left(\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}}\right)^{\frac{\theta}{\theta+1}}
+b_n^{\frac{1}{\theta+1}}\left(\overline{p}_{[n]}g_{[n]}n^{\beta_{[n]}}\right)^{\frac{\theta}{\theta+1}}\Bigg\}. \tag{50}
$$

Minimizing Z is the same thing as minimizing $\overline{Z}$,

$$
\overline{Z}=\sum_{k=1}^{j_1-1}\left[(1+\alpha)\left(\sum_{j=1}^{k}a_j+\sum_{j=1}^{n}c_j\right)-a_k\right]^{\frac{1}{\theta}}\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}}
+\left(\alpha\sum_{j=1}^{n}d_j+\sum_{j=1}^{j_1-1}a_j+\sum_{j=1}^{n}c_j\right)^{\frac{1}{\theta}}\overline{p}_{[j_1]}g_{[j_1]}j_1{}^{\beta_{[j_1]}}
+\sum_{k=j_1+1}^{j_2-1}\left[(1+\alpha)\sum_{j=1}^{n}d_j\right]^{\frac{1}{\theta}}\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}}+\left(\alpha\sum_{j=j_2+1}^{n}b_j+\sum_{j=1}^{n}d_j\right)^{\frac{1}{\theta}}\overline{p}_{[j_2]}g_{[j_2]}j_2{}^{\beta_{[j_2]}}
+\sum_{k=j_2+1}^{n-1}\left[(1+\alpha)\sum_{j=k+1}^{n}b_j+b_k\right]^{\frac{1}{\theta}}\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}}+b_n^{\frac{1}{\theta}}\overline{p}_{[n]}g_{[n]}n^{\beta_{[n]}}. \tag{51}
$$

$\overline{Z}$ could be computed by the assignment problem.

$$
\min \sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_{ik}z_{ik}
\quad s.t.\quad \begin{cases}\sum_{i=1}^{n}z_{ik}=1, k=1,\ldots,n;\\ \sum_{k=1}^{n}z_{ik}=1, i=1,\ldots,n;\\ z_{ik}=0 \; or \; 1, i,k=1,\ldots,n,\end{cases} \tag{52}
$$

where

$$
\gamma_{ik}=\left[(1+\alpha)\left(\sum_{j=1}^{k}a_j+\sum_{j=1}^{n}c_j\right)-a_k\right]^{\frac{1}{\theta}}\overline{p}_i g_i k^{\beta_i}, k=1,\ldots,j_1-1;
$$

$$\gamma_{ik} = \left(\alpha \sum_{j=1}^{n} d_j + \sum_{j=1}^{j_1-1} a_j + \sum_{j=1}^{n} c_j\right)^{\frac{1}{\theta}} \overline{p}_i g_i {j_1}^{\beta_i}, k = j_1;$$

$$\gamma_{ik} = \left[(1+\alpha) \sum_{j=1}^{n} d_j\right]^{\frac{1}{\theta}} \overline{p}_i g_i k^{\beta_i}, k = j_1+1, \ldots, j_2-1;$$

$$\gamma_{ik} = \left(\alpha \sum_{j=j_2+1}^{n} b_j + \sum_{j=1}^{n} d_j\right)^{\frac{1}{\theta}} \overline{p}_i g_i {j_2}^{\beta_i}, k = j_2;$$

$$\gamma_{ik} = \left[(1+\alpha) \sum_{j=k+1}^{n} b_j + b_k\right]^{\frac{1}{\theta}} \overline{p}_i g_i k^{\beta_i}, k = j_2+1, \ldots, n-1;$$

$$\gamma_{ik} = b_n^{\frac{1}{\theta}} \overline{p}_i g_i n^{\beta_i}, k = n. \quad (53)$$

The algorithm is summarized as follows:

Algorithm 1 $1|p_{[k]} = (\frac{\overline{p}_i k^{\beta_i}}{u_i})^{\theta}, q_{psd}, CONW| \sum_{k=1}^{n} [a_k E_{[k]} + b_k T_{[k]} + c_k \overline{d} + d_k D] + e \sum_{k=1}^{n} g_{[k]} u_{[k]}$

Input: $a_k, b_k, c_k, d_k, e, g_i, \alpha, \beta_i, \theta, \overline{p}_i, n$
Output: resource allocation, the optimal sequence
1: **First step :** Calculate γ_{ik} by (53);
2: **Second step :** Determine the optimal sequence by assignment problem;
3: **Third step:** Calculate the optimal resource allocation by (45).

Theorem 3.1 *For the problem* $1|p_{[k]} = (\frac{\overline{p}_i k^{\beta_i}}{u_i})^{\theta}, q_{psd}, CONW| \sum_{k=1}^{n} [a_k E_{[k]} + b_k T_{[k]} + c_k \overline{d} + d_k D] + e \sum_{k=1}^{n} g_{[k]} u_{[k]}$, *the complexity of the algorithm is* $O(n^3)$.

Proof The first step requires $O(n^2)$ time. The second step requires $O(n^3)$ time. The third step requires $O(n)$ time. So the complexity of the algorithm is $O(n^3)$. □

4 The problem $1|p_{[k]} = (\frac{\overline{p}_i k^{\beta_i}}{u_i})^{\theta}, q_{psd}, CONW, \sum_{k=1}^{n} g_{[k]} u_{[k]} \leq W| \sum_{k=1}^{n} [a_k E_{[k]} + b_k T_{[k]} + c_k \overline{d} + d_k D]$

Lemma 4.1 *For the optimal scheduling,* $\overline{d}$ *and* $\overline{\overline{d}}$ *are the completion time of some job.*

Lemma 4.2 *For the optimal scheduling, $\overline{d}$ is equal to the j_1th job completion time $C_{[j_1]}$, j_1 satisfies $\sum_{k=1}^{j_1-1} a_k \le \sum_{k=1}^{n} d_k - \sum_{k=1}^{n} c_k \le \sum_{k=1}^{j_1} a_k$; $\overline{\overline{d}}$ is equal to the j_2th job completion time $C_{[j_2]}$, j_2 satisfies $\sum_{k=j_2+1}^{n} b_k \le \sum_{k=1}^{n} d_k \le \sum_{k=j_2}^{n} b_k$.*

Lemma 4.3 *For the problem $1|p_{[k]} = (\frac{\overline{p}_i k^{\beta_i}}{u_i})^\theta, q_{psd}, CONW, \sum_{k=1}^{n} g_{[k]}u_{[k]} \le W| \sum_{k=1}^{n} [a_k E_{[k]} + b_k T_{[k]} + c_k \overline{d} + d_k D]$, the optimal resource allocation is*

$$
\begin{aligned}
u^*_{[k]} &= \frac{W\left[(1+\alpha)\left(\sum_{j=1}^{k} a_j + \sum_{j=1}^{n} c_j\right) - a_k\right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{I g_{[k]}^{\frac{1}{\theta+1}}}, k = 1, \ldots, j_1 - 1;\\
u^*_{[j_1]} &= \frac{W\left(\alpha \sum_{j=1}^{n} d_j + \sum_{j=1}^{j_1-1} a_j + \sum_{j=1}^{n} c_j\right)^{\frac{1}{\theta+1}} (\overline{p}_{[j_1]} j_1^{\beta_{[j_1]}})^{\frac{\theta}{\theta+1}}}{I g_{[j_1]}^{\frac{1}{\theta+1}}}, k = j_1;\\
u^*_{[k]} &= \frac{W\left[(1+\alpha) \sum_{j=1}^{n} d_j\right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{I g_{[k]}^{\frac{1}{\theta+1}}}, k = j_1 + 1, \ldots, j_2 - 1;\\
u^*_{[j_2]} &= \frac{W\left(\alpha \sum_{j=j_2+1}^{n} b_j + \sum_{j=1}^{n} d_j\right)^{\frac{1}{\theta+1}} (\overline{p}_{[j_2]} j_2^{\beta_{[j_2]}})^{\frac{\theta}{\theta+1}}}{I g_{[j_2]}^{\frac{1}{\theta+1}}}, k = j_2;\\
u^*_{[k]} &= \frac{W\left[(1+\alpha) \sum_{j=k+1}^{n} b_j + b_k\right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{I g_{[k]}^{\frac{1}{\theta+1}}}, k = j_2 + 1, \ldots, n - 1;\\
u^*_{[n]} &= \frac{W b_n^{\frac{1}{\theta+1}} (\overline{p}_{[n]} n^{\beta_{[n]}})^{\frac{\theta}{\theta+1}}}{I g_{[n]}^{\frac{1}{\theta+1}}}, k = n.
\end{aligned}
\tag{54}
$$

Proof When $\overline{d} = C_{[j_1]}$ and $\overline{\overline{d}} = C_{[j_2]}$,

$$
\begin{aligned}
H = &- \sum_{k=1}^{j_1-1} a_k C_{[k]} + \sum_{k=j_2+1}^{n} b_k C_{[k]} + \left(\sum_{k=1}^{j_1-1} a_k\right.\\
&\left. + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k\right) C_{[j_1]} + \left(\sum_{k=1}^{n} d_k - \sum_{k=j_2+1}^{n} b_k\right) C_{[j_2]}\\
&+ \lambda \left(\sum_{k=1}^{n} g_{[k]} u_{[k]} - W\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{j_1-1} \left[(1+\alpha) \left(\sum_{j=1}^{k} a_j + \sum_{j=1}^{n} c_j \right) - a_k \right] \left(\frac{\overline{p}_{[k]} k^{\beta_{[k]}}}{u_{[k]}} \right)^{\theta} \\
&+ \left(\alpha \sum_{j=1}^{n} d_j + \sum_{j=1}^{j_1-1} a_j + \sum_{j=1}^{n} c_j \right) \left(\frac{\overline{p}_{[j_1]} {j_1}^{\beta_{[j_1]}}}{u_{[j_1]}} \right)^{\theta} \\
&+ \sum_{k=j_1+1}^{j_2-1} (1+\alpha) \sum_{j=1}^{n} d_j (\frac{\overline{p}_{[k]} k^{\beta_{[k]}}}{u_{[k]}})^{\theta} + \left(\alpha \sum_{j=j_2+1}^{n} b_j + \sum_{j=1}^{n} d_j \right) \left(\frac{\overline{p}_{[j_2]} {j_2}^{\beta_{[j_2]}}}{u_{[j_2]}} \right)^{\theta} \\
&+ \sum_{k=j_2+1}^{n-1} \left[(1+\alpha) \sum_{j=k+1}^{n} b_j + b_k \right] \left(\frac{\overline{p}_{[k]} k^{\beta_{[k]}}}{u_{[k]}} \right)^{\theta} + b_n \left(\frac{\overline{p}_{[n]} n^{\beta_{[n]}}}{u_{[n]}} \right)^{\theta} \\
&+ \lambda \left(\sum_{k=1}^{n} g_{[k]} u_{[k]} - W \right),
\end{aligned}
\tag{55}
$$

where λ is the Lagrangian multiplier.

$$
\frac{\partial H}{\partial \lambda} = \sum_{k=1}^{n} g_{[k]} u_{[k]} - W = 0, \tag{56}
$$

$$
\begin{aligned}
&\frac{\partial H}{\partial u_{[k]}} = \lambda g_{[k]} - \theta \left[(1+\alpha) \left(\sum_{j=1}^{k} a_j + \sum_{j=1}^{n} c_j \right) - a_k \right] \frac{(\overline{p}_{[k]} k^{\beta_{[k]}})^{\theta}}{u_{[k]}^{\theta+1}} = 0, k = 1, \ldots, j_1 - 1; \\
&\frac{\partial H}{\partial u_{[j_1]}} = \lambda g_{[j_1]} - \theta \left(\alpha \sum_{j=1}^{n} d_j + \sum_{j=1}^{j_1-1} a_j + \sum_{j=1}^{n} c_j \right) \frac{(\overline{p}_{[j_1]} {j_1}^{\beta_{[j_1]}})^{\theta}}{u_{[j_1]}^{\theta+1}} = 0, k = j_1; \\
&\frac{\partial H}{\partial u_{[k]}} = \lambda g_{[k]} - \theta (1+\alpha) \sum_{j=1}^{n} d_j \frac{(\overline{p}_{[k]} k^{\beta_{[k]}})^{\theta}}{u_{[k]}^{\theta+1}} = 0, k = j_1 + 1, \ldots, j_2 - 1; \\
&\frac{\partial H}{\partial u_{[j_2]}} = \lambda g_{[j_2]} - \theta \left(\alpha \sum_{j=j_2+1}^{n} b_j + \sum_{j=1}^{n} d_j \right) \frac{(\overline{p}_{[j_2]} {j_2}^{\beta_{[j_2]}})^{\theta}}{u_{[j_2]}^{\theta+1}} = 0, k = j_2; \\
&\frac{\partial H}{\partial u_{[k]}} = \lambda g_{[k]} - \theta \left[(1+\alpha) \sum_{j=k+1}^{n} b_j + b_k \right] \frac{(\overline{p}_{[k]} k^{\beta_{[k]}})^{\theta}}{u_{[k]}^{\theta+1}} = 0, k = j_2 + 1, \ldots, n - 1; \\
&\frac{\partial H}{\partial u_{[n]}} = \lambda g_{[n]} - \theta b_n \frac{(\overline{p}_{[n]} n^{\beta_{[n]}})^{\theta}}{u_{[n]}^{\theta+1}} = 0, k = n;
\end{aligned}
\tag{57}
$$

We have

$$
u_{[k]}^{*} = \frac{\theta^{\frac{1}{\theta+1}} \left[(1+\alpha) \left(\sum_{j=1}^{k} a_j + \sum_{j=1}^{n} c_j \right) - a_k \right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{(\lambda g_{[k]})^{\frac{1}{\theta+1}}}, k = 1, \ldots, j_1 - 1;
$$

$$
\begin{aligned}
u^*_{[j_1]} &= \frac{\theta^{\frac{1}{\theta+1}}\left(\alpha\sum_{j=1}^{n}d_j+\sum_{j=1}^{j_1-1}a_j+\sum_{j=1}^{n}c_j\right)^{\frac{1}{\theta+1}}(\overline{p}_{[j_1]}{j_1}^{\beta_{[j_1]}})^{\frac{\theta}{\theta+1}}}{(\lambda g_{[j_1]})^{\frac{1}{\theta+1}}}, k=j_1;\\
u^*_{[k]} &= \frac{\theta^{\frac{1}{\theta+1}}\left[(1+\alpha)\sum_{j=1}^{n}d_j\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]}k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{(\lambda g_{[k]})^{\frac{1}{\theta+1}}}, k=j_1+1,\ldots,j_2-1;\\
u^*_{[j_2]} &= \frac{\theta^{\frac{1}{\theta+1}}\left(\alpha\sum_{j=j_2+1}^{n}b_j+\sum_{j=1}^{n}d_j\right)^{\frac{1}{\theta+1}}(\overline{p}_{[j_2]}{j_2}^{\beta_{[j_2]}})^{\frac{\theta}{\theta+1}}}{(\lambda g_{[j_2]})^{\frac{1}{\theta+1}}}, k=j_2;\\
u^*_{[k]} &= \frac{\theta^{\frac{1}{\theta+1}}\left[(1+\alpha)\sum_{j=k+1}^{n}b_j+b_k\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]}k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{(\lambda g_{[k]})^{\frac{1}{\theta+1}}}, k=j_2+1,\ldots,n-1;\\
u^*_{[n]} &= \frac{\theta^{\frac{1}{\theta+1}}b_n^{\frac{1}{\theta+1}}(\overline{p}_{[n]}n^{\beta_{[n]}})^{\frac{\theta}{\theta+1}}}{(\lambda g_{[n]})^{\frac{1}{\theta+1}}}, k=n.
\end{aligned}
\tag{58}
$$

Substitute the optimal resource allocation (58) into (56),

$$
\begin{aligned}
\lambda^{\frac{1}{\theta+1}} =& W^{-1}\theta^{\frac{1}{\theta+1}}\left\{\sum_{k=1}^{j_1-1}\left[(1+\alpha)\left(\sum_{j=1}^{k}a_j+\sum_{j=1}^{n}c_j\right)-a_k\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}\right.\\
&+\left(\alpha\sum_{j=1}^{n}d_j+\sum_{j=1}^{j_1-1}a_j+\sum_{j=1}^{n}c_j\right)^{\frac{1}{\theta+1}}(\overline{p}_{[j_1]}g_{[j_1]}{j_1}^{\beta_{[j_1]}})^{\frac{\theta}{\theta+1}}\\
&+\sum_{k=j_1+1}^{j_2-1}\left[(1+\alpha)\sum_{j=1}^{n}d_j\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}\\
&+\left(\alpha\sum_{j=j_2+1}^{n}b_j+\sum_{j=1}^{n}d_j\right)^{\frac{1}{\theta+1}}(\overline{p}_{[j_2]}g_{[j_2]}{j_2}^{\beta_{[j_2]}})^{\frac{\theta}{\theta+1}}\\
&+\sum_{k=j_2+1}^{n-1}\left[(1+\alpha)\sum_{j=k+1}^{n}b_j+b_k\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}\\
&\left.+b_n^{\frac{1}{\theta+1}}(\overline{p}_{[n]}g_{[n]}n^{\beta_{[n]}})^{\frac{\theta}{\theta+1}}\right\}\\
=& W^{-1}\theta^{\frac{1}{\theta+1}}I.
\end{aligned}
\tag{59}
$$

From (58), we have

$$u^*_{[k]} = \frac{W\left[(1+\alpha)\left(\sum_{j=1}^{k} a_j + \sum_{j=1}^{n} c_j\right) - a_k\right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{I g_{[k]}^{\frac{1}{\theta+1}}}, k = 1, \ldots, j_1 - 1;$$

$$u^*_{[j_1]} = \frac{W\left(\alpha \sum_{j=1}^{n} d_j + \sum_{j=1}^{j_1-1} a_j + \sum_{j=1}^{n} c_j\right)^{\frac{1}{\theta+1}} (\overline{p}_{[j_1]} {j_1}^{\beta_{[j_1]}})^{\frac{\theta}{\theta+1}}}{I g_{[j_1]}^{\frac{1}{\theta+1}}}, k = j_1;$$

$$u^*_{[k]} = \frac{W\left[(1+\alpha)\sum_{j=1}^{n} d_j\right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{I g_{[k]}^{\frac{1}{\theta+1}}}, k = j_1 + 1, \ldots, j_2 - 1;$$

$$u^*_{[j_2]} = \frac{W\left(\alpha \sum_{j=j_2+1}^{n} b_j + \sum_{j=1}^{n} d_j\right)^{\frac{1}{\theta+1}} (\overline{p}_{[j_2]} {j_2}^{\beta_{[j_2]}})^{\frac{\theta}{\theta+1}}}{I g_{[j_2]}^{\frac{1}{\theta+1}}}, k = j_2;$$

$$u^*_{[k]} = \frac{W\left[(1+\alpha)\sum_{j=k+1}^{n} b_j + b_k\right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{I g_{[k]}^{\frac{1}{\theta+1}}}, k = j_2 + 1, \ldots, n - 1;$$

$$u^*_{[n]} = \frac{W b_n^{\frac{1}{\theta+1}} (\overline{p}_{[n]} n^{\beta_{[n]}})^{\frac{\theta}{\theta+1}}}{I g_{[n]}^{\frac{1}{\theta+1}}}, k = n. \tag{60}$$

□

Substitute the optimal resource allocation (54) into the objective function

$$Z = W^{-\theta}\left\{\sum_{k=1}^{j_1-1}\left[(1+\alpha)\left(\sum_{j=1}^{k} a_j + \sum_{j=1}^{n} c_j\right) - a_k\right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} g_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}\right.$$
$$+ \left(\alpha\sum_{j=1}^{n} d_j + \sum_{j=1}^{j_1-1} a_j + \sum_{j=1}^{n} c_j\right)^{\frac{1}{\theta+1}} (\overline{p}_{[j_1]} g_{[j_1]} {j_1}^{\beta_{[j_1]}})^{\frac{\theta}{\theta+1}}$$
$$+ \sum_{k=j_1+1}^{j_2-1}\left[(1+\alpha)\sum_{j=1}^{n} d_j\right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} g_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}$$
$$+ \left(\alpha\sum_{j=j_2+1}^{n} b_j + \sum_{j=1}^{n} d_j\right)^{\frac{1}{\theta+1}} (\overline{p}_{[j_2]} g_{[j_2]} {j_2}^{\beta_{[j_2]}})^{\frac{\theta}{\theta+1}}$$

$$
\begin{aligned}
&+\sum_{k=j_2+1}^{n-1}\left[(1+\alpha)\sum_{j=k+1}^{n}b_j+b_k\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}\\
&\left.+b_n^{\frac{1}{\theta+1}}(\overline{p}_{[n]}g_{[n]}n^{\beta_{[n]}})^{\frac{\theta}{\theta+1}}\right\}^{\theta+1}\\
&=W^{-\theta}I^{\theta+1}.
\end{aligned}
\tag{61}
$$

Minimizing Z is the same thing as minimizing $\overline{Z}$,

$$
\begin{aligned}
\overline{Z}=&\sum_{k=1}^{j_1-1}\left[(1+\alpha)\left(\sum_{j=1}^{k}a_j+\sum_{j=1}^{n}c_j\right)-a_k\right]^{\frac{1}{\theta}}\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}}\\
&+\left(\alpha\sum_{j=1}^{n}d_j+\sum_{j=1}^{j_1-1}a_j+\sum_{j=1}^{n}c_j\right)^{\frac{1}{\theta}}\overline{p}_{[j_1]}g_{[j_1]}j_1{}^{\beta_{[j_1]}}\\
&+\sum_{k=j_1+1}^{j_2-1}\left[(1+\alpha)\sum_{j=1}^{n}d_j\right]^{\frac{1}{\theta}}\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}}+\left(\alpha\sum_{j=j_2+1}^{n}b_j+\sum_{j=1}^{n}d_j\right)^{\frac{1}{\theta}}\overline{p}_{[j_2]}g_{[j_2]}j_2{}^{\beta_{[j_2]}}\\
&+\sum_{k=j_2+1}^{n-1}\left[(1+\alpha)\sum_{j=k+1}^{n}b_j+b_k\right]^{\frac{1}{\theta}}\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}}+b_n^{\frac{1}{\theta}}\overline{p}_{[n]}g_{[n]}n^{\beta_{[n]}}.
\end{aligned}
\tag{62}
$$

$\overline{Z}$ could be computed by the assignment problem.

$$
\begin{aligned}
&\min\ \sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_{ik}z_{ik}\\
&s.t.\quad\begin{cases}\sum_{i=1}^{n}z_{ik}=1,k=1,\ldots,n;\\ \sum_{k=1}^{n}z_{ik}=1,i=1,\ldots,n;\\ z_{ik}=0\ or\ 1,i,k=1,\ldots,n,\end{cases}
\end{aligned}
\tag{63}
$$

where

$$
\gamma_{ik}=\left[(1+\alpha)\left(\sum_{j=1}^{k}a_j+\sum_{j=1}^{n}c_j\right)-a_k\right]^{\frac{1}{\theta}}\overline{p}_i g_i k^{\beta_i},k=1,\ldots,j_1-1;
$$

$$
\gamma_{ik}=\left(\alpha\sum_{j=1}^{n}d_j+\sum_{j=1}^{j_1-1}a_j+\sum_{j=1}^{n}c_j\right)^{\frac{1}{\theta}}\overline{p}_i g_i j_1{}^{\beta_i},k=j_1;
$$

$$\gamma_{ik} = \left[(1+\alpha)\sum_{j=1}^{n} d_j\right]^{\frac{1}{\theta}} \overline{p}_i g_i k^{\beta_i}, k = j_1+1, \ldots, j_2-1;$$

$$\gamma_{ik} = \left(\alpha \sum_{j=j_2+1}^{n} b_j + \sum_{j=1}^{n} d_j\right)^{\frac{1}{\theta}} \overline{p}_i g_i j_2^{\beta_i}, k = j_2;$$

$$\gamma_{ik} = \left[(1+\alpha)\sum_{j=k+1}^{n} b_j + b_k\right]^{\frac{1}{\theta}} \overline{p}_i g_i k^{\beta_i}, k = j_2+1, \ldots, n-1;$$

$$\gamma_{ik} = b_n^{\frac{1}{\theta}} \overline{p}_i g_i n^{\beta_i}, k = n. \tag{64}$$

The algorithm is summarized as follows:

Algorithm 2 $1|p_{[k]} = (\frac{\overline{p}_i k^{\beta_i}}{u_i})^\theta, q_{psd}, CONW, \sum_{k=1}^{n} g_{[k]}u_{[k]} \leq W|\sum_{k=1}^{n}[a_k E_{[k]} + b_k T_{[k]} + c_k \overline{d} + d_k D]$

Input: $a_k, b_k, c_k, d_k, g_i, \alpha, \beta_i, \theta, \overline{p}_i, n, W$
Output: resource allocation, the optimal sequence
1: **First step :** Calculate γ_{ik} by (64);
2: **Second step :** Determine the optimal sequence by assignment problem;
3: **Third step:** Calculate the optimal resource allocation by (54).

Theorem 4.1 *For the problem* $1|p_{[k]} = (\frac{\overline{p}_i k^{\beta_i}}{u_i})^\theta, q_{psd}, CONW, \sum_{k=1}^{n} g_{[k]}u_{[k]} \leq W|\sum_{k=1}^{n}[a_k E_{[k]} + b_k T_{[k]} + c_k \overline{d} + d_k D]$*, the complexity of the algorithm is* $O(n^3)$*.*

Proof The first step requires $O(n^2)$ time. The second step requires $O(n^3)$ time. The third step requires $O(n)$ time. So the complexity of the algorithm is $O(n^3)$. □

5 The problem $1|p_{[k]} = (\frac{\overline{p}_i k^{\beta_i}}{u_i})^\theta, q_{psd}, CONW, \sum_{k=1}^{n}[a_k E_{[k]} + b_k T_{[k]} + c_k \overline{d} + d_k D] \leq M|\sum_{k=1}^{n} g_{[k]}u_{[k]}$

Lemma 5.1 *For the optimal scheduling,* $\overline{d}$ *and* $\overline{\overline{d}}$ *are the completion time of some job.*

Lemma 5.2 *For the optimal scheduling,* $\overline{d}$ *is equal to the* j_1*th job completion time* $C_{[j_1]}$*,* j_1 *satisfies* $\sum_{k=1}^{j_1-1} a_k \leq \sum_{k=1}^{n} d_k - \sum_{k=1}^{n} c_k \leq \sum_{k=1}^{j_1} a_k$*;* $\overline{\overline{d}}$ *is equal to the* j_2*th job completion time* $C_{[j_2]}$*,* j_2 *satisfies* $\sum_{k=j_2+1}^{n} b_k \leq \sum_{k=1}^{n} d_k \leq \sum_{k=j_2}^{n} b_k$*.*

Lemma 5.3 *For the problem* $1|p_{[k]}=(\frac{\overline{p}_i k^{\beta_i}}{u_i})^{\theta}, q_{psd}, CONW, \sum_{k=1}^{n}[a_k E_{[k]}+b_k T_{[k]}+c_k\overline{d}+d_k D]\leq M|\sum_{k=1}^{n} g_{[k]}u_{[k]}$, *the optimal resource allocation is*

$$u_{[k]}^{*}=\frac{I^{\frac{1}{\theta}}\left[(1+\alpha)\left(\sum_{j=1}^{k}a_j+\sum_{j=1}^{n}c_j\right)-a_k\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]}k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{M^{\frac{1}{\theta}}g_{[k]}^{\frac{1}{\theta+1}}}, k=1,\ldots,j_1-1;$$

$$u_{[j_1]}^{*}=\frac{I^{\frac{1}{\theta}}\left(\alpha\sum_{j=1}^{n}d_j+\sum_{j=1}^{j_1-1}a_j+\sum_{j=1}^{n}c_j\right)^{\frac{1}{\theta+1}}(\overline{p}_{[j_1]}{j_1}^{\beta_{[j_1]}})^{\frac{\theta}{\theta+1}}}{M^{\frac{1}{\theta}}g_{[j_1]}^{\frac{1}{\theta+1}}}, k=j_1;$$

$$u_{[k]}^{*}=\frac{I^{\frac{1}{\theta}}\left[(1+\alpha)\sum_{j=1}^{n}d_j\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]}k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{M^{\frac{1}{\theta}}g_{[k]}^{\frac{1}{\theta+1}}}, k=j_1+1,\ldots,j_2-1;$$

$$u_{[j_2]}^{*}=\frac{I^{\frac{1}{\theta}}\left(\alpha\sum_{j=j_2+1}^{n}b_j+\sum_{j=1}^{n}d_j\right)^{\frac{1}{\theta+1}}(\overline{p}_{[j_2]}{j_2}^{\beta_{[j_2]}})^{\frac{\theta}{\theta+1}}}{M^{\frac{1}{\theta}}g_{[j_2]}^{\frac{1}{\theta+1}}}, k=j_2;$$

$$u_{[k]}^{*}=\frac{I^{\frac{1}{\theta}}\left[(1+\alpha)\sum_{j=k+1}^{n}b_j+b_k\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]}k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{M^{\frac{1}{\theta}}g_{[k]}^{\frac{1}{\theta+1}}}, k=j_2+1,\ldots,n-1;$$

$$u_{[n]}^{*}=\frac{I^{\frac{1}{\theta}}b_n^{\frac{1}{\theta+1}}(\overline{p}_{[n]}n^{\beta_{[n]}})^{\frac{\theta}{\theta+1}}}{M^{\frac{1}{\theta}}g_{[n]}^{\frac{1}{\theta+1}}}, k=n. \tag{65}$$

Proof When $\overline{d}=C_{[j_1]}$ and $\overline{\overline{d}}=C_{[j_2]}$,

$$\begin{aligned}H=&\sum_{k=1}^{n}g_{[k]}u_{[k]}+\lambda\left[-\sum_{k=1}^{j_1-1}a_kC_{[k]}+\sum_{k=j_2+1}^{n}b_kC_{[k]}+\left(\sum_{k=1}^{j_1-1}a_k+\sum_{k=1}^{n}c_k-\sum_{k=1}^{n}d_k\right)C_{[j_1]}\right.\\&\left.+\left(\sum_{k=1}^{n}d_k-\sum_{k=j_2+1}^{n}b_k\right)C_{[j_2]}-M\right]\\=&\sum_{k=1}^{n}g_{[k]}u_{[k]}+\lambda\{\sum_{k=1}^{j_1-1}\left[(1+\alpha)\left(\sum_{j=1}^{k}a_j+\sum_{j=1}^{n}c_j\right)-a_k\right]\left(\frac{\overline{p}_{[k]}k^{\beta_{[k]}}}{u_{[k]}}\right)^{\theta}\\&+\left(\alpha\sum_{j=1}^{n}d_j+\sum_{j=1}^{j_1-1}a_j+\sum_{j=1}^{n}c_j\right)\left(\frac{\overline{p}_{[j_1]}{j_1}^{\beta_{[j_1]}}}{u_{[j_1]}}\right)^{\theta}\\&+\sum_{k=j_1+1}^{j_2-1}(1+\alpha)\sum_{j=1}^{n}d_j\left(\frac{\overline{p}_{[k]}k^{\beta_{[k]}}}{u_{[k]}}\right)^{\theta}+\left(\alpha\sum_{j=j_2+1}^{n}b_j+\sum_{j=1}^{n}d_j\right)\left(\frac{\overline{p}_{[j_2]}{j_2}^{\beta_{[j_2]}}}{u_{[j_2]}}\right)^{\theta}\end{aligned}$$

$$+ \sum_{k=j_2+1}^{n-1} \left[(1+\alpha) \sum_{j=k+1}^{n} b_j + b_k \right] \left(\frac{\overline{p}_{[k]} k^{\beta_{[k]}}}{u_{[k]}} \right)^{\theta} + b_n \left(\frac{\overline{p}_{[n]} n^{\beta_{[n]}}}{u_{[n]}} \right)^{\theta} - M\}, \tag{66}$$

where λ is the Lagrangian multiplier.

$$\begin{aligned} \frac{\partial H}{\partial \lambda} = & \sum_{k=1}^{j_1-1} \left[(1+\alpha) \left(\sum_{j=1}^{k} a_j + \sum_{j=1}^{n} c_j \right) - a_k \right] \left(\frac{\overline{p}_{[k]} k^{\beta_{[k]}}}{u_{[k]}} \right)^{\theta} \\ & + \left(\alpha \sum_{j=1}^{n} d_j + \sum_{j=1}^{j_1-1} a_j + \sum_{j=1}^{n} c_j \right) \left(\frac{\overline{p}_{[j_1]} {j_1}^{\beta_{[j_1]}}}{u_{[j_1]}} \right)^{\theta} \\ & + \sum_{k=j_1+1}^{j_2-1} (1+\alpha) \sum_{j=1}^{n} d_j \left(\frac{\overline{p}_{[k]} k^{\beta_{[k]}}}{u_{[k]}} \right)^{\theta} + \left(\alpha \sum_{j=j_2+1}^{n} b_j + \sum_{j=1}^{n} d_j \right) \left(\frac{\overline{p}_{[j_2]} {j_2}^{\beta_{[j_2]}}}{u_{[j_2]}} \right)^{\theta} \\ & + \sum_{k=j_2+1}^{n-1} \left[(1+\alpha) \sum_{j=k+1}^{n} b_j + b_k \right] \left(\frac{\overline{p}_{[k]} k^{\beta_{[k]}}}{u_{[k]}} \right)^{\theta} + b_n \left(\frac{\overline{p}_{[n]} n^{\beta_{[n]}}}{u_{[n]}} \right)^{\theta} - M = 0, \end{aligned} \tag{67}$$

$$\begin{aligned} & \frac{\partial H}{\partial u_{[k]}} = g_{[k]} - \lambda\theta \left[(1+\alpha) \left(\sum_{j=1}^{k} a_j + \sum_{j=1}^{n} c_j \right) - a_k \right] \frac{(\overline{p}_{[k]} k^{\beta_{[k]}})^{\theta}}{u_{[k]}^{\theta+1}} = 0, k = 1, \ldots, j_1 - 1; \\ & \frac{\partial Z}{\partial u_{[j_1]}} = g_{[j_1]} - \lambda\theta \left(\alpha \sum_{j=1}^{n} d_j + \sum_{j=1}^{j_1-1} a_j + \sum_{j=1}^{n} c_j \right) \frac{(\overline{p}_{[j_1]} {j_1}^{\beta_{[j_1]}})^{\theta}}{u_{[j_1]}^{\theta+1}} = 0, k = j_1; \\ & \frac{\partial Z}{\partial u_{[k]}} = g_{[k]} - \lambda\theta(1+\alpha) \sum_{j=1}^{n} d_j \frac{(\overline{p}_{[k]} k^{\beta_{[k]}})^{\theta}}{u_{[k]}^{\theta+1}} = 0, k = j_1 + 1, \ldots, j_2 - 1; \\ & \frac{\partial Z}{\partial u_{[j_2]}} = g_{[j_2]} - \lambda\theta \left(\alpha \sum_{j=j_2+1}^{n} b_j + \sum_{j=1}^{n} d_j \right) \frac{\left(\overline{p}_{[j_2]} {j_2}^{\beta_{[j_2]}}\right)^{\theta}}{u_{[j_2]}^{\theta+1}} = 0, k = j_2; \\ & \frac{\partial Z}{\partial u_{[k]}} = g_{[k]} - \lambda\theta \left[(1+\alpha) \sum_{j=k+1}^{n} b_j + b_k \right] \frac{(\overline{p}_{[k]} k^{\beta_{[k]}})^{\theta}}{u_{[k]}^{\theta+1}} \\ & = 0, k = j_2 + 1, \ldots, n - 1; \\ & \frac{\partial Z}{\partial u_{[n]}} = g_{[n]} - \lambda\theta b_n \frac{(\overline{p}_{[n]} n^{\beta_{[n]}})^{\theta}}{u_{[n]}^{\theta+1}} = 0, k = n; \end{aligned} \tag{68}$$

We have

$$u^*_{[k]} = \frac{\lambda^{\frac{1}{\theta+1}} \theta^{\frac{1}{\theta+1}} \left[(1+\alpha) \left(\sum_{j=1}^{k} a_j + \sum_{j=1}^{n} c_j \right) - a_k \right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{g_{[k]}^{\frac{1}{\theta+1}}}, k = 1, \ldots, j_1 - 1;$$

$$u^*_{[j_1]} = \frac{\lambda^{\frac{1}{\theta+1}} \theta^{\frac{1}{\theta+1}} (\alpha \sum_{j=1}^{n} d_j + \sum_{j=1}^{j_1-1} a_j + \sum_{j=1}^{n} c_j)^{\frac{1}{\theta+1}} (\overline{p}_{[j_1]} {j_1}^{\beta_{[j_1]}})^{\frac{\theta}{\theta+1}}}{g_{[j_1]}^{\frac{1}{\theta+1}}}, k = j_1;$$

$$u_{[k]}^{*}=\frac{\lambda^{\frac{1}{\theta+1}}\theta^{\frac{1}{\theta+1}}\left[(1+\alpha)\sum_{j=1}^{n}d_j\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]}k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{g_{[k]}^{\frac{1}{\theta+1}}},k=j_1+1,\ldots,j_2-1;$$

$$u_{[j_2]}^{*}=\frac{\lambda^{\frac{1}{\theta+1}}\theta^{\frac{1}{\theta+1}}\left(\alpha\sum_{j=j_2+1}^{n}b_j+\sum_{j=1}^{n}d_j\right)^{\frac{1}{\theta+1}}(\overline{p}_{[j_2]}j_2^{\beta_{[j_2]}})^{\frac{\theta}{\theta+1}}}{g_{[j_2]}^{\frac{1}{\theta+1}}},k=j_2;$$

$$u_{[k]}^{*}=\frac{\lambda^{\frac{1}{\theta+1}}\theta^{\frac{1}{\theta+1}}\left[(1+\alpha)\sum_{j=k+1}^{n}b_j+b_k\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]}k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{g_{[k]}^{\frac{1}{\theta+1}}},k=j_2+1,\ldots,n-1;$$

$$u_{[n]}^{*}=\frac{\lambda^{\frac{1}{\theta+1}}\theta^{\frac{1}{\theta+1}}b_n^{\frac{1}{\theta+1}}(\overline{p}_{[n]}n^{\beta_{[n]}})^{\frac{\theta}{\theta+1}}}{g_{[n]}^{\frac{1}{\theta+1}}},k=n. \tag{69}$$

Substitute the optimal resource allocation (69) into (67),

$$\begin{aligned}
\lambda^{\frac{1}{\theta+1}}=&M^{-\frac{1}{\theta}}\theta^{-\frac{1}{\theta+1}}\left\{\sum_{k=1}^{j_1-1}\left[(1+\alpha)\left(\sum_{j=1}^{k}a_j+\sum_{j=1}^{n}c_j\right)-a_k\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}\right.\\
&+\left(\alpha\sum_{j=1}^{n}d_j+\sum_{j=1}^{j_1-1}a_j+\sum_{j=1}^{n}c_j\right)^{\frac{1}{\theta+1}}(\overline{p}_{[j_1]}g_{[j_1]}j_1^{\beta_{[j_1]}})^{\frac{\theta}{\theta+1}}\\
&+\sum_{k=j_1+1}^{j_2-1}\left[(1+\alpha)\sum_{j=1}^{n}d_j\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}\\
&+\left(\alpha\sum_{j=j_2+1}^{n}b_j+\sum_{j=1}^{n}d_j\right)^{\frac{1}{\theta+1}}(\overline{p}_{[j_2]}g_{[j_2]}j_2^{\beta_{[j_2]}})^{\frac{\theta}{\theta+1}}\\
&+\sum_{k=j_2+1}^{n-1}\left[(1+\alpha)\sum_{j=k+1}^{n}b_j+b_k\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}\\
&\left.+b_n^{\frac{1}{\theta+1}}(\overline{p}_{[n]}g_{[n]}n^{\beta_{[n]}})^{\frac{\theta}{\theta+1}}\right\}^{\frac{1}{\theta}}\\
=&M^{-\frac{1}{\theta}}\theta^{-\frac{1}{\theta+1}}I^{\frac{1}{\theta}}.
\end{aligned} \tag{70}$$

From (70), we have

$$u^*_{[k]} = \frac{I^{\frac{1}{\theta}}\left[(1+\alpha)\left(\sum_{j=1}^{k} a_j + \sum_{j=1}^{n} c_j\right) - a_k\right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{M^{\frac{1}{\theta}} g_{[k]}^{\frac{1}{\theta+1}}}, k = 1, \dots, j_1 - 1;$$

$$u^*_{[j_1]} = \frac{I^{\frac{1}{\theta}}\left(\alpha \sum_{j=1}^{n} d_j + \sum_{j=1}^{j_1-1} a_j + \sum_{j=1}^{n} c_j\right)^{\frac{1}{\theta+1}} (\overline{p}_{[j_1]} j_1{}^{\beta_{[j_1]}})^{\frac{\theta}{\theta+1}}}{M^{\frac{1}{\theta}} g_{[j_1]}^{\frac{1}{\theta+1}}}, k = j_1;$$

$$u^*_{[k]} = \frac{I^{\frac{1}{\theta}}\left[(1+\alpha) \sum_{j=1}^{n} d_j\right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{M^{\frac{1}{\theta}} g_{[k]}^{\frac{1}{\theta+1}}}, k = j_1 + 1, \dots, j_2 - 1;$$

$$u^*_{[j_2]} = \frac{I^{\frac{1}{\theta}}\left(\alpha \sum_{j=j_2+1}^{n} b_j + \sum_{j=1}^{n} d_j\right)^{\frac{1}{\theta+1}} (\overline{p}_{[j_2]} j_2{}^{\beta_{[j_2]}})^{\frac{\theta}{\theta+1}}}{M^{\frac{1}{\theta}} g_{[j_2]}^{\frac{1}{\theta+1}}}, k = j_2;$$

$$u^*_{[k]} = \frac{I^{\frac{1}{\theta}}\left[(1+\alpha) \sum_{j=k+1}^{n} b_j + b_k\right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}}{M^{\frac{1}{\theta}} g_{[k]}^{\frac{1}{\theta+1}}}, k = j_2 + 1, \dots, n - 1;$$

$$u^*_{[n]} = \frac{I^{\frac{1}{\theta}} b_n^{\frac{1}{\theta+1}} (\overline{p}_{[n]} n^{\beta_{[n]}})^{\frac{\theta}{\theta+1}}}{M^{\frac{1}{\theta}} g_{[n]}^{\frac{1}{\theta+1}}}, k = n. \tag{71}$$

□

Substitute the optimal resource allocation (65) into the objective function

$$Z = M^{-\frac{1}{\theta}} \left\{ \sum_{k=1}^{j_1-1} \left[(1+\alpha)\left(\sum_{j=1}^{k} a_j + \sum_{j=1}^{n} c_j\right) - a_k\right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} g_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}} \right.$$

$$+ \left(\alpha \sum_{j=1}^{n} d_j + \sum_{j=1}^{j_1-1} a_j + \sum_{j=1}^{n} c_j\right)^{\frac{1}{\theta+1}} (\overline{p}_{[j_1]} g_{[j_1]} j_1{}^{\beta_{[j_1]}})^{\frac{\theta}{\theta+1}}$$

$$+ \sum_{k=j_1+1}^{j_2-1} \left[(1+\alpha) \sum_{j=1}^{n} d_j\right]^{\frac{1}{\theta+1}} (\overline{p}_{[k]} g_{[k]} k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}$$

$$+ \left(\alpha \sum_{j=j_2+1}^{n} b_j + \sum_{j=1}^{n} d_j\right)^{\frac{1}{\theta+1}} (\overline{p}_{[j_2]} g_{[j_2]} j_2{}^{\beta_{[j_2]}})^{\frac{\theta}{\theta+1}}$$

$$
+\sum_{k=j_2+1}^{n-1}\left[(1+\alpha)\sum_{j=k+1}^{n}b_j+b_k\right]^{\frac{1}{\theta+1}}(\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}})^{\frac{\theta}{\theta+1}}
$$

$$
\left.+b_n^{\frac{1}{\theta+1}}(\overline{p}_{[n]}g_{[n]}n^{\beta_{[n]}})^{\frac{\theta}{\theta+1}}\right\}^{1+\frac{1}{\theta}}
$$

$$
=M^{-\frac{1}{\theta}}I^{1+\frac{1}{\theta}}. \tag{72}
$$

Minimizing Z is the same thing as minimizing $\overline{Z}$,

$$
\begin{aligned}
\overline{Z}=&\sum_{k=1}^{j_1-1}\left[(1+\alpha)\left(\sum_{j=1}^{k}a_j+\sum_{j=1}^{n}c_j\right)-a_k\right]^{\frac{1}{\theta}}\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}}\\
&+\left(\alpha\sum_{j=1}^{n}d_j+\sum_{j=1}^{j_1-1}a_j+\sum_{j=1}^{n}c_j\right)^{\frac{1}{\theta}}\overline{p}_{[j_1]}g_{[j_1]}{j_1}^{\beta_{[j_1]}}\\
&+\sum_{k=j_1+1}^{j_2-1}\left[(1+\alpha)\sum_{j=1}^{n}d_j\right]^{\frac{1}{\theta}}\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}}+\left(\alpha\sum_{j=j_2+1}^{n}b_j+\sum_{j=1}^{n}d_j\right)^{\frac{1}{\theta}}\overline{p}_{[j_2]}g_{[j_2]}{j_2}^{\beta_{[j_2]}}\\
&+\sum_{k=j_2+1}^{n-1}\left[(1+\alpha)\sum_{j=k+1}^{n}b_j+b_k\right]^{\frac{1}{\theta}}\overline{p}_{[k]}g_{[k]}k^{\beta_{[k]}}+b_n^{\frac{1}{\theta}}\overline{p}_{[n]}g_{[n]}n^{\beta_{[n]}}.
\end{aligned} \tag{73}
$$

$\overline{Z}$ could be computed by the assignment problem.

$$
\begin{aligned}
&\min \sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_{ik}z_{ik}\\
&s.t.\quad \begin{cases}\sum_{i=1}^{n}z_{ik}=1, k=1,\ldots,n;\\ \sum_{k=1}^{n}z_{ik}=1, i=1,\ldots,n;\\ z_{ik}=0\ or\ 1, i,k=1,\ldots,n,\end{cases}
\end{aligned} \tag{74}
$$

where

$$
\gamma_{ik}=\left[(1+\alpha)\left(\sum_{j=1}^{k}a_j+\sum_{j=1}^{n}c_j\right)-a_k\right]^{\frac{1}{\theta}}\overline{p}_i g_i k^{\beta_i}, k=1,\ldots,j_1-1;
$$

$$
\gamma_{ik}=\left(\alpha\sum_{j=1}^{n}d_j+\sum_{j=1}^{j_1-1}a_j+\sum_{j=1}^{n}c_j\right)^{\frac{1}{\theta}}\overline{p}_i g_i {j_1}^{\beta_i}, k=j_1;
$$

$$\gamma_{ik} = \left[(1+\alpha) \sum_{j=1}^{n} d_j \right]^{\frac{1}{\theta}} \overline{p}_i g_i k^{\beta_i}, k = j_1 + 1, \ldots, j_2 - 1;$$

$$\gamma_{ik} = \left(\alpha \sum_{j=j_2+1}^{n} b_j + \sum_{j=1}^{n} d_j \right)^{\frac{1}{\theta}} \overline{p}_i g_i j_2^{\beta_i}, k = j_2;$$

$$\gamma_{ik} = \left[(1+\alpha) \sum_{j=k+1}^{n} b_j + b_k \right]^{\frac{1}{\theta}} \overline{p}_i g_i k^{\beta_i}, k = j_2 + 1, \ldots, n-1;$$

$$\gamma_{ik} = b_n^{\frac{1}{\theta}} \overline{p}_i g_i n^{\beta_i}, k = n. \quad (75)$$

The algorithm is summarized as follows:

Algorithm 3 $1|p_{[k]} = (\frac{\overline{p}_i k^{\beta_i}}{u_i})^{\theta}, q_{psd}, CONW, \sum_{k=1}^{n} [a_k E_{[k]} + b_k T_{[k]} + c_k \overline{d} + d_k D] \leq M| \sum_{k=1}^{n} g_{[k]} u_{[k]}$

Input: $a_k, b_k, c_k, d_k, g_i, \alpha, \beta_i, \theta, \overline{p}_i, n, M$
Output: resource allocation, the optimal sequence
1: **First step :** Calculate γ_{ij} by (75);
2: **Second step :** Determine the optimal sequence by assignment problem;
3: **Third step:** Calculate the optimal resource allocation by (65).

Theorem 5.1 *For the problem* $1|p_{[k]} = (\frac{\overline{p}_i k^{\beta_i}}{u_i})^{\theta}, q_{psd}, CONW, \sum_{k=1}^{n} [a_k E_{[k]} + b_k T_{[k]} + c_k \overline{d} + d_k D] \leq M| \sum_{k=1}^{n} g_{[k]} u_{[k]}$, *the complexity of the algorithm is* $O(n^3)$.

Proof The first step requires $O(n^2)$ time. The second step requires $O(n^3)$ time. The third step requires $O(n)$ time. So the complexity of the algorithm is $O(n^3)$. □

6 Example

We give an example to demonstrate the solution process.

Example 6.1 4 jobs are processed. The parameter values are as follows: $e = 2, \alpha = 1$, $\theta = 2$, $W = 10$, $M = 20$. $\beta_i, \overline{p}_i$ and g_i are in Table 1. a_k, b_k, c_k and d_k are in Table 2.

By Lemma 3.3, $j_1 = 2$, $j_2 = 3$.
By assignment problem, the optimal sequence of the jobs is J1 → J4 → J2 → J3 (Table 3).

(1) For the problem $1|p_{[k]} = (\frac{\overline{p}_i k^{\beta_i}}{u_i})^{\theta}, q_{psd}, CONW| \sum_{k=1}^{n} [a_k E_{[k]} + b_k T_{[k]} + c_k \overline{d} + d_k D] + e \sum_{k=1}^{n} g_{[k]} u_{[k]}$, the waiting time, delivery time, actual processing time, completion time and resource are in Table 4.

Table 1 β_i, $\overline{p}_i$ and g_i

J_i	J_1	J_2	J_3	J_4
β_i	−3	−0.5	−0.5	−0.25
$\overline{p}_i$	1	2	3	5
g_i	1	2	2	1

Table 2 a_k, b_k, c_k and d_k

k	1	2	3	4
a_k	3	2	2	1
b_k	2	1	3	8
c_k	1	2	2	1
d_k	1	3	4	2

Table 3 Assignment problem

$i\backslash k$	1	2	3	4
1	3.8729	0.5449	0.1571	0.0442
2	15.4919	12.3288	9.7979	5.6569
3	23.2379	18.4932	14.6969	8.4853
4	19.3649	15.4110	16.1185	10

Table 4 Waiting time, delivery time, actual processing time, completion time and resource

Position	1	2	3	4
$w_{[k]}$	0	0.1644	0.5303	0.7847
$q_{[k]}$	0	0.1644	0.5303	0.7847
$p_{[k]}$	0.1644	0.3659	0.2544	0.5201
$C_{[k]}$	0.1644	0.6947	1.3150	2.0894
$u^*_{[k]}$	2.4662	6.9512	2.2894	2.0801

The due window is $D=[d_1,d_2]=[0.6947, 1.315]$. The total cost is $\sum_{k=1}^{4}[a_k E_{[k]} + b_k T_{[k]} + c_k\overline{d} + d_k D] + e\sum_{k=1}^{4} g_{[k]}u_{[k]} = 54.4695$.

(2) For the problem $1|p_{[k]} = (\frac{\overline{p}_i k^{\beta_i}}{u_i})^{\theta}, q_{psd}, CONW, \sum_{k=1}^{n} g_{[k]}u_{[k]} \leq W|\sum_{k=1}^{n}[a_k E_{[k]} + b_k T_{[k]} + c_k\overline{d} + d_k D]$, the waiting time, delivery time, actual processing time, completion time and resource are in Table 5.

The due window is $D=[d_1,d_2]=[2.2901, 4.3348]$. The total cost is $\sum_{k=1}^{4}[a_k E_{[k]} + b_k T_{[k]} + c_k\overline{d} + d_k D] = 59.8544$.

(3) For the problem $1|p_{[k]} = (\frac{\overline{p}_i k^{\beta_i}}{u_i})^{\theta}, q_{psd}, CONW, \sum_{k=1}^{n}[a_k E_{[k]} + b_k T_{[k]} + c_k\overline{d} + d_k D] \leq M|\sum_{k=1}^{n} g_{[k]}u_{[k]}$, the waiting time, delivery time, actual processing time, completion time and resource are in Table 6.

The due window is $D=[d_1,d_2]=[0.7652, 1.4485]$. The total cost is $\sum_{k=1}^{4} g_{[k]}u_{[k]} = 17.2995$.

Table 5 Waiting time, delivery time, actual processing time, completion time and resource

Position	1	2	3	4
$w_{[k]}$	0	0.5420	1.7481	2.5867
$q_{[k]}$	0	0.5420	1.7481	2.5867
$p_{[k]}$	0.5420	1.2061	0.8386	1.7141
$C_{[k]}$	0.5420	2.2901	4.3348	6.8875
$u^*_{[k]}$	1.3583	3.8285	1.2609	1.1457

Table 6 Waiting time, delivery time, actual processing time, completion time and resource

Position	1	2	3	4
$w_{[k]}$	0	0.1811	0.5841	0.8643
$q_{[k]}$	0	0.1811	0.5841	0.8643
$p_{[k]}$	0.1811	0.4030	0.2802	0.5728
$C_{[k]}$	0.1811	0.7652	1.4485	2.3014
$u^*_{[k]}$	2.3498	6.6231	2.1813	1.9819

7 Conclusion

A single machine scheduling problem with learning effect, delivery time and resource allocation is considered under common due window assignment. The actual processing time is related to normal processing time, job-dependent learning effect and allocated resources. There are three objective functions are considered. The first objective function is to minimize the total costs of earliness, tardiness, start time of window, window size and resource allocation; the second objective function is to minimize the total costs of earliness, tardiness, start time of window and window size subject to $\sum_{k=1}^{n} g_{[k]}u_{[k]} \leq W$; the third objective function is to minimize the cost of resource allocation subject to $\sum_{k=1}^{n}[a_k E_{[k]}+b_k T_{[k]}+c_k\overline{d}+d_k D] \leq M$. The goal is to determine the optimal sequence and resource allocation. All three problems are given polynomial time algorithms. The complexity of the algorithms are $O(n^3)$. In the future, the maintenance activity environment can be considered to expand the research. In addition, the resource allocation scheduling with deterioration effect can also be considered (see Huang [43], Huang et al. [44], Zhang et al. [45], and Lv et al. [46]).

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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